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POSITIVE REGIONS OF COMPUTABLE BINARY RELATIONS

A b s t r a c t. Positive region plays a fundamental role in rough set-based attribute reduction. We study positive regions of decision systems and of binary relations in rough set theory within the framework of reverse mathematics and computability theory. First, we propose the notion of infinite decision systems and prove that the existence of positive regions of decision systems is equivalent to arithmetic comprehension over the weak base theory RCA_0. We also show that the complexity of positive regions of computable decision systems lies exactly in Π^0_2 of the arithmetic hierarchy. Next, we study positive regions of equivalence relations and binary relations. We show that the existence of each of the two positive regions is equivalent to arithmetic comprehension over RCA_0; however, the exact complexity of positive regions of computable equivalence relations lies in Π^0_1 and the exact complexity of positive regions of computable binary relations lies in Σ^0_2 of the arithmetic hierarchy.

1 Introduction

In 1970s, Friedman initiated the program of reverse mathematics with the goal to find minimal axioms to prove theorems in ordinary mathematics [10, 11]. The subject has made

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great developments in the past fifty years, with much work done on various branches of mathematics, for example, real analysis and limited topology, countable algebra and combinatorics, refer to Simpson [25], Dzhafarov and Mummert [7]. For a general introduction of the subject, see also Stillwell [29]. In reverse mathematics, we often take RCA_0 (Recursive Comprehension Axiom with Σ^0_1-induction) as a base theory whose principle axioms include basic axioms that define basic operations on natural numbers, Σ^0_1-induction and Δ^0_1-comprehension, where Σ^0_1-induction is an induction axiom on Σ^0_1 formulas, i.e., $(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1))) \to \forall n \varphi(n)$ with $\varphi(n)$ a Σ^0_1 formula; Δ^0_1-comprehension is a set existence axiom

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

with $\varphi(n)$ a Σ^0_1 formula and $\psi(n)$ a Π^0_1 formula such that $\varphi(n)$ and $\psi(n)$ do not contain X as a free variable. That is, Δ^0_1-comprehension says that sets that can be defined by both Σ^0_1 formulas and Π^0_1 formulas exist.

Over the weak base theory RCA_0, many important theorems of mathematics are equivalent to only five subsystems of second order arithmetic, which are listed as RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1 - \mathrm{CA}_0$ in the order of proof strength, see [7, 25] for more details. For instance, the statement "every vector space over a field has a basis" is equivalent to ACA_0 (Arithmetic Comprehension Axiom with Σ^0_1-induction) over the weak system RCA_0. The system WKL_0 contains RCA_0 and Weak König's Lemma: every infinite binary tree has an infinite path. The system ACA_0 contains RCA_0 and Σ^0_n-comprehension for all n, where Σ^0_n-comprehension ensures the existence of sets defined by Σ^0_n formulas. For more background on arithmetic sets and arithmetic hierarchy, see Ash and Knight's book [1] or books on computability theory [19, 26, 27]. We also review basic notions of computability theory in Section 2.

In 1980s, Pawlak proposed rough set theory to model the vagueness and uncertainty of real world [20]. As a generalization of set theory, rough set theory has been an important tool for data analysis with applications to data mining, knowledge representation, pattern recognition, and so on. Rough set theory deals with various data structures, for instance, information systems, decision systems, generalized rough sets, refer to [13, 15, 16, 17, 21, 22, 24]. For practical applications of rough set theory in decision-making systems or granular computing, see, for instance, [2, 23, 30, 32]. Similar to the study of algebraic structures in the framework of reverse and computable mathematics [3, 4, 5, 6, 8, 9, 12, 14, 18, 28], we initiate the study of the complexity of data structures in rough set theory by methods of reverse mathematics and computability theory.

Positive region approach is a useful tool for attribute reduction in rough set theory. In this paper, we first extend finite decision systems in rough set theory to infinite case and examine the complexity of positive regions of infinite decision systems from the standpoint of reverse mathematics and computability theory. We then proceed to study positive regions of equivalence relations and of binary relations in a similar manner.

1.1 Positive regions of decision systems

We start with basic notions of rough set theory. A *binary relation* R on a possibly infinite universal set U is a set $R \subseteq U \times U$. A binary relation R is an *equivalence relation* if the following axioms hold:

(1) reflexivity: $(\forall x \in U)[(x, x) \in R]$;

(2) symmetry: $(\forall x, y \in U)[(x, y) \in U \to (y, x) \in U]$;

(3) transitivity: $(\forall x, y, z \in U)[(x, y) \in R \wedge (y, z) \in R \to (x, z) \in R]$.

The right R-relative set $r_R(x)$ of x and the left R-relative set $l_R(x)$ of x are respectively defined as

$$r_R(x) = \{y \in U : (x, y) \in R\} \text{ and } l_R(x) = \{y \in U : (y, x) \in R\}.$$

When R is an equivalence relation, $[x]_R = r_R(x) = l_R(x)$ is the equivalence class of x and the quotient set $U/R = \{[x]_R : x \in U\}$ contains all equivalence classes of R.

Definition 1.1. [31] Let R be a binary relation on a possibly infinite universal set U. For a set $X \subseteq U$, the set

$$\underline{R}(X) = \{x \in U : r_R(x) \subseteq X\}$$

is called the lower approximation of X; the set

$$\overline{R}(X) = \{x \in U : r_R(x) \cap X \neq \emptyset\}$$

is called the upper approximation of X.

Pawlak rough sets are ordered pairs of the form $(\underline{R}(X), \overline{R}(X))$ with R an equivalence relation. While the generalized rough sets, proposed by Yao [31], are ordered pairs of the form $(\underline{R}(X), \overline{R}(X))$ with R an arbitrary binary relation.

In classical rough set theory, an *information system* (or *information table*) is a pair (U, A) with U a finite set of objects, A a finite set of attributes. A *decision system* (or *decision table*) is an information system (U, A) such that the set of attributes is a disjoint union of two sets, namely, $A = C \cup D$, with C a finite set of conditional attributes and D a finite set of decisional attributes. We write (U, C, D) for a decision system. Each column of a decision system with an attribute a is viewed as a map

$$a : U \to V_a; x \mapsto a(x).$$

While each row of a decision system is viewed as a decision rule.

Example 1.2. Table 1 illustrates a decision system (U, C, D) with $U = \{1, 2, 3, 4\}$, $C = \{c_1, c_2, c_3\}$ and $D = \{d\}$.

Table 1: Decision system

Objects	c_1	c_2	c_3	d
1	1	0	1	1
2	0	0	1	0
3	1	0	1	1
4	0	0	1	1

For example, c_1 is viewed as a map $c_1 : U \to \{0, 1\}$ such that $c_1(1) = c_1(3) = 1$ and $c_1(2) = c_1(4) = 0$; d is viewed as a map $d : U \to \{0, 1\}$ such that $d(1) = d(3) = d(4) = 1$ and $d(2) = 0$.

Let (U, C, D) be a decision system and let $B \subseteq C \cup D$. We can define an equivalence relation

$$R_B = \{(x, y) \in U : (\forall a \in B)[a(x) = a(y)]\},$$

known as the *indiscernibility relation* with respect to B. The *positive region* of a decision system (U, C, D) is the following set

$$POS_{R_C}(R_D) = \bigcup_{X \in U/R_D} \underline{R_C}(X),$$

where R_C and R_D are indiscernibility relations with respect to C and D, respectively; $U/R_D = \{[x]_{R_D} : x \in U\}$ and $\underline{R_C}(X) = \{y \in U : [y]_{R_C} \subseteq X\}$.

Example 1.3. In Table 1, the equivalence classes of the indiscernibility relation R_C are $\{1, 3\}$ and $\{2, 4\}$; the equivalence classes of the indiscernibility relation R_D are $X_1 = \{1, 3, 4\}$ and $X_2 = \{2\}$. Then the positive region of the decision system is just the set

$$\begin{aligned} POS_{R_C}(R_D) &= \underline{R_C}(X_1) \cup \underline{R_C}(X_2) \\ &= \{y \in U : [y]_{R_C} \subseteq \{1, 3, 4\}\} \cup \{y \in U : [y]_{R_C} \subseteq \{2\}\} \\ &= \{1, 3\}. \end{aligned}$$

By viewing attributes on an infinite set U of objects (encoded by natural numbers) as functions on U, we propose the following infinite decision systems.

Definition 1.4. A decision system is a triple (U, f, g) with U a subset of $\mathbb{N}$, f and g binary functions from $\mathbb{N} \times U$ to $\mathbb{N}$. For each $n \in \mathbb{N}$, $f_n(x) = f(n, x)$ is called a conditional function on U, $g_n(x) = g(n, x)$ is called a decisional function on U.

In an infinite decision system (U, f, g), the binary functions $f, g : \mathbb{N} \times U \to \mathbb{N}$ represent a sequence of conditional functions $\langle f_n : n \in \mathbb{N}\rangle$ and a sequence of decisional functions

$\langle g_n : n \in \mathbb{N}\rangle$, respectively. f and g naturally determine the following *indiscernibility relations* R_f and R_g on U, respectively, where for all $x, y \in U$,

$$(x,y) \in R_f \Leftrightarrow (\forall n \in \mathbb{N})[f_n(x) = f_n(y)] \Leftrightarrow (\forall n \in \mathbb{N})[f(n,x) = f(n,y)],$$
$$(x,y) \in R_g \Leftrightarrow (\forall n \in \mathbb{N})[g_n(x) = g_n(y)] \Leftrightarrow (\forall n \in \mathbb{N})[g(n,x) = g(n,y)].$$

Based on the equivalence relations R_f and R_g, we naturally formalize the positive region of an infinite decision system as follows.

Definition 1.5. The positive region of a decision system (U, f, g) is

$$\begin{aligned} POS_{R_f}(R_g) &= \bigcup_{X \in U/R_g} \underline{R_f}(X) = \bigcup_{y \in U} \underline{R_f}([y]_{R_g}) \\ &= \bigcup_{y \in U} \{x \in U : [x]_{R_f} \subseteq [y]_{R_g}\} \\ &= \bigcup_{y \in U} \{x \in U : (\forall z \in U)[(x,z) \in R_f \to (y,z) \in R_g]\} \\ &= \{x \in U : (\exists y \in U)(\forall z \in U)[(x,z) \in R_f \to (y,z) \in R_g]\}. \\ &= \{x \in U : (\exists y \in U)(\forall k \in \mathbb{N}, z \in U)(\exists l \in \mathbb{N})[g(k,y) = g(k,z) \vee f(l,x) \neq f(l,z)]\}. \end{aligned}$$

In Section 3, we study the complexity of positive regions of infinite decision systems and obtain the following results in reverse and computable mathematics:

(1) The statement "for any decision system (U, f, g), $POS_{R_f}(R_g)$ exists" is equivalent to ACA_0 over RCA_0.

(2) For any decision system $(\mathbb{N}, f, g)$, the positive region $POS_{R_f}(R_g)$ is $\Pi_2^{f,g}$, i.e., Π^0_2 relative to the parameters f and g.

(3) There is a computable decision system $(\mathbb{N}, f, g)$ such that $POS_{R_f}(R_g)$ is Π^0_2-complete.

Our results reveal that the exact complexity of positive regions of computable decision systems lies in Π^0_2.

1.2 Positive regions of binary relations

The positive region $POS_{R_f}(R_g)$ of a decision system (U, f, g) is defined based on two specific equivalence relations, namely, the indiscernibility relations R_f and R_g. This motivates us to define positive regions of general equivalence relations. Given two equivalence relations E, F on a universal set U, we can define the positive region of F over E by setting

$$POS_E(F) = \bigcup_{X \in U/F} \underline{E}(X) = \bigcup_{y \in U} \underline{E}([y]_F) = \bigcup_{y \in U} \{x \in U : [x]_E \subseteq [y]_F\}.$$

By weakening equivalence classes to right relative sets, we can further define positive regions of two arbitrary binary relations as follows.

Definition 1.6. Let E, F be binary relations on a universal set $U \subseteq \mathbb{N}$. The positive region of F over E is the set

$$\begin{aligned} POS_E(F) &= \bigcup_{y\in U} \underline{E}(r_F(y)) = \bigcup_{y\in U} \{x \in U : r_E(x) \subseteq r_F(y)\} \\ &= \{x \in U : (\exists y \in U)(\forall z \in U)[(x,z) \in E \to (y,z) \in F]\}. \end{aligned}$$

The positive region $POS_E(F)$ is a Σ^0_2 set relative to E, F. It consists of those objects $x \in U$ such that the right E-relative set of x is contained in some right F-relative set.

In Section 4, we study positive regions of equivalence relations and of arbitrary binary relations in the context of reverse and computable mathematics, obtaining the following results:

(1) The statement "for any two binary relations E, F, $POS_E(F)$ exists" is equivalent to ACA_0 over RCA_0.

(2) The statement "for any two equivalence relations E, F, $POS_E(F)$ exists" is equivalent to ACA_0 over RCA_0.

(3) There are computable equivalence relations E, F such that $POS_E(F)$ is Π^0_1-complete.

(4) There are computable reflexive and symmetric relations E, F such that $POS_E(F)$ is Σ^0_2-complete.

Both the existence of positive regions of equivalence relations and the existence of positive regions of arbitrary binary relations are equivalent to arithmetic comprehension over RCA_0. However, the exact arithmetic complexity of the two positive regions are different. The exact complexity of positive regions of equivalence relations lies in Π^0_1, whereas the exact complexity of positive regions of arbitrary binary relations lies in Σ^0_2.

The rest of the paper is structured as follows. In Section 2, we review necessary notions of computability theory. In Section 3, we prove reverse mathematics results and computability results on positive regions of decision systems. In Section 4, we prove reverse mathematics results and computability results on positive regions of equivalence relations and of arbitrary binary relations.

2 Preliminaries

We provide basic notions and results of computability theory that will be used to prove main theorems in Sections 3, 4. For more details, refer to books such as [1, 19, 26, 27].

Let $f : A \to \mathbb{N}$ be a function with domain $A \subseteq \mathbb{N}$. f is *partial computable* if there exists a Turing program that outputs the value $f(n)$ for all input $n \in A$. The function f is *total computable* (or simply *computable*) if it is partial computable with domain $A = \mathbb{N}$. A set

X is *computable* if the characteristic function of it is computable. A set X is *computably enumerable* (often abbreviated as c.e.) if it is the domain of a partial computable function. All partial computable functions are often listed as $\varphi_0, \varphi_1, \dots, \varphi_e, \dots$ with φ_e the function computed by the e-th Turing program. All c.e. sets are often listed as $W_0, W_1, \dots, W_e, \dots$ with W_e the domain of φ_e. One important property is that a set X is computable if and only if both X and the complement of X are computably enumerable.

Proposition 2.1. *The Halting set $K = \{e \in \mathbb{N} : \varphi_e(e)$ converges$\}$ is a c.e. set but not computable.*

Arithmetic formulas are formulas containing only number quantifiers. Sets that can be defined by arithmetic formulas are called *arithmetic sets*. According to the appearing of number quantifiers in formulas, we have the following hierarchy.

(1) A formula is Σ_0^0 if it has only bounded number quantifiers.

(2) A formula is Σ_1^0 if it is of the form $\exists x\varphi(x)$ with $\varphi(x)$ a Σ_0^0 formula; a formula is Π_1^0 if it is of the form $\forall x\varphi(x)$ with $\varphi(x)$ a Σ_0^0 formula.

(3) For $n \geq 2$, a formula is Σ_n^0 if it is of the form $\exists x\varphi(x)$ with $\varphi(x)$ a Π_{n-1}^0 formula; a formula is Π_n^0 if it is of the form $\forall x\varphi(x)$ with $\varphi(x)$ a Σ_{n-1}^0 formula.

A set X is Σ_n^0 (resp., Π_n^0) if it can be defined by a Σ_n^0 (resp., Π_n^0) formula. We also use Σ_n^0 (resp., Π_n^0) to denote the class of Σ_n^0 (resp., Π_n^0) sets. In particular, Σ_2^0 sets are those defined by formulas of the form $\exists x\forall y\varphi(x, y)$ with $\varphi(x, y)$ containing only bounded number quantifiers; Π_2^0 sets are those defined by formulas of the form $\forall x\exists y\varphi(x, y)$ with $\varphi(x, y)$ containing only bounded number quantifiers. We also have the relativized arithmetic hierarchy with set parameters. For instance, a set X is $\Sigma_n^{A,B}$ (resp., $\Pi_n^{A,B}$) if it can be defined by a Σ_n^0 (resp., Π_n^0) formula with set parameters A, B.

Example 2.2. The positive region of a binary relation F over a binary relation E

$$POS_E(F) = \{x \in U : (\exists y \in U)(\forall z \in U)[(x, z) \in E \to (y, z) \in F]\}.$$

is a $\Sigma_2^{E,F}$ set.

Let $A, B \subseteq \mathbb{N}$. A is called *many-one reducible* to B if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $n \in A \Leftrightarrow f(n) \in B$ for all n. We now define the hardest Σ_n^0 or Π_n^0 sets under many-one reducibility.

Definition 2.3. Let Γ be a complexity class like Σ_n^0 or Π_n^0. A set A is m-complete Γ (or simply Γ-complete) if $A \in \Gamma$ and any Γ set B is m-reducible to A.

To prove main theorems on the complexity of positive regions of decision systems, equivalence relations and binary relations, we will use the following Γ-complete sets:

(1) The Halting set K is Σ^0_1-complete.

(2) The complement of the Halting set K is Π^0_1-complete.

(3) The index set $\mathrm{Inf} = \{e \in \mathbb{N} : W_e \text{ is infinite}\}$ is Π^0_2-complete.

(4) The index set $\mathrm{Fin} = \{e \in \mathbb{N} : W_e \text{ is finite}\}$ is Σ^0_2-complete.

3 Positive regions of decision systems

Let (U, f, g) be a decision system. The positive region of (U, f, g) is defined as the following $\Sigma^{U,f,g}_3$ set:

$$POS_{R_f}(R_g) = \{x \in U : (\exists y \in U)(\forall k \in \mathbb{N}, z \in U)(\exists l \in \mathbb{N}) \\ [g(k, y) = g(k, z) \vee f(l, x) \neq f(l, z)]\}.$$

We first discover that $POS_{R_f}(R_g)$ can be actually described by a much weak Π^0_2 formula with parameters U, f, g.

Proposition 3.1. *The positive region $POS_{R_f}(R_g)$ of a decision system (U, f, g) is a $\Pi^{U,f,g}_2$ set.*

Proof. Let $x \in U$. If $x \in POS_{R_f}(R_g)$, by definition, there is a $y \in U$ such that for all $k \in \mathbb{N}, z \in U$, either $g(k, y) = g(k, z)$ or $f(l, x) \neq f(l, z)$ for some $l \in \mathbb{N}$. In particular, for $z = x$, we have $f(l, x) = f(l, z)$ for all l; this implies that $g(k, y) = g(k, x)$ for all $k \in \mathbb{N}$. Now x satisfies the following Π^0_2 condition relative to U, f, g:

$$(\forall k \in \mathbb{N}, z \in U)(\exists l \in \mathbb{N})[g(k, x) = g(k, z) \vee f(l, x) \neq f(l, z)].$$

If $x \notin POS_{R_f}(R_g)$, by definition, for all $y \in U$, there are $k_y \in \mathbb{N}$, $z_y \in U$ such that $g(k_y, y) \neq g(k_y, z_y)$ and $f(l, x) = f(l, z_y)$ for all $l \in \mathbb{N}$. Again, for the particular number $y = x$, there are k_x, z_x such that $g(k_x, x) \neq g(k_x, z_x)$ and $f(l, x) = f(l, z_x)$ for all $l \in \mathbb{N}$. That is, x satisfies the following condition

$$(\exists k \in \mathbb{N}, z \in U)(\forall l \in \mathbb{N})[g(k, x) \neq g(k, z) \wedge f(l, x) = f(l, z)].$$

We have seen that $x \in POS_{R_f}(R_g)$ if and only if the $\Pi^{U,f,g}_2$ condition

$$(\forall k \in \mathbb{N}, z \in U)(\exists l \in \mathbb{N})[g(k, x) = g(k, z) \vee f(l, x) \neq f(l, z)]$$

is true. So $POS_{R_f}(R_g)$ is a $\Pi^{U,f,g}_2$ set. □

We obtain the following reverse mathematics results on the existence of positive regions of decision systems.

Theorem 3.2. *The following are equivalent over* RCA_0.

(1) ACA_0.

(2) For any decision system (U, f, g)*, the positive region* $POS_{R_f}(R_g)$ *exists.*

Proof. (1) $\Rightarrow$ (2). By Proposition 3.1, $POS_{R_f}(R_g)$ is a Π^0_2 set. So it exists by arithmetic comprehension.

(2) $\Rightarrow$ (1). Let $\alpha : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. We construct a decision system $(\mathbb{N}, f, g)$ such that the positive region of it encodes

$$range(\alpha) = \{n \in \mathbb{N} : (\exists m \in \mathbb{N})[n = \alpha(m)]\}.$$

Assume that the complement of the range of α is infinite. Define $f, g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as follows:

- For all $k, z \in \mathbb{N}$, let $g(k, z) = 2^k 3^z$. Then $g(k, z) = g(k', z')$ implies that $k = k'$ and $z = z'$ for all $k, z, k', z' \in \mathbb{N}$.
- For all $l, z \in \mathbb{N}$, let
$$f(l, z) := \begin{cases} 1, & \text{if } z = \alpha(l) \\ 0, & \text{otherwise.} \end{cases}$$

f, g are Σ^0_0 functions on $\mathbb{N} \times \mathbb{N}$. So $(\mathbb{N}, f, g)$ is a decision system in RCA_0.

We show that $n \in range(\alpha) \Leftrightarrow n \in POS_{R_f}(R_g)$ for all $n \in \mathbb{N}$.

(i) If $n \in range(\alpha)$, let $n = \alpha(m)$ for some m. For any $k, z \in \mathbb{N}$, if $z \neq n$, then $g(0, z) \neq g(0, n)$ and $f(m, n) = 1 \neq f(m, z) = 0$. By definition, $n \in POS_{R_f}(R_g)$.

(ii) If $n \notin range(\alpha)$, then $f(l, n) = 0$ for all l. Take a number z such that $z \neq n$ and $z \notin range(\alpha)$. Then $g(0, z) \neq g(0, n)$ and $f(l, z) = f(l, n) = 0$ for all l. By definition, $n \notin POS_{R_f}(R_g)$.

Now $range(\alpha)$ exists by Δ^0_1-comprehension with parameter $POS_{R_f}(R_g)$. □

A decision system (U, f, g) is *computable* if U is a computable set, f, g are computable functions from $\mathbb{N} \times U$ to $\mathbb{N}$. In the proof of Theorem 3.2 above, by taking the one-to-one function $\alpha : \mathbb{N} \to \mathbb{N}$ to be a computable function that enumerates the Halting set K, we get the following result in computable mathematics.

Corollary 3.3. *There is a computable decision system* $(\mathbb{N}, f, g)$ *such that the positive region* $POS_{R_f}(R_g)$ *of it computes the Halting set* K.

We further obtain that the Π^0_2 description of $POS_{R_f}(R_g)$ is optimal.

Theorem 3.4. *There is a computable decision system* $(\mathbb{N}, f, g)$ *such that the positive region* $POS_{R_f}(R_g)$ *of it is* Π^0_2*-complete.*

Proof. The positive region of a decision system $(\mathbb{N}, f, g)$ can be written as

$$POS_{R_f}(R_g) = \{x \in \mathbb{N} : (\forall k, z \in \mathbb{N})(\exists l \in \mathbb{N})[g(k,x) = g(k,z) \vee f(l,x) \neq f(l,z)]\}.$$

Let $g(k,z) = 2^k 3^z$ for all k, z. Then for $z \neq z'$, we have $g(k,z) \neq g(k,z')$. For any binary function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, the positive region of the decision system (U, f, g) is

$$POS_{R_f}(R_g) = \{x \in \mathbb{N} : (\forall z \in \mathbb{N})(\exists l \in \mathbb{N})[x = z \vee f(l,x) \neq f(l,z)]\}.$$

The index set $\mathrm{Inf} = \{e \in \mathbb{N} : W_e \text{ is infinite}\}$ is a typical Π^0_2-complete set. For the function $g(k,z) = 2^k 3^z$, we will construct a computable binary function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$, the following requirements are satisfied:

$\mathcal{P}_e$: If $e \in \mathrm{Inf}$, then $2e \in POS_{R_f}(R_g)$.

$\mathcal{Q}_e$: If $e \notin \mathrm{Inf}$, then $2e \notin POS_{R_f}(R_g)$.

Then $POS_{R_f}(R_g)$ will be Π^0_2-complete.

Fix a standard enumeration of the sequence of c.e. sets $\langle W_e : e \in \mathbb{N}\rangle$ as $\langle W_{e,s} : e, s \in \mathbb{N}\rangle$, where $W_e = \bigcup_{s \in \mathbb{N}} W_{e,s}$, $W_{e,0} = \emptyset$, $W_{e,s} \subseteq W_{e,s+1}$ and at most one number $x < s$ enumerates into at most one W_e with $e \leq s$ at every stage $s \geq 1$. We build the binary function f by stages according to the enumeration of c.e. sets. To make f computable, we ensure that f is defined at least on all pairs (l, z) with $l, z < s$ at every stage s.

To satisfy $\mathcal{P}_e$, at every stage s at which some number goes into $W_{e,s}$, for all $z \neq 2e$ with $z \leq 2s+1$, we pick a large number $l_{z,e}$ and define $f(l_{z,e}, z) \neq f(l_{z,e}, 2e)$. To satisfy $\mathcal{Q}_e$, at any stage s at which no number goes into $W_{e,s}$, we pick a large number z_e and define $f(l, 2z_e + 1) = f(l, 2e)$ for all $l \leq 2s+1$; the witness z_e chosen at the stage s will be updated when new numbers enumerate into W_e later. The formal construction of f proceeds as follows.

Construction of f.

Stage 0. Define $f(l,z) = 0$ for all $l, z \leq 3$. Set $M_0 = 3$.

For $s \geq 1$, assume that we have defined $f(l,z)$ for all $l, z \leq M_{s-1}$ with $M_{s-1} \geq 2s+1$ by the end of stage $s-1$. In the following, when we choose a large number x, then x is strictly larger than all z with one of $f(l,z)$, $f(z,l')$ already defined for some l, l'.

Stage $s \geq 1$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ step by step according to the enumeration of W_e $(0 \leq e \leq s)$ at current stage s. There are $s+2$ steps with one Step e for each $0 \leq e \leq s$ and a final Step $s+1$. At the final Step $s+1$, we reserve a large number $M_s > 2s+1$ and ensure that f, g are defined on $[0, M_s] \times [0, M_s]$.

Step e $(0 \leq e \leq s)$**.** There are two cases depending on whether $W_{e,s} = W_{e,s-1}$ or not.

(1) If $W_{e,s} \neq W_{e,s-1}$, then some number enumerates into $W_{e,s}$. For each $z \neq 2e$ with $z \leq M_{s-1}$, check whether $f(l,z) \neq f(l,2e)$ for some $l \leq M_{s-1}$ and act as follows, starting with $z = 0$ and ending with $z = M_{s-1}$.

(1.1) If the answer is yes, do nothing.

(1.2) Otherwise, $f(l,z) = f(l,2e)$ for all $l \leq M_{s-1}$, choose a large number $l_{z,e}$ and define

$$f(l_{z,e}, 2e) = 2^{l_{z,e}}, f(l_{z,e}, z) = 2^{l_{z,e}} + 1.$$

For $z < z' \leq M_{s-1}$, if both $l_{z,e}$ and $l_{z',e}$ are chosen at current stage s, then we have $l_{z,e} < l_{z',e}$.

(2) If $W_{e,s} = W_{e,s-1}$, then no number enumerates into $W_{e,s}$. There are two subcases.

(2.1) If $s = 1$, or $e = s > 1$, or some number enumerates into $W_{e,s-1}$ with $0 \leq e < s$, choose a large number z_e and define

$$f(l, 2z_e + 1) = f(l, 2e)$$

for all $l \leq M_{s-1}$.

Note that $2e \leq 2s < 2s+1 \leq M_{s-1}$. For all $l \leq M_{s-1}$, $f(l,2e)$ has been defined by stage $s-1$.

(2.2) Otherwise, there was a largest stage $s_e < s$ with $W_{s_e-1} = W_{s_e} = \cdots = W_{s-1} = W_s$, we have chosen a large witness z_e at stage s_e and have defined $f(l, 2z_e+1) = f(l, 2e)$ for all $l \leq M_{s-1}$, do nothing.

Step $s+1$**.** Let N_s be the largest number z such that one of $f(l,z)$, $f(z,l')$ has been defined for some l, l' by the end of Step s of current stage s. Set $M_s = 2N_s + 1$.

For all $l, z \leq M_s$ with $f(l,z)$ undefined before, define $f(l,z)$ as follows.

(1) For each $e \leq s$ with $W_{e,s} = W_{e,s-1}$, let z_e be the appointed witness.

(1.1) If z_e was appointed at Step e of current stage s, then $z_e > M_{s-1}$. For any $l \in (M_{s-1}, M_s]$, $f(l, 2z_e+1)$ cannot be defined at any Step e' with $e' \leq s$,

- if $f(l,2e)$ was defined before, define $f(l, 2z_e+1) = f(l,2e)$;
- if $f(l,2e)$ was undefined before, define $f(l, 2z_e+1) = f(l,2e) = 0$.

(1.2) If z_e was appointed at some previous stage $s_e < s$, then for $z = 2e$ or $2z_e + 1$, we may have defined

$$f(l_{z,e'}, 2e') = 2^{l_{z,e'}}, f(l_{z,e'}, z) = 2^{l_{z,e'}} + 1$$

for a large number $l_{z,e'}$ at Step $e' \neq e$ with $e' \leq s$ of current stage s. So we may have defined

$$f(l_{2e,e'}, 2e) = 2^{l_{2e,e'}} + 1 \text{ or } f(l_{2z_e+1,e'}, 2z_e+1) = 2^{l_{2z_e+1,e'}} + 1.$$

If both $f(l_{2e,e'}, 2e)$ and $f(l_{2z_e+1,e'}, 2z_e+1)$ are defined, as $e < z_e$, we have $2e < 2z_e + 1$ and $l_{2e,e'} < l_{2z_e+1,e'}$. We cannot define $f(l, 2e)$ and $f(l, 2z_e+1)$ for the same l at Step e'.

For each $l \in (M_{s-1}, M_s]$, at most one of $f(l, 2e)$ and $f(l, 2z_e + 1)$ was defined before,

* if $f(l, 2e)$ was defined before, define $f(l, 2z_e + 1) = f(l, 2e)$;
* if $f(l, 2z_e + 1)$ was defined before, define $f(l, 2e) = f(l, 2z_e + 1)$;
* otherwise, just define $f(l, 2z_e + 1) = f(l, 2e) = 0$.

(2) If there are other numbers $l, z \leq M_s$ with $f(l, z)$ undefined before, define $f(l, z) = 0$.

This ends the construction of f.

We maintained a partial domain $[0, M_s] \times [0, M_s]$ with $M_s \geq 2(s+1)+1$ for f at each stage s of the construction. For all numbers k, z, the value of $f(k, z)$ has been defined by a stage s with $s > k$ and $s > z$, and $f(k, z)$ is unchanged once it was defined. So f is a computable function on $\mathbb{N} \times \mathbb{N}$. Together with the computable function $g(k, z) = 2^k 3^z$, the triple $(\mathbb{N}, f, g)$ forms a computable decision system whose positive region is the set

$$POS_{R_f}(R_g) = \{x \in \mathbb{N} : (\forall z \in \mathbb{N})(\exists l \in \mathbb{N})[x = z \vee f(l, x) \neq f(l, z)]\}.$$

Finally, we verify that $e \in \text{Inf}$ if and only if $2e \in POS_{R_f}(R_g)$ for all e.

Lemma 3.5. *For all $e \in \mathbb{N}$, $\mathcal{P}_e$ is satisfied.*

Proof. If $e \in \text{Inf}$, then there are infinitely many stages at which some number enumerates into W_e. Let $s_1 < s_2 < \cdots < s_i < \cdots$ be consecutive stages at which some number enumerates into W_e. At stage s_i of the construction, for each $z \leq M_{s_i-1}$, if $z \neq 2e$, then either $f(l, z) \neq f(l, 2e)$ for some $l \leq M_{s_i-1}$ or we have chosen a large number $l_{z,e}$ and have defined

$$f(l_{z,e}, 2e) = 2^{l_{z,e}}, f(l_{z,e}, z) = 2^{l_{z,e}} + 1.$$

Then f satisfies the condition

$$(\forall i \in \mathbb{N})(\forall z \leq M_{s_i-1})(\exists l \in \mathbb{N})[2e = z \vee f(l, 2e) \neq f(l, z)].$$

Note that $M_{s_i-1} \geq 2s_i + 1 > i$ for all i. This implies that

$$(\forall z \in \mathbb{N})(\exists l \in \mathbb{N})[2e = z \vee f(l, 2e) \neq f(l, z)],$$

and we have $2e \in POS_{R_f}(R_g)$. □

Lemma 3.6. *For all $e \in \mathbb{N}$, $\mathcal{Q}_e$ is satisfied.*

Proof. If $e \notin \mathrm{Inf}$, then W_e is finite, let s_e be the least stage $\geq e$ such that $W_{e,s} = W_{e,s_e} = W_{e,s_e-1}$ for all $s \geq s_e$. At stage s_e of the construction, we have chosen a large witness z_e and have defined $f(l, 2z_e+1) = f(l, 2e)$ for all $l \leq M_{s_e}$. At every stage $s > s_e$, as no number enumerates into $W_{e,s}$, we maintained $f(l, 2z_e+1) = f(l, 2e)$ for all $l \leq M_s$ at the final Step $s+1$ of stage s of the construction. Then for the appointed witness z_e, the constructed function f meets the condition

$$(\forall l \in \mathbb{N})[f(l, 2z_e + 1) = f(l, 2e)],$$

and we have $2e \notin POS_{R_f}(R_g)$. □

We have constructed a computable decision system $(\mathbb{N}, f, g)$ such that the positive region $POS_{R_f}(R_g)$ of it is Π^0_2-complete. This completes the proof of Theorem 3.4. □

4 Positive regions of equivalence and binary relations

In this section, we study the complexity of positive regions of equivalence relations and of arbitrary binary relations. For two binary relations E, F on a universal set U, the positive region of F over E is the following $\Sigma_2^{E,F}$ set:

$$POS_E(F) = \{x \in U : (\exists y \in U)(\forall z \in U)[(x, z) \in E \to (y, z) \in F]\}.$$

We first obtain reverse mathematics results about the existence of positive regions of equivalence relations and of binary relations.

Theorem 4.1. *The following are equivalent over* RCA_0.

(1) ACA_0.

(2) For any two binary relations E, F, the positive region $POS_E(F)$ exists.

(3) For any two equivalence relations E, F, the positive region $POS_E(F)$ exists.

Proof. (1) $\Rightarrow$ (2). For two binary relations E, F, $POS_E(F)$ exists by arithmetic comprehension. So (2) holds.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Fix a one-to-one function $\alpha : \mathbb{N} \to \mathbb{N}$. We define two equivalence relations E, F such that $POS_E(F)$ encodes the range of α. Let $F = \{(x, x) : x \in \mathbb{N}\}$. That is, F is the equivalence relation on $\mathbb{N}$ such that the equivalence class of x is the singleton set $\{x\}$ for all $x \in \mathbb{N}$.

Define E as follows: for any x, $(x, x) \in E$; for any x, y with $x \neq y$, $(x, y) \in E$ if and only if one of the following conditions hold:

- $(\exists n, m \in \mathbb{N})[x = 2n \land y = 2m + 1 \land n = \alpha(m)]$;

- $(\exists n, m \in \mathbb{N})[x = 2m + 1 \wedge y = 2n \wedge n = \alpha(m)]$.

E is an equivalence relation such that the equivalence class of $2m+1$ equals $\{2m+1, 2\alpha(m)\}$ for all m.

We show that for all $n \in \mathbb{N}$, $n \in range(\alpha) \Leftrightarrow 2n \notin POS_E(F)$.

(i) If $n \in range(\alpha)$, let $n = \alpha(m)$ for some m, then we have $(2n, 2n) \in E$ and $(2n, 2m+1) \in E$. Note that $(y, z) \notin F$ for $y \neq z$.

For all y,

* if $y = 2n$, there is a $z = 2m + 1$ such that $(2n, z) \in E$ and $(y, z) \notin F$;
* if $y \neq 2n$, there is a $z = 2n \neq y$ such that $(2n, z) \in E$ and $(y, z) \notin F$.

By definition, we have $2n \notin POS_E(F)$.

(ii) If $n \notin range(\alpha)$, then $n \neq \alpha(m)$ for all m. For any z, $(2n, z) \in E$ implies that $z = 2n$, and we have $(2n, 2n) \in F$. So $2n \in POS_E(F)$.

Now $range(\alpha)$ exists by Δ^0_1-comprehension with parameter $POS_E(F)$.

This finishes the proof of Theorem 4.1. □

Although the positive region $POS_E(F)$ of a binary relation F over a binary relation E is defined as a $\Sigma^{E,F}_2$ set, when E, F are restricted to equivalence relations, $POS_E(F)$ can be described by a much easier Π^0_1 formula with parameters E, F.

Proposition 4.2. *For two equivalence relations E, F on a universal set U, the positive region $POS_E(F)$ is a $\Pi^{E,F}_1$ set.*

Proof. Let $x \in U$. If $x \in POS_E(F)$, then there is a $y \in U$ such that for all $z \in U$, $(x, z) \in E$ implies that $(y, z) \in F$. In particular, for this $y \in U$, as $(x, x) \in E$, we have $(y, x) \in F$ and thus $(x, y) \in F$ because of the symmetry of F. Now for any z, if $(x, z) \in E$, then we have both $(x, y) \in F$ and $(y, z) \in F$. By the transitivity of F, we have $(x, z) \in F$. So x satisfies the $\Pi^{E,F}_1$ property:

$$(\forall z \in U)[(x, z) \in E \to (x, z) \in F].$$

If $x \notin POS_E(F)$, then for all $y \in U$, there is a $z \in U$ such that $(x, z) \in E$ and $(y, z) \notin F$. In particular, for $y = x$, there is a $z \in U$ such that $(x, z) \in E$ but $(x, z) \notin F$. That is, x satisfies the $\Sigma^{E,F}_1$ property:

$$(\exists z \in U)[(x, z) \in E \wedge (x, z) \notin F].$$

Then $POS_E(F) = \{x \in U : (\forall z \in U)[(x, z) \in E \to (x, z) \in F]\}$ is a $\Pi^{E,F}_1$ set. □

Theorem 4.1 implies that the Π^0_1 definition is optimal for the positive region of a computable equivalence relation over a computable equivalence relation.

Corollary 4.3. *There are computable equivalence relations* E, F *on* $\mathbb{N}$ *such that* $POS_E(F)$ *is* Π^0_1*-complete.*

Proof. In the proof of (3) $\Rightarrow$ (1) in Theorem 4.1, let $\alpha : \mathbb{N} \to \mathbb{N}$ be a computable function that enumerates the Halting set K. For the constructed equivalence relations E, F, we have $n \notin K \Leftrightarrow 2n \in POS_E(F)$ for all n. Since the complement of K is Π^0_1-complete, so is $POS_E(F)$. □

Unlike positive regions of equivalence relations, the Σ^0_2 definition of positive regions of arbitrary binary relations cannot be weakened.

Theorem 4.4. *There are computable reflexive and symmetric relations* E, F *on* $\mathbb{N}$ *such that* $POS_E(F)$ *is* Σ^0_2*-complete.*

Proof. For binary relations E, F on $\mathbb{N}$, the positive region of F over E is the set

$$POS_E(F) = \{x \in \mathbb{N} : (\exists y \in \mathbb{N})(\forall z \in \mathbb{N})[(x, z) \in E \to (y, z) \in F]\}.$$

Consider the Σ^0_2-complete set $\mathrm{Fin} = \{e \in \mathbb{N} : W_e \text{ is finite}\}$. We construct two computable reflexive and symmetric relations $E, F \subseteq \mathbb{N} \times \mathbb{N}$ meeting the requirements:

$\mathcal{P}_e$: If $e \in \mathrm{Fin}$, then $e \in POS_E(F)$.

$\mathcal{Q}_e$: If $e \notin \mathrm{Fin}$, then $e \notin POS_E(F)$.

Then $POS_E(F) = \mathrm{Fin}$, which is Σ^0_2-complete.

Fix a standard enumeration of c.e. sets as $\langle W_{e,s} : e, s \in \mathbb{N}\rangle$ with $W_{e,0} = \emptyset$ and for $s \geq 1$, at most one element $x < s$ enumerates into at most one W_e at stage s. For $e, s \in \mathbb{N}$, $W_{e,s}$ contains numbers enumerated into W_e by stage s, we have $W_{e,s} \subseteq W_{e,s+1}$ and $W_e = \bigcup_{s\in\mathbb{N}} W_{e,s}$. In the construction, we enumerate approximations E_s, F_s of the desired binary relations E, F such that $E_s \subseteq E_{s+1}$ and $F_s \subseteq F_{s+1}$; finally, set $E := \bigcup_{s\in\mathbb{N}} E_s, F := \bigcup_{s\in\mathbb{N}} F_s$. To ensure that E and F are computable subsets of $\mathbb{N} \times \mathbb{N}$, we only put pairs of the form (x, y) with either $x > s$ or $y > s$ into E, F at every stage $s \geq 1$ of the construction.

To satisfy $\mathcal{P}_e$, at any stage s at which no number enumerates into $W_{e,s}$, we pick a large number y_e and put (y_e, z) into F if (e, z) has already been added into E; the witness y_e will be chosen again if some number enumerates into W_e later. To satisfy $\mathcal{Q}_e$, at every stage s at which some number goes into $W_{e,s}$, we pick a large number z_s and put (e, z_s) into E; however, for each $i \leq s$, (i, z_s) will not be added into F later. The formal construction of E, F proceeds as follows.

Construction of E, F**.**

Stage 0. Let $(x, x) \in E_0$ and $(x, x) \in F_0$ for all $x \in \mathbb{N}$.

For $s \geq 1$, assume that we have obtained E_{s-1} and F_{s-1} by the end of stage $s - 1$. In the construction, when we say choosing a large new number z, then z is strictly larger than all x, y appeared in those pairs (x, y) with $x \neq y$ that have already been enumerated into E or F. For convenience, we use $max(x, y)$ to denote the bigger number of x, y.

Stage $s \geq 1$. Add pairs (x, y) with $x \neq y$ into E_s, F_s according to the enumeration of W_e $(0 \leq e \leq s)$. There are $s+1$ steps, one step for each $0 \leq e \leq s$. Set $E_{-1,s} := E_{s-1}$, $F_{-1,s} := F_{s-1}$. At the end of each Step e with $(0 \leq e \leq s)$, we will obtain $E_{e,s}$ and $F_{e,s}$. Finally, set $E_s := E_{s,s}$, $F_s := F_{s,s}$.

Assume that we have obtained $E_{e-1,s}$ and $F_{e-1,s}$. Step e with $0 \leq e \leq s$ proceeds as follows.

Step e. There are two cases depending on whether $W_{e,s} = W_{e,s-1}$.

(1) If $W_{e,s} = W_{e,s-1}$, then no number goes into W_e at current stage. Check whether there is a stage t with $e \leq t \leq s-1$ and $W_{e,t-1} = W_{e,t} = \cdots = W_{e,s-1} = W_{e,s}$.

(1.1) If the answer is no, then either $s = 1$, or $e = s > 1$, or $W_{e,s} = W_{e,s-1} \neq W_{e,s-2}$ with $0 \leq e < s$. Say that $\mathcal{P}_e$ *requires attention* at stage s. Define E, F as follows.

(a) Choose a large new number $y_e > s$ for the requirement $\mathcal{P}_e$.

(b) Put (y_e, z) and (z, y_e) into $F_{e,s}$ for all z with $(e, z) \in E_{e-1,s}$.

(c) Let $F_{e,s}$ be the union of $F_{e-1,s}$ and the newly added pairs of numbers at Step e. Let $E_{e,s} = E_{e-1,s}$. Declare that $\mathcal{P}_e$ is *satisfied*.

In this case (1.1), we have $(\forall z \in \mathbb{N})[(e, z) \in E_{e,s} \to (y_e, z) \in F_{e,s}]$.

(1.2) If the answer is yes, then just set $E_{e,s} = E_{e-1,s}$, $F_{e,s} = F_{e-1,s}$.

In this case (1.2), there was a largest stage s_e with $e \leq s_e \leq s-1$ and $W_{e,s_e-1} = W_{e,s_e} = W_{e,s}$, we have chosen a large witness y_e at previous stage s_e for $\mathcal{P}_e$. So $\mathcal{P}_e$ has been satisfied since stage s_e.

(2) If $W_{e,s} \neq W_{e,s-1}$, then some number enumerates into W_e at current stage s.

- Enumerate pairs of numbers into E, F as follows:

(a) Choose a large new number $z_s > s$.

(b) Add (e, z_s) and (z_s, e) into $E_{e,s}$.

(c) Let $E_{e,s}$ be the union of $E_{e-1,s}$ and the newly added pairs of numbers at Step e. Let $F_{e,s} = F_{e-1,s}$. Declare that $\mathcal{P}_e$ is *injured*.

We only enumerate pairs (x, y) with $max(x, y)$ large enough into F later. In particular, for the pairs (i, z_s) with $0 \leq i \leq s$, we never enumerate them into F later. So we have $(\forall i \leq s)[(e, z_s) \in E \wedge (i, z_s) \notin F]$.

This ends the construction of E, F.

Let $E = \bigcup_{s \in \mathbb{N}} E_s$ and $F = \bigcup_{s \in \mathbb{N}} F_s$. For any pair $(x, y) \in \mathbb{N} \times \mathbb{N}$, $(x, y) \in E$ if and only if $(x, y) \in E_s$ for some stage $s \leq max(x, y)$, so E is a computable binary relation on

$\mathbb{N}$. Similarly, F is a computable binary relation on $\mathbb{N}$. In the construction, we put (x, x) into E, F for all x, and for $x \neq y$, we put (x, y) and (y, x) into E or F simultaneously. So E, F are reflexive and symmetric relations on $\mathbb{N}$.

Lemma 4.5. *For all $e \in \mathbb{N}$, $\mathcal{P}_e$ is satisfied.*

Proof. When $e \in$ Fin, there is a least stage $s_e \geq e$ such that no new number enumerates into W_e at any stage $s \geq s_e$. That is, $W_{e,s} = W_{e,s_e} = W_{e,s_e-1}$ for all $s \geq s_e$. At stage s_e of the construction, we have picked a large enough number y_e and have added (y_e, z) into F_{s_e} for all z with $(e, z) \in E_{s_e}$. Furthermore, at every stage $t > s_e$, we add no new pairs of the form (e, x) into E for some x. Hence, there is a big enough y_e such that $(e, z) \in E$ implies that $(y_e, z) \in F$ for all z, and we have $e \in POS_E(F)$. □

Lemma 4.6. *For all $e \in \mathbb{N}$, $\mathcal{Q}_e$ is satisfied.*

Proof. When $e \notin$ Fin, there are infinitely many stages at which some number enumerates into W_e. Let $s_0 < s_1 < \cdots < s_i < \cdots$ be all consecutive stages at which some number enumerates into W_e, i.e., $W_{e,s_i} \neq W_{e,s_i-1}$ for all i. At stage s_i of the construction, we have chosen a large number z_{s_i} and have added (e, z_{s_i}) into $E_{s_i} \subseteq E$. At any stage $s > s_i$, we only add pairs of the form (x', y') with $max(x', y') > z_{s_i}$ into F. This implies that all (y, z_{s_i}) with $y \leq s_i$ cannot be added into F at any stage $s > s_i$; in particular, we have $(i, z_{s_i}) \notin F$. Therefore, for all i, there is a large witness z_{s_i} such that $(e, z_{s_i}) \in E$ and $(i, z_{s_i}) \notin F$, that is, $e \notin POS_E(F)$. □

We have constructed two computable reflexive and symmetric relations E, F such that the positive region $POS_E(F)$ is Σ^0_2-complete. This finishes the proof of Theorem 4.4. □

According to Proposition 4.2 above, the reflexive and symmetric relations E, F constructed in Theorem 4.4 cannot be strengthened into equivalence relations.

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