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SELECTION METHOD FOR INQUISITIVE MODAL LOGIC

A b s t r a c t. The selection method is one of the methods to prove that various modal logics have the finite model property. For a given formula that is satisfiable in some model, we select a finite tree-like submodel, while preserving the satisfiability of the observed formula. In this paper, we adapt the selection method for the inquisitive modal logic $\mathrm{InqML}_{\boxplus}$. We first define a tree-like model in the inquisitive setting and show that each satisfiable formula is satisfiable in a tree-like model. Then, using the notions of n-bisimulation and characteristic formulas, we show that $\mathrm{InqML}_{\boxplus}$ has the finite tree model property, i.e., each satisfiable formula is satisfiable in a finite tree-like model. Furthermore, we show analogous results for the inquisitive modal logic $\mathrm{InqML}_{\Rrightarrow}$, and as a consequence we obtain the decidability of $\mathrm{InqML}_{\Rrightarrow}$.

1 Introduction

Inquisitive logic is a generalization of classical logic that can express questions. The language of inquisitive logic is obtained by adding a connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, called the inquisitive disjunction, to the language of classical logic. This connective allows us to write down

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formulas that represent questions. For example, a formula $\varphi \vee\!\!\!\vee \psi$ can be read as "is φ or ψ the case?". Although it is not clear what would it mean to say that a question is true or false, we can talk about whether a question is supported by a given information. Therefore, semantics is not defined using a relation between worlds and formulas, but using a relation between sets of worlds (information states) and formulas.

The inquisitive modal logic $\mathrm{InqML}_{\boxplus}$ is a generalization of standard modal logic, whose language is extended by the modal operator $\boxplus$, called window, which is a natural analogue of the operator $\Box$ in standard modal logic. In the epistemic context, in this logic it is possible to express not only the information that agents have, but also the information in which they are interested. For example, the formula $\boxplus ?\varphi$ can be read as "the agent wonders whether φ", where $?\varphi$ is an abbreviation for $\varphi \vee\!\!\!\vee \neg\varphi$.

The inquisitive modal logic $\mathrm{InqML}_{\boxplus}$ was first introduced in [5], and systematically elaborated in [4]. In [6] and [8] the first results in the model theory of $\mathrm{InqML}_{\boxplus}$ were obtained, such as the notion of bisimulation, an analogue of Ehrenfeucht-Fraïssé theorem, the standard translation from $\mathrm{InqML}_{\boxplus}$ to two-sorted first-order logic and an analogue of van Benthem characterization theorem.

A further contribution to the model theory of $\mathrm{InqML}_{\boxplus}$ we made in [7], where we proved that $\mathrm{InqML}_{\boxplus}$ has the finite model property, and consequently, that it is decidable. The finite model property in modal logic is usually proved by two distinct methods: the selection method and the filtration method (cf. e.g. [1]). Since in [7] we use the latter method, it is natural to consider the former method as well. While filtration identifies worlds using some equivalence relation, selection takes finitely many worlds from the starting model, and disregards other worlds. The selection method is arguably more natural, in sense that it is intuitively clear why we select worlds which we select. Also, it proves a stronger property: the finite tree model property. On the other hand, the resulting tree-like structure can differ greatly from the starting model, while filtration preserves some structural similarity to the starting model. This can be important, in particular when we want to prove the finite model property w.r.t. a class of models satisfying certain constraints. Having in mind these advantages and disadvantages, it is useful to have both methods developed for a given logic.

So, in this paper we present the selection method for the inquisitive modal logic $\mathrm{InqML}_{\boxplus}$. Let φ be a formula that is satisfiable in a model $\mathfrak{M}$. The main idea of the selection method is to obtain a finite tree-like submodel of $\mathfrak{M}$, in which φ is also satisfiable. To obtain such a model, we first define a tree-like model in the inquisitive setting and show that φ is also satisfiable in this model. Since a tree-like model is generally of infinite height, we reduce it to a finite height by using the notion of n-bisimulation. The resulting structure can still be infinite, since it can have infinitely many branches. Finally, in order to get a finite structure, unnecessary branches are removed without affecting the satisfiability of the formula φ. From this we can conclude that each satisfiable formula is satisfiable in a finite tree-like model, i.e., $\mathrm{InqML}_{\boxplus}$ has the finite tree model property. In particular, $\mathrm{InqML}_{\boxplus}$ has the finite model property.

In [2] the logic $\mathrm{InqML}_{\boxplus}$ was generalized to an inquisitive modal logic $\mathrm{InqML}_{\Rrightarrow}$ (also cf. [3]). The language of $\mathrm{InqML}_{\Rrightarrow}$ is based on a binary modal operator $\Rrightarrow$, while the formulas of $\mathrm{InqML}_{\Rrightarrow}$ are interpreted over models in which a downward closure condition is not required. In this paper, we use the selection method to prove that the logic $\mathrm{InqML}_{\Rrightarrow}$ has the finite model property.

In contrast to the logic $\mathrm{InqML}_{\boxplus}$, for which decidability has been shown (cf. [7]), the decidability of the logic $\mathrm{InqML}_{\Rrightarrow}$ has not yet been tested. As an important consequence of the finite model property, we show that the logic $\mathrm{InqML}_{\Rrightarrow}$ is also decidable.

In Section 2 we give an overview of basic definitions and results about $\mathrm{InqML}_{\boxplus}$. In Section 3, we define a tree-like structure and show that $\mathrm{InqML}_{\boxplus}$ has the tree model property. In Section 4, we show that the logic $\mathrm{InqML}_{\boxplus}$ has the finite tree model property by applying the selection method. In Section 5, we use the selection method to show the finite model property and thus the decidability of the logic $\mathrm{InqML}_{\Rrightarrow}$.

2 Preliminaries

Definitions and facts from this section are taken from [4] and [8].

Let W be a non-empty set of elements called the *worlds*. An *information state* (or simply, a *state*) over W is any subset $s \subseteq W$. Furthermore, an *inquisitive state* over W is a non-empty set of information states $\Pi \subseteq \mathcal{P}(W)$ that is downward closed, i.e., $s \in \Pi$ implies $t \in \Pi$ for all $t \subseteq s$.

Definition 2.1. Let P be a set of propositional variables. An **inquisitive modal model** is a triple $\mathfrak{M} = (W, \Sigma, V)$, where W is a set of worlds, $\Sigma : W \to \mathcal{P}(\mathcal{P}(W))$ a function that maps each world w to an inquisitive state $\Sigma(w)$ and $V : \mathrm{P} \to \mathcal{P}(W)$ a function that maps each propositional variable to a set of worlds.

In this paper we will also need an auxiliary structure in which the sets $\Sigma(w)$ are not necessarily downward closed.

Definition 2.2. An **inquisitive modal pseudo-model** is a triple $\mathfrak{M} = (W, \Sigma, V)$, where W and V are defined as in the case of an inquisitive modal model and $\Sigma : W \to \mathcal{P}(\mathcal{P}(W))$ is a function that maps each world to a non-empty set of information states.

The syntax of $\mathrm{InqML}_{\boxplus}$ is given as follows:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \vee\!\!\!\vee \varphi) \mid \boxplus\varphi.$$

We take negation and disjunction as defined connectives in the usual way:

$$\neg\varphi := \varphi \to \bot, \quad \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi).$$

Definition 2.3 (Semantics of $\mathrm{InqML}_{\boxplus}$)**.** Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model or a pseudo-model. The relation of **support** between states in $\mathfrak{M}$ and formulas is defined as follows:

- $\mathfrak{M}, s \models p \iff s \subseteq V(p)$;
- $\mathfrak{M}, s \models \bot \iff s = \emptyset$;
- $\mathfrak{M}, s \models \varphi \wedge \psi \iff \mathfrak{M}, s \models \varphi$ and $\mathfrak{M}, s \models \psi$;
- $\mathfrak{M}, s \models \varphi \to \psi \iff$ for all $t \subseteq s$, if $\mathfrak{M}, t \models \varphi$ then $\mathfrak{M}, t \models \psi$;
- $\mathfrak{M}, s \models \varphi \mathbin{\vee\!\!\!\vee} \psi \iff \mathfrak{M}, s \models \varphi$ or $\mathfrak{M}, s \models \psi$;
- $\mathfrak{M}, s \models \boxplus\varphi \iff$ for all $w \in s$, for all $t \in \Sigma(w)$, $\mathfrak{M}, t \models \varphi$.

A state s is *compatible* with a formula φ if there is $t \subseteq s, t \neq \emptyset$, such that $\mathfrak{M}, t \models \varphi$. The support conditions for the connectives $\neg$ and $\vee$ can then be expressed as follows:

- $\mathfrak{M}, s \models \neg\varphi \iff s$ is not compatible with φ;
- $\mathfrak{M}, s \models \varphi \vee \psi \iff$ for all $t \subseteq s$, $t \neq \emptyset$, we have that t is compatible with φ or t is compatible with ψ.

Remark 2.4. The modality $\Box$ is usually included in the alphabet with the following semantics:

$$\mathfrak{M}, s \models \Box\varphi \iff \text{ for all } w \in s \text{ we have } \mathfrak{M}, \bigcup \Sigma(w) \models \varphi.$$

In [4], however, it is proved that every occurrence of $\Box$ can be paraphrased away using $\boxplus$. The elimination of $\Box$ is not uniform, but depends on a formula, so the question whether $\Box$ is included in the alphabet or not has an impact on the proof theory of $\mathrm{InqML}_{\boxplus}$. Since this paper only deals with the model theory of $\mathrm{InqML}_{\boxplus}$, it does not matter whether $\Box$ is included in the alphabet or not. Therefore, in accordance with our previous work (see [7]), we have not included $\Box$ in the alphabet.

Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model or a pseudo-model. The following properties hold generally in $\mathrm{InqML}_{\boxplus}$:

- *Persistence:* if $\mathfrak{M}, s \models \varphi$ and $t \subseteq s$, then $\mathfrak{M}, t \models \varphi$;
- *Empty state property:* $\mathfrak{M}, \emptyset \models \varphi$.

Definition 2.5. A formula φ is **satisfiable** if there exist an inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$ and $s \subseteq W, s \neq \emptyset$, such that $\mathfrak{M}, s \models \varphi$.

A formula φ is **refutable** if there exist an inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$ and $s \subseteq W$ such that $\mathfrak{M}, s \not\models \varphi$.

A formula φ is *true* at a world w in an inquisitive modal model or a pseudo-model $\mathfrak{M}$, denoted $\mathfrak{M}, w \models \varphi$, if $\mathfrak{M}, \{w\} \models \varphi$.

Definition 2.6. A formula φ is **truth-conditional** if, for all $\mathfrak{M}, s$, the following holds:

$$\mathfrak{M}, s \models \varphi \text{ if and only if for all } w \in s \text{ we have } \mathfrak{M}, w \models \varphi.$$

Remark 2.7. From the support condition for $\boxplus$, it is obvious that every formula of the form $\boxplus\varphi$ is truth-conditional.

Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal pseudo-model. We define an associated inquisitive modal model $\mathfrak{M}^{\downarrow} = (W, \Sigma^{\downarrow}, V)$, where $\Sigma^{\downarrow} : W \to \mathcal{P}(\mathcal{P}(W))$ is defined by $\Sigma^{\downarrow}(w) = \{t \in \mathcal{P}(W) : t \subseteq s \text{ for some } s \in \Sigma(w)\}$, for $w \in W$, i.e., $\Sigma^{\downarrow}(w)$ is obtained by taking the downward closure of the set $\Sigma(w)$.

It is easy to see ([8], Proposition 2.6) that the following proposition holds.

Proposition 2.8. *Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal pseudo-model and $s \subseteq W$. Then for every formula φ we have:*

$$\mathfrak{M}^{\downarrow}, s \models \varphi \text{ if and only if } \mathfrak{M}, s \models \varphi.$$

Notice that the previous proposition implies the following: a formula φ is satisfiable if and only if there exist an inquisitive modal pseudo-model $\mathfrak{M} = (W, \Sigma, V)$ and $s \subseteq W, s \neq \emptyset$, such that $\mathfrak{M}, s \models \varphi$.

3 Tree model property of $\mathrm{InqML}_{\boxplus}$

In this section we define a tree-like model, i.e., a model in which there is a unique path from a particular world (called the root) to every world of the model. At the end of the section, we prove that each satisfiable formula is satisfiable in a tree-like model.

First, we define the notion of a generated submodel.

Definition 3.1. Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model or a pseudo-model. A **submodel** of $\mathfrak{M}$ is an inquisitive modal model or a pseudo-model $\mathfrak{M}' = (W', \Sigma', V')$ if:

- $W' \subseteq W$;
- $\Sigma' = \Sigma \cap (W' \times \mathcal{P}(\mathcal{P}(W')))$;
- $V'(p) = V(p) \cap W'$, for every propositional variable p.

A submodel $\mathfrak{M}'$ of $\mathfrak{M}$ is a **generated submodel** of $\mathfrak{M}$ if the following condition is satisfied:

- if $w \in W'$ and $s \in \Sigma(w)$, then $s \subseteq W'$.

Let $A \subseteq W$. The **submodel generated by** A is the smallest generated submodel of $\mathfrak{M}$ whose set of worlds contains A.

In particular, a **rooted model** is a model generated by a singleton set whose element is called a **root**.

In the following proposition we show that support is preserved on a generated submodel.

Proposition 3.2. *Let $\mathfrak{M}$ be an inquisitive modal model or a pseudo-model and $\mathfrak{M}'$ a generated submodel of $\mathfrak{M}$. Then for every $s \subseteq W'$ and every formula φ we have:*

$$\mathfrak{M}', s \models \varphi \text{ if and only if } \mathfrak{M}, s \models \varphi.$$

Proof. We prove the claim by induction on the complexity of φ.

The cases in which φ is a propositional variable, $\bot$ or a formula of the form $\psi \wedge \chi$, $\psi \to \chi$ or $\psi \vee\!\!\!\vee \chi$ are trivial to prove.

Let $\varphi = \boxplus\psi$ and $s \subseteq W'$. First suppose that $\mathfrak{M}', s \not\models \boxplus\psi$. Then there are $w \in s$ and $t \in \Sigma'(w)$ such that $\mathfrak{M}', t \not\models \psi$. By induction hypothesis we have $\mathfrak{M}, t \not\models \psi$. Since $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$, it follows that $t \in \Sigma(w)$, and therefore $\mathfrak{M}, s \not\models \boxplus\psi$. Conversely, let $\mathfrak{M}, s \not\models \boxplus\psi$. Then there are $w \in s$ and $t \in \Sigma(w)$ such that $\mathfrak{M}, t \not\models \psi$. By induction hypothesis we have $\mathfrak{M}', t \not\models \psi$. Since $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$, $w \in W'$ and $t \in \Sigma(w)$ imply $t \subseteq W'$, so we get $t \in \Sigma'(w)$. Thus, $\mathfrak{M}', s \not\models \boxplus\psi$. □

The next goal is to show that every world in a rooted model is reachable from the root. First, we define the notion of a path.

Definition 3.3. Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model or a pseudo-model and let $w, u \in W$. A **path** from w to u is a finite sequence $(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u)$ consisting of worlds and states such that $s_1 \in \Sigma(w), s_{i+1} \in \Sigma(u_i), u_i \in s_i$ and $u \in s_n$, for all $i = 1, \ldots, n-1$.

Proposition 3.4. *Let $\mathfrak{M} = (W, \Sigma, V)$ be a rooted model with the root w. Then for every $u \in W$ there is a path from w to u.*

Proof. Let $\mathfrak{M}' = (W', \Sigma', V')$ be the submodel of $\mathfrak{M}$, where W' is the set of worlds from W for which there is a path from w. We claim that $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$. Suppose $u \in W'$ and $s \in \Sigma(u)$. We need to show that $s \subseteq W'$. Let $v \in s$. Since $u \in W'$, there is a path from w to u, say $(w, s_1, u_1, s_2, u_2, \ldots, s_n, u)$. Then $(w, s_1, u_1, s_2, u_2, \ldots, s_n, u, s, v)$ is a path from w to v, so we get $v \in W'$. Therefore, $s \subseteq W'$.

Obviously, W' contains w, so since $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$ whose set of worlds contains w and $\mathfrak{M}$ is the model generated by $\{w\}$, we get $\mathfrak{M}' = \mathfrak{M}$. There is therefore a path from w to every world of the model $\mathfrak{M}$. □

The notion of a bounded morphism is crucial for transition from a rooted model to a tree-like model.

For $s \subseteq W$, denote $f(s) = \{f(w) : w \in s\}$.

Definition 3.5. Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models or pseudo-models. A function $f : W \to W'$ is a **bounded morphism** from $\mathfrak{M}$ to $\mathfrak{M}'$ if the following conditions are satisfied:

(at_{bm}) for every propositional variable p we have: $w \in V(p)$ if and only if $f(w) \in V'(p)$;

$(forth_{bm})$ if $s \in \Sigma(w)$, then $f(s) \in \Sigma'(f(w))$;

$(back_{bm})$ if $s' \in \Sigma'(f(w))$, then there is $s \in \Sigma(w)$ such that $f(s) = s'$.

In the following proposition we show that support is preserved when there is a bounded morphism from one model to another.

Proposition 3.6. *Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models or pseudo-models and let $f : W \to W'$ be a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$. Then for every $s \subseteq W$ and every formula φ we have:*

$$\mathfrak{M}, s \models \varphi \text{ if and only if } \mathfrak{M}', f(s) \models \varphi.$$

Proof. We prove the claim by induction on the complexity of φ.

If $\varphi = p$, then the claim follows from condition (at_{bm}). The cases in which φ is $\bot$ or a formula of the form $\psi \wedge \chi$ or $\psi \vee\!\!\!\vee \chi$ are trivial to prove.

Let $\varphi = \psi \to \chi$ and $s \subseteq W$. First suppose that $\mathfrak{M}, s \not\models \psi \to \chi$. Then there is $t \subseteq s$ such that $\mathfrak{M}, t \models \psi$ and $\mathfrak{M}, t \not\models \chi$. By induction hypothesis, we have $\mathfrak{M}', f(t) \models \psi$ and $\mathfrak{M}', f(t) \not\models \chi$. Since $f(t) \subseteq f(s)$, we get $\mathfrak{M}', f(s) \not\models \psi \to \chi$.

Conversely, let $\mathfrak{M}', f(s) \not\models \psi \to \chi$. Then there is $t' \subseteq f(s)$ such that $\mathfrak{M}', t' \models \psi$ and $\mathfrak{M}', t' \not\models \chi$. Since $t' \subseteq f(s)$, there is $t \subseteq s$ such that $f(t) = t'$. By induction hypothesis, we have $\mathfrak{M}, t \models \psi$ and $\mathfrak{M}, t \not\models \chi$. Since $t \subseteq s$, we get $\mathfrak{M}, s \not\models \psi \to \chi$.

Let $\varphi = \boxplus\psi$ and $s \subseteq W$. First suppose that $\mathfrak{M}, s \not\models \boxplus\psi$. Then there are $w \in s$ and $t \in \Sigma(w)$ such that $\mathfrak{M}, t \not\models \psi$. By induction hypothesis we get $\mathfrak{M}', f(t) \not\models \psi$. By condition $(forth_{bm})$, we have $f(t) \in \Sigma'(f(w))$. Then $f(w) \in f(s), f(t) \in \Sigma'(f(w))$ and $\mathfrak{M}', f(t) \not\models \psi$ imply $\mathfrak{M}', f(s) \not\models \boxplus\psi$.

Conversely, let $\mathfrak{M}', f(s) \not\models \boxplus\psi$. Then there are $w' \in f(s)$ and $t' \in \Sigma'(w')$ such that $\mathfrak{M}', t' \not\models \psi$. Since $w' \in f(s)$, there is $w \in s$ such that $f(w) = w'$. Since $t' \in \Sigma'(f(w))$, by condition $(back_{bm})$, there is $t \in \Sigma(w)$ such that $f(t) = t'$. By induction hypothesis we get $\mathfrak{M}, t \not\models \psi$. Now $w \in s, t \in \Sigma(w)$ and $\mathfrak{M}, t \not\models \psi$ imply $\mathfrak{M}, s \not\models \boxplus\psi$. □

Now, we define a tree-like structure in the inquisitive modal logic $\mathrm{InqML}_{\boxplus}$.

Definition 3.7. An inquisitive modal model $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ is a **downward-tree** if the following conditions are satisfied:

(i) there is a unique world $w \in W_S$, called the root, such that there are no $u \in W_S$ and $s \in \Sigma_S(u)$ with $w \in s$;

(ii) for every $u \in W_S$ there are unique worlds $u_1, \ldots, u_{n-1}$ such that $(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u)$ is a path from w to u;

(iii) for any path $(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u)$ there is another path $(w, s_1^*, u_1, s_2^*, u_2, \ldots, u_{n-1}, s_n^*, u)$ such that for every path $(w, t_1, u_1, t_2, u_2, \ldots, u_{n-1}, t_n, u)$ we have $t_i \subseteq s_i^*$ for all $i = 1, \ldots, n$.

In the previous definition, we require that a downward-tree $\mathfrak{M}_S$ is an inquisitive modal model, i.e., that the sets $\Sigma_S(v)$ are downward closed. Thus, if w is the root of $\mathfrak{M}_S$, then a path from w to u is not necessarily unique, as it might be possible to choose different states as part of the desired path. In order to obtain a tree-like structure in this setting, condition (iii) must also be fulfilled, since otherwise we could obtain a structure that we do not want to have as a tree. Let us illustrate this with the following example.

Example 3.8. Let $\mathfrak{M} = (W, \Sigma, V)$ be an inquisitive modal model such that $W = \{w, u_1, u_2, u_3\}$, $\Sigma(w) = \{\emptyset, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_2, u_3\}\}, \Sigma(u_1) = \Sigma(u_2) = \Sigma(u_3) = \{\emptyset\}$, and $V(p) = \emptyset$ for each propositional variable p. Then $\mathfrak{M}$ satisfies conditions (i) and (ii), but not (iii). Indeed, the only paths from w to u_2 are $(w, \{u_2\}, u_2)$, $(w, \{u_1, u_2\}, u_2)$ and $(w, \{u_2, u_3\}, u_2)$, so condition (iii) obviously fails. Therefore, $\mathfrak{M}$ is not a downward-tree as desired.

Remark 3.9. An inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$ can be associated with an underlying Kripke model $\mathfrak{M}_K = (W, \sigma, V)$, where $\sigma : W \to \mathcal{P}(W)$ is defined by $\sigma(u) := \bigcup \Sigma(u)$. If $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ is a downward-tree, then it is obvious that the underlying Kripke model $\mathfrak{M}_{S_K} = (W_S, \sigma_S, V_S)$ is a tree in the usual sense. On the other hand, an inquisitive modal model from the previous example is not a downward-tree, while the underlying Kripke model is a tree.

If we want a unique path from the root to each world, we need a tree-like structure in which the sets $\Sigma_S(u)$ are not necessarily downward closed.

Definition 3.10. An inquisitive modal pseudo-model $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ is a **tree** if the following conditions are satisfied:

(i) there is a unique world $w \in W_S$, called the root, such that there are no $u \in W_S$ and $s \in \Sigma_S(u)$ with $w \in s$;

(ii) for every $u \in W_S$ there is a unique path from w to u.

Remark 3.11. Let $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ be a tree. Then an inquisitive modal model $\mathfrak{M}_S^{\downarrow} = (W_S, \Sigma_S^{\downarrow}, V_S)$ is a downward-tree.

Theorem 3.12. *Let $\mathfrak{M} = (W, \Sigma, V)$ be a rooted model. Then there exists a tree $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ such that $\mathfrak{M}$ is an image of $\mathfrak{M}_S$ by a bounded morphism.*

Proof. Let w be the root of a model $\mathfrak{M}$. We define a structure $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ as follows. Let W_S be the set of all finite sequences, whose elements are worlds and states of the model $\mathfrak{M}$, of the form

$(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u_n)$, where $s_1 \in \Sigma(w), s_{i+1} \in \Sigma(u_i)$, for all $i = 1, \ldots, n-1$, and $u_j \in s_j$, for all $j = 1, \ldots, n$.

Let $w, u_1, \ldots, u_{n-1} \in W$ and $s_1, \ldots, s_n \subseteq W$ be such that $s_1 \in \Sigma(w), s_{i+1} \in \Sigma(u_i)$ and $u_i \in s_i$, for all $i = 1, \ldots, n-1$. Denote by $S_{(w,s_1,u_1,s_2,u_2,\ldots,u_{n-1},s_n)}$ the set

$$S_{(w,s_1,u_1,s_2,u_2,\ldots,u_{n-1},s_n)} := \{(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u_n) : u_n \in s_n\} \subseteq W_S.$$

We define a function Σ_S as follows:

$$s_S \in \Sigma_S(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u_n) \text{ if and only if}$$
$$s_S = S_{(w,s_1,u_1,s_2,u_2,\ldots,u_{n-1},s_n,u_n,s_{n+1})}, \text{ for some } s_{n+1} \in \Sigma(u_n).$$

Furthermore, we define a valuation V_S as follows:

$$(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u_n) \in V_S(p) \text{ if and only if } u_n \in V(p).$$

It is obvious that $\mathfrak{M}_S$ is a tree with the root (w).

Now, we define a function $f : W_S \to W$ with

$$f(w, s_1, u_1, s_2, u_2, \ldots, u_{n-1}, s_n, u_n) = u_n.$$

Using Proposition 3.4, it is easy to show that the function f is surjective. It is also easy to show that the function f satisfies the conditions $(at_{bm}), (forth_{bm})$ and $(back_{bm})$, i.e., f is a bounded morphism.

Therefore, f is the required surjective bounded morphism. □

Finally, we prove that $\text{InqML}_{\boxplus}$ has the tree model property.

Theorem 3.13 (Tree model property of $\text{InqML}_{\boxplus}$)**.** *Each satisfiable formula φ is satisfiable at the root of a downward-tree.*

Proof. Suppose that a formula φ is satisfiable in an inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$. This means that there is some $s \subseteq W, s \neq \emptyset$, such that $\mathfrak{M}, s \models \varphi$. By persistence, for every $u \in s$ we have $\mathfrak{M}, u \models \varphi$. Let $w \in s$. Then $\mathfrak{M}, w \models \varphi$. Let $\mathfrak{M}' = (W', \Sigma', V')$ be a submodel of $\mathfrak{M}$ generated by $\{w\}$, i.e., $\mathfrak{M}'$ is a rooted model with the root w. By Proposition 3.2, it holds $\mathfrak{M}', w \models \varphi$. By the previous theorem there is a tree $\mathfrak{M}'_S = (W'_S, \Sigma'_S, V'_S)$ such that $\mathfrak{M}'$ is the image of $\mathfrak{M}'_S$ by a bounded morphism f with $f((w)) = w$, where (w) is the root of $\mathfrak{M}'_S$. From Proposition 3.6, it follows that $\mathfrak{M}'_S, (w) \models \varphi$. Let $\mathfrak{M}'^{\downarrow}_S = (W'_S, \Sigma'^{\downarrow}_S, V'_S)$. Then $\mathfrak{M}'^{\downarrow}_S$ is a downward-tree and, using Proposition 2.8, we get $\mathfrak{M}'^{\downarrow}_S, (w) \models \varphi$. □

4 Finite tree model property of $\mathrm{InqML}_{\boxplus}$ via selection

In this section, we use results on n-bisimulation in the inquisitive setting to prove the main theorem.

As usual, the modal depth of a formula φ is defined as the maximal number of nested modalities in φ. As mentioned in [6], it is easy to see that the class of all formulas with the modal depth of at most k, assuming that we have finitely many propositional variables, is finite up to logical equivalence.

The notion of inquisitive bisimulation was first defined and studied in [6]. The purpose of bisimulation is to establish a relation between two models in which related worlds satisfy the same propositional variables and have corresponding inquisitive states. In this paper, we only need the notion of n-bisimulation, a finite approximation of bisimulation, which was also defined in [6].

First, for a relation $Z \subseteq W \times W'$, we define the relation $\overline{Z} \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$ as follows: $s\overline{Z}s'$ if and only if

- for every $w \in s$ there is $w' \in s'$ such that wZw';
- for every $w' \in s'$ there is $w \in s$ such that wZw'.

Definition 4.1. Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models and $n \in \mathbb{N}$. A family $(Z_i, i \leq n)$ of non-empty relations $Z_i \subseteq W \times W'$ is called n-**bisimulation** if, for all $i \leq n$, the following conditions are satisfied:

(at_n) if wZ_iw', then for every propositional variable p we have: $w \in V(p)$ if and only if $w' \in V'(p)$;

$(forth_n)$ if $wZ_iw', i > 0$, and $s \in \Sigma(w)$, then there is $s' \in \Sigma'(w')$ such that $s\overline{Z_{i-1}}s'$;

$(back_n)$ if $wZ_iw', i > 0$, and $s' \in \Sigma'(w')$, then there is $s \in \Sigma(w)$ such that $s\overline{Z_{i-1}}s'$.

Worlds w and w' are n-**bisimilar**, which is denoted by $\mathfrak{M}, w \sim_n \mathfrak{M}', w'$, if there is an n-bisimulation $(Z_i, i \leq n)$ such that wZ_nw'.

States s and s' are n-**bisimilar**, which is denoted by $\mathfrak{M}, s \sim_n \mathfrak{M}', s'$, if there is an n-bisimulation $(Z_i, i \leq n)$ such that $s\overline{Z_n}s'$.

Remark 4.2. The notion of n-bisimulation between inquisitive modal models $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ can equivalently be defined as a family of non-empty relations $Z_i \subseteq W \times W' \cup \mathcal{P}(W) \times \mathcal{P}(W')$ (cf. [6] for more details). In this paper, we have chosen to define n-bisimulation as a family of relations defined exclusively on the worlds of the observed models.

Definition 4.3. Let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models and $n \in \mathbb{N}$. The states s and s' are **modally n-equivalent**, which is denoted by $\mathfrak{M}, s \equiv_n \mathfrak{M}', s'$, if for every formula φ with the modal depth of at most n the following holds:

$$\mathfrak{M}, s \models \varphi \text{ if and only if } \mathfrak{M}', s' \models \varphi.$$

In particular, if $\mathfrak{M}, \{w\} \equiv_n \mathfrak{M}', \{w'\}$, then the worlds w and w' are **modally n-equivalent**, which is denoted by $\mathfrak{M}, w \equiv_n \mathfrak{M}', w'$.

It is easy to see that the following holds:

$$\text{if } \mathfrak{M}, s \sim_n \mathfrak{M}', s', \text{ then } \mathfrak{M}, s \equiv_n \mathfrak{M}', s'.$$

The next result about characteristic formulas is proved in [6].

Proposition 4.4. *Let the set of propositional variables be finite, let $n \in \mathbb{N}$, and let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inquisitive modal models. Then for every $w \in W$ there is a formula $\chi^n_{\mathfrak{M},w}$, which is called a characteristic formula, with the modal depth n, such that for every $w' \in W'$ we have:*

$$\mathfrak{M}', w' \models \chi^n_{\mathfrak{M},w} \textit{ if and only if } \mathfrak{M}', w' \sim_n \mathfrak{M}, w.$$

Let $\mathfrak{M}$ be a rooted model with the root w. If a formula φ with the modal depth k is true at w, then the submodel of $\mathfrak{M}$, which only contains worlds that are reached in at most k steps from w, is sufficient to prove that the formula φ is true at w. We prove this in the next lemma. First we define the height of a world and the k-restriction of a rooted model.

Definition 4.5. Let $\mathfrak{M} = (W, \Sigma, V)$ be a rooted model with the root w. The **height** h of worlds of $\mathfrak{M}$ is recursively defined as follows:

- $h(w) = 0$;
- $h(u) = n + 1$ if $u \in s$, where $s \in \Sigma(v)$ and $h(v) = n$, with the condition that the world u has not already been assigned a height lower than $n + 1$.

Definition 4.6. For a natural number k, the k-**restriction** of a rooted model $\mathfrak{M} = (W, \Sigma, V)$, denoted by $\mathfrak{M}|_k = (W|_k, \Sigma|_k, V|_k)$, is the submodel of $\mathfrak{M}$ that contains only worlds whose height is at most k, i.e., $W|_k = \{u \in W : h(u) \leq k\}$, $\Sigma|_k = \Sigma \cap (W|_k \times \mathcal{P}(\mathcal{P}(W|_k)))$, and for every propositional variable p, $V|_k(p) = V(p) \cap W|_k$.

Lemma 4.7. *Let $\mathfrak{M} = (W, \Sigma, V)$ be a rooted model with the root w and let $k \in \mathbb{N}$. Then for every world $u \in W|_k$, we have $\mathfrak{M}|_k, u \sim_l \mathfrak{M}, u$, where $l = k - h(u)$.*

Proof. Let $u \in W|_k$. Let $(Z_i, i \leq l)$ be a family of relations $Z_i \subseteq W|_k \times W$ defined by $Z_i = \{(v,v) : v \in W|_{k-i}\}$ for all $i = 0, 1, \ldots, l$. Relations Z_i are clearly non-empty. Furthermore, it is obvious that $uZ_l u$. We claim that $(Z_i, i \leq l)$ is an l-bisimulation.

Let $vZ_i v$, where $v \in W|_{k-i}$, $0 \leq i \leq l$. Then for every propositional variable p we have $v \in V|_k(p)$ if and only if $v \in V(p)$, so condition (at_l) holds. Let $vZ_i v$, where $v \in W|_{k-i}, 0 < i \leq l$, and let $s \in \Sigma|_k(v)$. Then we have $s \in \Sigma(v)$. Since $s \subseteq W|_{k-i+1}$, we get $s\overline{Z_{i-1}}s$, which proves condition $(forth_l)$. Now let $vZ_i v$, where $v \in W|_{k-i}, 0 < i \leq l$, and let $s \in \Sigma(v)$. Then we have $s \subseteq W|_{k-i+1}$, and therefore $s \subseteq W|_k$. This implies $s \in \Sigma|_k(v)$, so, since $s \subseteq W|_{k-i+1}$, we get $s\overline{Z_{i-1}}s$, which proves condition $(back_l)$.

Hence, $(Z_i, i \leq l)$ is an l-bisimulation. □

We are now ready to prove the finite tree model property of $\mathrm{InqML}_{\boxplus}$

Theorem 4.8 (Finite tree model property of $\mathrm{InqML}_{\boxplus}$)**.** *Each satisfiable formula φ is satisfiable in a finite downward-tree.*

Proof. Let a formula φ be satisfiable in an inquisitive modal model $\mathfrak{M} = (W, \Sigma, V)$. Furthermore, let k be the modal depth of the formula φ and let P be the set of all propositional variables occurring in φ.

By Theorem 3.13 there is a downward-tree $\mathfrak{M}_1$ with the root w_0 such that $\mathfrak{M}_1, w_0 \models \varphi$. Denote with $\mathfrak{M}_2 = \mathfrak{M}_1|_k$. From the previous lemma, it follows that $\mathfrak{M}_2, w_0 \sim_k \mathfrak{M}_1, w_0$. Then we have $\mathfrak{M}_2, w_0 \equiv_k \mathfrak{M}_1, w_0$ and thus $\mathfrak{M}_2, w_0 \models \varphi$.

Now we define sets of states $S_0, \ldots, S_k$. Define S_0 as $S_0 = \{\{w_0\}\}$, and suppose that $S_1, \ldots, S_i, i < k$, have already been defined. Let us now define the set S_{i+1}.

Choose any $u \in s$, where $s \in S_i$. Since P is finite, there are finitely many non-equivalent formulas with the modal depth of at most $k - i$. For every such formula of the form $\boxplus\psi$ such that $\mathfrak{M}_2, u \not\models \boxplus\psi$, choose a state $s_u \in \Sigma_2(u)$ such that $\mathfrak{M}_2, s_u \not\models \psi$. If the state s_u is finite, then we add s_u and all subsets of s_u to S_{i+1}.

If the state s_u is infinite, then we reduce s_u to a finite state in the following way:

Let $s_u = \{v_j : j \in J\}$, where J is an infinite set. Now we select all non-equivalent characteristic formulas of the worlds from s_u. Since there are finitely many non-equivalent formulas with the modal depth of at most $k - i - 1$, there are finitely many such characteristic formulas; without loss of generality, let these formulas be $\chi^{k-i-1}_{\mathfrak{M}_2, v_{j_1}}, \ldots, \chi^{k-i-1}_{\mathfrak{M}_2, v_{j_m}}$. Denote $s'_u = \{v_{j_1}, \ldots, v_{j_m}\} \subseteq s_u$. We claim that $\mathfrak{M}_2, s_u \sim_{k-i-1} \mathfrak{M}_2, s'_u$. If $v \in s_u$, then there exists $v' \in s'_u$ such that $\chi^{k-i-1}_{\mathfrak{M}_2,v} \equiv \chi^{k-i-1}_{\mathfrak{M}_2,v'}$. Thus, since $\mathfrak{M}_2, v' \models \chi^{k-i-1}_{\mathfrak{M}_2,v}$, by Proposition 4.4 we have $\mathfrak{M}_2, v' \sim_{k-i-1} \mathfrak{M}_2, v$. Conversely, for $v' \in s'_u$, we choose the same $v' \in s_u$ and obtain $\mathfrak{M}_2, v' \sim_{k-i-1} \mathfrak{M}_2, v'$. Hence, $\mathfrak{M}_2, s_u \sim_{k-i-1} \mathfrak{M}_2, s'_u$.

An infinite state s_u is thus reduced to a finite state s'_u in such a way that $\mathfrak{M}_2, s_u \sim_{k-i-1} \mathfrak{M}_2, s'_u$. Then we have $\mathfrak{M}_2, s_u \equiv_{k-i-1} \mathfrak{M}_2, s'_u$, so we obtain $\mathfrak{M}_2, s'_u \not\models \psi$. Now we add the state s'_u and all subsets of s'_u to S_{i+1}.

We repeat this selection process for every world u from every state $s \in S_i$, and we define S_{i+1} as the set of all states selected in this way.

Notice that every set $\bigcup S_i$ is finite, and the height of every world of every state belonging to S_i is i.

Let $\mathfrak{M}_3 = (W_3, \Sigma_3, V_3)$ be the inquisitive modal model, where $W_3 = \bigcup_{\substack{u \in s \\ s \in S_0 \cup \dots \cup S_k}} u$, and the functions Σ_3 and V_3 are restrictions of the functions Σ_2 and V_2 of the model $\mathfrak{M}_2 = (W_2, \Sigma_2, V_2)$ to the set W_3. Notice that the model $\mathfrak{M}_3$ is a finite downward-tree.

Let us now prove the following claim.

$(*)$ Let $i \leq k$. Then for every state $s \in S_0 \cup \dots \cup S_i$ we have $\mathfrak{M}_3, s \equiv_{k-i} \mathfrak{M}_2, s$.

Let $i \leq k$ and $s \in S_0 \cup \dots \cup S_i$. Furthermore, let φ be a formula with the modal depth of at most $k - i$. We need to show the following: $\mathfrak{M}_3, s \models \varphi$ if and only if $\mathfrak{M}_2, s \models \varphi$. We prove this by induction on the complexity of φ.

Let $\varphi = p$, where p is a propositional variable. Since the valuation V_3 is a restriction of the valuation V_2 on W_3, the claim immediately holds.

For $\varphi = \bot$, the claim obviously holds.

The cases $\varphi = \psi \wedge \chi$ and $\varphi = \psi \vee\!\!\!\vee \chi$ are trivial.

Let $\varphi = \psi \to \chi$ and suppose $\mathfrak{M}_3, s \not\models \psi \to \chi$, where $s \in S_l$ for some $l \leq i \leq k$. This means that there is $t \subseteq s$ such that $\mathfrak{M}_3, t \models \psi$ and $\mathfrak{M}_3, t \not\models \chi$. Since $t \in S_l \subseteq S_0 \cup \dots \cup S_l$ and the modal depth of the formulas ψ and χ is at most $k - i \leq k - l$, by the induction hypothesis we have $\mathfrak{M}_2, t \models \psi$ and $\mathfrak{M}_2, t \not\models \chi$, i.e., $\mathfrak{M}_2, s \not\models \psi \to \chi$. The converse is proved analogously.

Let $\varphi = \boxplus\psi$ and suppose $\mathfrak{M}_3, s \not\models \boxplus\psi$, where $s \in S_l$ for some $l \leq i < k$. Since the formula $\boxplus\psi$ is truth-conditional, there is $u \in s$ such that $\mathfrak{M}_3, u \not\models \boxplus\psi$. Then there is $t \in S_{l+1} \subseteq W_3$ such that $t \in \Sigma_3(u)$ and $\mathfrak{M}_3, t \not\models \psi$. From $t \in S_{l+1} \subseteq S_0 \cup \dots \cup S_{l+1}$ and the fact that the modal depth of ψ is at most $k - i - 1 \leq k - l - 1$, it now follows, using the induction hypothesis, that $\mathfrak{M}_2, t \not\models \psi$. Since Σ_3 is the restriction of Σ_2 to W_3, we get $t \in \Sigma_2(u)$, so $\mathfrak{M}_2, u \not\models \boxplus\psi$, and thus $\mathfrak{M}_2, s \not\models \boxplus\psi$.

Conversely, suppose $\mathfrak{M}_2, s \not\models \boxplus\psi$, where $s \in S_l$ for some $l \leq i < k$. Then there is $u \in s$ such that $\mathfrak{M}_2, u \not\models \boxplus\psi$. By the definition of the set S_{l+1}, there is $t \in S_{l+1} \subseteq S_0 \cup \dots \cup S_{l+1}$ such that $t \in \Sigma_2(u)$ and $\mathfrak{M}_2, t \not\models \psi$. Since the modal depth of ψ is at most $k - i - 1 \leq k - l - 1$, using the induction hypothesis, it follows that $\mathfrak{M}_3, t \not\models \psi$. Since Σ_3 is the restriction of Σ_2 to W_3, $u \in W_3$ and $t \subseteq W_3$ imply $t \in \Sigma_3(u)$. Thus we get $\mathfrak{M}_3, u \not\models \boxplus\psi$, and therefore $\mathfrak{M}_3, s \not\models \boxplus\psi$.

Hence, the claim $(*)$ is proved.

This implies $\mathfrak{M}_3, w_0 \equiv_k \mathfrak{M}_2, w_0$, so since $\mathfrak{M}_2, w_0 \models \varphi$, it follows that $\mathfrak{M}_3, w_0 \models \varphi$, which proves the theorem. $\square$

5 Selection method for $\mathrm{InqML}_{\Rrightarrow}$

In $\mathrm{InqML}_{\boxplus}$ formulas are interpreted over inquisitive modal models, i.e., downward closed neighborhood models[1]. If we drop the downward closure condition, it is shown in [2] that the modal operator $\boxplus$ is no longer sufficient to have an expressively adequate language. To solve this problem, a binary modal operator $\Rrightarrow$, read "yields", is used instead of $\boxplus$. The resulting inquisitive modal logic $\mathrm{InqML}_{\Rrightarrow}$, which was first introduced in [2], can thus be seen as a generalization of the logic $\mathrm{InqML}_{\boxplus}$, in which formulas are interpreted over neighborhood models that are not necessarily downward closed. For more details on the motivation for $\mathrm{InqML}_{\Rrightarrow}$ see [2] and [3].

Let P be a set of propositional variables.

The syntax of $\mathrm{InqML}_{\Rrightarrow}$ is given as follows:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \vee\!\!\!\vee \varphi) \mid (\varphi \Rrightarrow \varphi).$$

Definition 5.1. An **inhabited neighborhood model** is a triple $\mathfrak{M} = (W, \Sigma, V)$, where W is a set of worlds, $\Sigma : W \to \mathcal{P}(\mathcal{P}(W))$ a function such that for all $w \in W$ we have $\emptyset \notin \Sigma(w)$, and $V : \mathrm{P} \to \mathcal{P}(W)$ a valuation function.

For the sake of simplicity, we assume in the previous definition that for all $w \in W$ we have $\emptyset \notin \Sigma(w)$, i.e., the neighborhoods are non-empty. If we allow empty neighborhoods, we have to add a modal constant to the language to determine whether or not $\emptyset \in \Sigma(w)$ (cf. [3], Section 8).

The support conditions for propositional variables, $\bot$, and the formulas of the form $\varphi \wedge \psi, \varphi \to \psi$ and $\varphi \vee\!\!\!\vee \psi$ are the same as in the logic $\mathrm{InqML}_{\boxplus}$. The only novelty concerns the interpretation of $\Rrightarrow$.

Definition 5.2 (Support for $\Rrightarrow$)**.** Let $\mathfrak{M} = (W, \Sigma, V)$ be an inhabited neighborhood model. The relation of **support** between states in $\mathfrak{M}$ and formulas of the form $\varphi \Rrightarrow \psi$ is defined as follows:

$$\mathfrak{M}, s \models \varphi \Rrightarrow \psi \iff \text{for all } w \in s \text{ and } t \in \Sigma(w), \mathfrak{M}, t \models \varphi \text{ implies } \mathfrak{M}, t \models \psi.$$

A unary modal operator $\boxplus$ is defined as an abbreviation:

$$\boxplus\varphi := \top \Rrightarrow \varphi.$$

Remark 5.3. Note that every formula of the form $\varphi \Rrightarrow \psi$ is truth-conditional.

The *set of declarative formulas* is defined as follows:

$$\alpha ::= p \mid \bot \mid (\alpha \wedge \alpha) \mid (\alpha \to \alpha) \mid (\varphi \Rrightarrow \varphi),$$

[1] A **neighborhood model** is a structure in which each world is associated with a set of neighborhoods, where the **neighborhood** is defined as a set of worlds.

where φ is an arbitrary formula.

The set of declarative formulas is, up to logical equivalence, exactly the set of truth-conditional formulas (cf. [2] and [3]).

Definition 5.4. To every formula φ we can assign a set of declarative formulas, called the **set of resolutions** $\mathcal{R}(\varphi)$, as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ if α is a propositional variable, $\bot$, or a formula $\varphi \Rrightarrow \psi$;
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \to \psi) = \{\gamma_f \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$, where $\gamma_f := \bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha))$;
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

Every formula has a normal form, as the following theorem shows (cf. [2] and [3]).

Theorem 5.5 (Normal form of $\mathrm{InqML}_{\Rrightarrow}$)**.** *Every formula φ is equivalent to the inquisitive disjunction of its resolutions, i.e.,*

$$\varphi \equiv \alpha_1 \vee\!\!\!\vee \ \ldots \ \vee\!\!\!\vee \alpha_n,$$

where $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$.

For other basic properties of $\mathrm{InqML}_{\Rrightarrow}$ see [2] and [3].

Our goal is to implement the selection method for $\mathrm{InqML}_{\Rrightarrow}$. The first step is to show that the logic $\mathrm{InqML}_{\Rrightarrow}$ has the tree model property. So, we define a tree-like structure in $\mathrm{InqML}_{\Rrightarrow}$, by using the notion of path in an inhabited neighborhood model, which is defined in the same way as in Definition 3.3.

Definition 5.6. An inhabited neighborhood model $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ is a **tree** if the following conditions are satisfied:

(i) there is a unique world $w \in W_S$, called the root, such that there are no $u \in W_S$ and $s \in \Sigma_S(u)$ with $w \in s$;

(ii) for every $u \in W_S$ there is a unique path from w to u.

Note that the previous definition is the same as the definition of a tree in $\mathrm{InqML}_{\boxplus}$, with the only difference that in this logic we observe inhabited neighborhood models, while in $\mathrm{InqML}_{\boxplus}$ we observe pseudo-models.

The definitions of generated submodel, rooted model and bounded morphism as well as analogues of Proposition 3.2, Proposition 3.4, Proposition 3.6 and Theorem 3.12 from Section 3 can easily be generalized to $\mathrm{InqML}_{\Rrightarrow}$.

The proof of the following theorem is then carried out using the previous results analogously to the proof of Theorem 3.13.

Theorem 5.7 (Tree model property of $\mathrm{InqML}_{\Rrightarrow}$)**.** *Each satisfiable formula φ is satisfiable at the root of a tree.*

The notion of n-bisimulation in $\mathrm{InqML}_{\Rrightarrow}$ is defined as in $\mathrm{InqML}_{\boxplus}$ (cf. [3]).

It is easy to see that n-bisimilar states support the same formulas with the modal depth[2] of at most n, i.e., if $\mathfrak{M}, s \sim_n \mathfrak{M}', s'$, then $\mathfrak{M}, s \equiv_n \mathfrak{M}', s'$ (cf. [3]).

Let the set of propositional variables be finite, let $n \in \mathbb{N}$, and let $\mathfrak{M} = (W, \Sigma, V)$ and $\mathfrak{M}' = (W', \Sigma', V')$ be inhabited neighborhood models. In [3] it is shown that for every $w \in W$ there is a characteristic formula $\chi^n_{\mathfrak{M},w}$ with the modal depth n, such that for every $w' \in W'$ we have:

$$\mathfrak{M}', w' \models \chi^n_{\mathfrak{M},w} \text{ if and only if } \mathfrak{M}', w' \sim_n \mathfrak{M}, w.$$

Furthermore, an analogue of Lemma 4.7 also holds in this logic.

Using the previous facts, we can adapt the proof of Theorem 4.8 to prove the following theorem.

Theorem 5.8 (Finite tree model property of $\mathrm{InqML}_{\Rrightarrow}$)**.** *Each satisfiable formula φ is satisfiable in a finite tree.*

Proof. Let a formula φ be satisfiable in an inhabited neighborhood model $\mathfrak{M} = (W, \Sigma, V)$. Furthermore, let k be the modal depth of the formula φ and let P be the set of all propositional variables occurring in φ.

By Theorem 5.7 there exists a tree $\mathfrak{M}_1$ with the root w_0 such that $\mathfrak{M}_1, w_0 \models \varphi$. Denote with $\mathfrak{M}_2 = \mathfrak{M}_1|_k$. Now we have $\mathfrak{M}_2, w_0 \sim_k \mathfrak{M}_1, w_0$ and therefore $\mathfrak{M}_2, w_0 \equiv_k \mathfrak{M}_1, w_0$. Hence, $\mathfrak{M}_2, w_0 \models \varphi$.

Now we define sets of states $S_0, \ldots, S_k$. In contrast to the proof of Theorem 4.8, we do not need to require that the sets S_i are downward closed. Let $S_0 = \{\{w_0\}\}$ and suppose that $S_1, \ldots, S_i, i < k$, have already been defined. We define the set S_{i+1}, observing the formulas of the form $\psi \Rrightarrow \chi$ as follows.

Choose any $u \in s$, where $s \in S_i$. The set P is finite, so there are finitely many non-equivalent formulas with the modal depth of at most $k - i$. For every such formula of the form $\psi \Rrightarrow \chi$ such that $\mathfrak{M}_2, u \not\models \psi \Rrightarrow \chi$, choose a state $s_u \in \Sigma_2(u)$ such that $\mathfrak{M}_2, s_u \models \psi$ and $\mathfrak{M}_2, s_u \not\models \chi$. If the state s_u is finite, then we add s_u to S_{i+1}.

If the state s_u is infinite, then we reduce s_u to a finite state s'_u using characteristic formulas in the same way as in the proof of Theorem 4.8, and add the resulting finite state s'_u to S_{i+1}.

As in the proof of Theorem 4.8, we define the inhabited neighborhood model $\mathfrak{M}_3 = (W_3, \Sigma_3, V_3)$, where $W_3 = \bigcup\limits_{\substack{u \in s \\ s \in S_0 \cup \cdots \cup S_k}} u$, and the functions Σ_3 and V_3 are restrictions of the

[2]The modal depth of the formula $\varphi \Rrightarrow \psi$ is the maximum of modal depths of formulas φ and ψ increased by 1.

functions Σ_2 and V_2 of the model $\mathfrak{M}_2 = (W_2, \Sigma_2, V_2)$ to the set W_3. The model $\mathfrak{M}_3$ is a finite tree.

Furthermore, we prove the following claim.

($*$) Let $i \leq k$ and let $T_i := \{t \subseteq W_3 : t \subseteq s \text{ for some } s \in S_i\}$. Then for every state $s \in T_0 \cup \cdots \cup T_i$ we have $\mathfrak{M}_3, s \equiv_{k-i} \mathfrak{M}_2, s$.

Let $i \leq k$ and $s \in T_0 \cup \cdots \cup T_i$, and let φ be a formula with the modal depth of at most $k-i$. By induction on the complexity of φ we prove: $\mathfrak{M}_3, s \models \varphi$ if and only if $\mathfrak{M}_2, s \models \varphi$.

We only consider cases in which φ is the formula of the form $\psi \to \chi$ and $\psi \Rrightarrow \chi$, since other cases are trivial.

Let $\varphi = \psi \to \chi$ and suppose $\mathfrak{M}_3, s \not\models \psi \to \chi$, where $s \in T_l$ for some $l \leq i \leq k$. Then there is $t \subseteq s$ such that $\mathfrak{M}_3, t \models \psi$ and $\mathfrak{M}_3, t \not\models \chi$. Since $t \in T_l \subseteq T_0 \cup \cdots \cup T_l$ and the modal depth of the formulas ψ and χ is at most $k-i \leq k-l$, by the induction hypothesis we have $\mathfrak{M}_2, t \models \psi$ and $\mathfrak{M}_2, t \not\models \chi$, and thus $\mathfrak{M}_2, s \not\models \psi \to \chi$. The converse is proved analogously.

Let $\varphi = \psi \Rrightarrow \chi$ and suppose $\mathfrak{M}_3, s \not\models \psi \Rrightarrow \chi$, where $s \in T_l$ for some $l \leq i < k$. Since the formula $\psi \Rrightarrow \chi$ is truth-conditional, there is $u \in s$ such that $\mathfrak{M}_3, u \not\models \psi \Rrightarrow \chi$. Then there is $t \in S_{l+1} \subseteq W_3$ such that $t \in \Sigma_3(u)$ for which we have $\mathfrak{M}_3, t \models \psi$ and $\mathfrak{M}_3, t \not\models \chi$. Now from $t \in S_{l+1} \subseteq T_{l+1} \subseteq T_0 \cup \cdots \cup T_{l+1}$ and the fact that the modal depth the formulas ψ and χ is at most $k-i-1 \leq k-l-1$, using the induction hypothesis, we obtain $\mathfrak{M}_2, t \models \psi$ and $\mathfrak{M}_2, t \not\models \chi$. Since Σ_3 is the restriction of Σ_2 to W_3, we have $t \in \Sigma_2(u)$. This implies $\mathfrak{M}_2, u \not\models \psi \Rrightarrow \chi$ and thus $\mathfrak{M}_2, s \not\models \psi \Rrightarrow \chi$.

Conversely, suppose $\mathfrak{M}_2, s \not\models \psi \Rrightarrow \chi$, where $s \in T_l$ for some $l \leq i < k$. Then there is $u \in s$ such that $\mathfrak{M}_2, u \not\models \psi \Rrightarrow \chi$. By the definition of the set S_{l+1}, there is $t \in S_{l+1} \subseteq T_{l+1} \subseteq T_0 \cup \cdots \cup T_{l+1}$ such that $t \in \Sigma_2(u)$ for which we have $\mathfrak{M}_2, t \models \psi$ and $\mathfrak{M}_2, t \not\models \chi$. Since the modal depth of the formulas ψ and χ is at most $k-i-1 \leq k-l-1$, using the induction hypothesis, we obtain $\mathfrak{M}_3, t \models \psi$ and $\mathfrak{M}_3, t \not\models \chi$. Since Σ_3 is the restriction of Σ_2 to W_3, from $u \in W_3$ and $t \subseteq W_3$ it follows that $t \in \Sigma_3(u)$. Now we have $\mathfrak{M}_3, u \not\models \psi \Rrightarrow \chi$, and thus $\mathfrak{M}_3, s \not\models \psi \Rrightarrow \chi$.

This proves the claim ($*$).

Finally, we have $\mathfrak{M}_3, w_0 \equiv_k \mathfrak{M}_2, w_0$. Then from $\mathfrak{M}_2, w_0 \models \varphi$, it follows that $\mathfrak{M}_3, w_0 \models \varphi$. □

The next goal is to show that the logic $\mathrm{InqML}_{\Rrightarrow}$ is decidable. To show this, we need the following claim: each refutable formula is refutable in a finite inhabited neighborhood model.

In inquisitive modal logic, for a model $\mathfrak{M} = (W, \Sigma, V)$, a state $s \subseteq W$ and a formula φ, the following generally does not hold:

$$\mathfrak{M}, s \not\models \varphi \text{ if and only if } \mathfrak{M}, s \models \neg\varphi.$$

From this, it follows that proving φ is refutable is not the same as proving that $\neg\varphi$

is satisfiable, so we need to adapt the selection method used for satisfiable formulas to obtain the required claim for refutable formulas as well.

Let us therefore suppose that a formula φ is refutable in an inhabited neighborhood model $\mathfrak{M} = (W, \Sigma, V)$, i.e., there is some $s \subseteq W$ such that $\mathfrak{M}, s \not\models \varphi$.[3] By Theorem 5.5, there are declarative formulas $\alpha_1, \dots, \alpha_n$ such that $\varphi \equiv \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$. Then we have $\mathfrak{M}, s \not\models \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$ and thus $\mathfrak{M}, s \not\models \alpha_i$ for every $i = 1, \dots, n$. Since the formulas α_i are truth-conditional, for every $i = 1, \dots n$, there is $w_i \in s$ such that $\mathfrak{M}, w_i \not\models \alpha_i$.

Now, in exactly the same way as for satisfiable formulas, we apply the selection method. Thus, for every $i = 1, \dots n$, there is a finite tree $\mathfrak{M}^i_S$ with the root w^i_0 such that $\mathfrak{M}^i_S, w^i_0 \not\models \alpha_i$.

We now need the following definition.

Definition 5.9. Let $(\mathfrak{M}_i = (W_i, \Sigma_i, V_i), i \in I)$ be a family of inhabited neighborhood models. An inhabited neighborhood model $\biguplus_{i \in I} \mathfrak{M}_i = (W, \Sigma, V)$ is a **disjoint union** of models $(\mathfrak{M}_i, i \in I)$ if:

- $W = \bigcup_{i \in I} (W_i \times \{i\})$;
- $(s, j) \in \Sigma(w, i)$ if and only if $i = j$ and $s \in \Sigma_i(w)$, where $(s, i) = \{(w, i) : w \in s\} \subseteq W_i \times \{i\}$, for $s \subseteq W_i$;
- for every propositional variable p we have: $(w, i) \in V(p)$ if and only if $w \in V_i(p)$.

By induction on the complexity of a formula φ, it is easy to show that for every $i \in I$ and $s \subseteq W_i$ the following holds:

$$\mathfrak{M}_i, s \models \varphi \text{ if and only if } \biguplus_{i \in I} \mathfrak{M}_i, (s, i) \models \varphi.$$

Let $\mathfrak{M}_S = (W_S, \Sigma_S, V_S)$ be the disjoint union of the trees $\mathfrak{M}^1_S, \dots, \mathfrak{M}^n_S$, i.e., $\mathfrak{M}_S = \biguplus_{i=1}^{n} \mathfrak{M}^i_S$. Notice that $\mathfrak{M}_S$ is a finite inhabited neighborhood model. Put $s' = \{(w^1_0, 1), \dots, (w^n_0, n)\} \subseteq W_S$. Since $\mathfrak{M}^i_S, w^i_0 \not\models \alpha_i$, for every $i = 1, \dots n$, we have $\mathfrak{M}_S, (w^i_0, i) \not\models \alpha_i$. By persistence, we obtain $\mathfrak{M}_S, s' \not\models \alpha_i$ for every $i = 1, \dots, n$. Now we have $\mathfrak{M}_S, s' \not\models \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$, and therefore $\mathfrak{M}_S, s' \not\models \varphi$.

Hence, each refutable formula is refutable in a finite inhabited neighborhood model.

Corollary 5.10. *Inquisitive modal logic* $\text{InqML}_{\Rightarrow}$ *has the finite model property.*

[3]Notice that we cannot conclude from this that there is a world $w \in s$ such that $\mathfrak{M}, w \not\models \varphi$.

Now, using the finite model property of $\mathrm{InqML}_{\Rrightarrow}$, we are able to show that $\mathrm{InqML}_{\Rrightarrow}$ is a decidable logic.

In [3], a Hilbert-style proof system for $\mathrm{InqML}_{\Rrightarrow}$ is defined, and the completeness of that system is proved. Using the finite model property and the completeness we can prove the decidability of $\mathrm{InqML}_{\Rrightarrow}$ by a standard argument (cf. [1], p. 343).

First notice that the following holds:

- the set of validities of $\mathrm{InqML}_{\Rrightarrow}$ is recursively enumerable;
- the set of all (up to isomorphism) finite inhabited neighborhood models is recursively enumerable.

So there is an algorithm that simultaneously enumerates all validities of $\mathrm{InqML}_{\Rrightarrow}$ that are compared with a given formula φ, and all finite inhabited neighborhood models in which the falsifiability of φ is checked.

Since $\mathrm{InqML}_{\Rrightarrow}$ has the finite model property, the algorithm will in finitely many steps either find an inhabited neighborhood model in which φ is refuted or establish that φ is a validity of $\mathrm{InqML}_{\Rrightarrow}$.

Thus, the following theorem holds.

Theorem 5.11 (Decidability of $\mathrm{InqML}_{\Rrightarrow}$)**.** *Inquisitive modal logic* $\mathrm{InqML}_{\Rrightarrow}$ *is decidable.*

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