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A SHORT WAY TO DIRECTED JÓNSSON TERMS

A b s t r a c t. We show that a variety with Jónsson terms $t_1, \dots, t_{n-1}$ has directed Jónsson terms $d_1, \dots, d_{n-1}$ for the same value of the indices, solving a problem raised by Kazda et al. Refined results are obtained for locally finite varieties.

1 Introduction

Congruence distributive varieties can be characterized by means of the existence of Jónsson terms. More recently, Kazda et al. [7] provided another characterization by means of a "directed" variant of Jónsson terms. This novel characterization and generalizations have found applications in computational complexity [1, 2, 9], as well as in classical universal algebra [1, 5, 10, 11].

The construction from [7] provides a rather large number of terms, as evaluated in [7, Section 7], where the problem is asked whether this value can be lowered. We answer the question, proving the quite unexpected result that if congruence distributivity of some variety $\mathcal{V}$ is witnessed by a certain number of Jónsson terms, then $\mathcal{V}$ has the very same

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(or, possibly, smaller) number of *directed* Jónsson terms. This is the best possible result; see [12, Theorem 5.2(i)].

We now recall the basic definitions. Given a variety $\mathcal{V}$, a sequence $t_0, t_1, \dots, t_{n-1}, t_n$ of Jónsson terms is a sequence satisfying the following equations in all algebras in $\mathcal{V}$.

$$x \approx t_i(x, y, x), \qquad \text{for } 0 < i < n, \tag{J1}$$

$$x \approx t_0(x, y, z), \tag{J2}$$

$$t_i(x, x, z) \approx t_{i+1}(x, x, z), \qquad \text{for } i \text{ even, } 0 \leqslant i < n, \tag{J3}$$

$$t_i(x, z, z) \approx t_{i+1}(x, z, z), \qquad \text{for } i \text{ odd, } 0 \leqslant i < n, \tag{J4}$$

$$t_n(x, y, z) \approx z. \tag{J5}$$

A sequence of *directed Jónsson terms* is a sequence of terms satisfying (J1), (J2) and (J5), as well as

$$t_i(x, z, z) \approx t_{i+1}(x, x, z), \qquad \text{for every } i,\ 0 \leqslant i < n. \tag{D}$$

In the case of directed terms there is no distinction between even and odd indices. To the best of our knowledge, directed Jónsson terms first appeared (unnamed) in [15], motivated by [13].

A variety with Jónsson terms (directed Jónsson terms) $t_0, t_1, \dots, t_{n-1}, t_n$ is said to be *n-distributive* (*n-directed distributive*). In both cases, the terms t_0 and t_n are projections, hence the conditions can be reformulated by talking only about $t_1, \dots, t_{n-1}$. For example, a variety has directed Jónsson terms if and only if $\mathcal{V}$ has terms $t_1, \dots, t_{n-1}$ satisfying (J1), (D), $x \approx t_1(x, x, z)$ and $t_{n-1}(x, z, z) \approx z$.

Jónsson [6] proved that a variety $\mathcal{V}$ is congruence distributive if and only if $\mathcal{V}$ has Jónsson terms, for some n. Kazda et al. [7] proved that a variety $\mathcal{V}$ has a sequence of Jónsson terms if and only if $\mathcal{V}$ has a sequence of directed Jónsson terms.

The proofs in [7] provide very long chains of terms, see [7, Section 7]. Here we show that if $\mathcal{V}$ is n-distributive, then $\mathcal{V}$ is n-directed distributive. Notice that, on the other hand, for $n \geqslant 3$, an n-directed distributive variety is not necessarily n distributive; see [12, Theorem 5.2].

The basic idea of our proof is actually very simple. Recall that a *Pixley* term for some variety is a ternary term t such that $x \approx t(x, z, z) \approx t(x, y, x)$ and $t(x, x, z) \approx z$. As well-known, a variety with a Pixley term is 2-distributive, as witnessed by the term $t^\diamondsuit(x, y, z) = t(x, t(x, y, z), z)$. Since, in the special case $n = 2$, n-distributivity is the same as n-directed distributivity, we get that a variety with a Pixley term is 2-directed distributive. Hence one could hope that, for arbitrary n and given terms witnessing n-distributivity, setting

$$t_i^\diamondsuit(x, y, z) = t_i(t_i(x, z, z)t_i(x, y, z), t_i(x, x, z)), \tag{1.1}$$

for even i, could produce a sequence of directed Jónsson terms. Of course, this naïve expectation, as it stands, is wrong, since also the the remaining terms should be modified, and some required identity might be missing.

The above procedure works in the case $n = 4$, by suitably modifying the terms t_1 and t_3, as exemplified in the proof of [12, Proposition 5.4]. For larger n, we do not know a simple way to obtain the result. The best we are able to do, so far, is to perform the substitution (1.1) on one even index at a time, each time suitably modifying the remaining terms. Compare the proof of Claim 4.4 below, but note that we will work with binary terms, as we are going to explain in the next paragraph. Already in the special case $n = 6$ our proof becomes rather involved, see Example 4.6 in the preprint version of this paper[1].

On the other hand, as just mentioned, we can indeed simplify the arguments a little by considering binary terms instead of ternary terms, using the well-known fact that the Jónsson and the directed identities can be written in a way involving just two variables. Thus we are lead to an accurate analysis of binary terms in $\mathcal{V}$ and of some ways of combining them, extending notions and ideas from [4, 7]. The main tool here is Proposition 3.2: clause (v)(d) there corresponds to the substitution (1.1), while clause (v)(b) helps to maintain the remaining identities. However, the most intricate aspect of the proof of the main result is the need of additional identities connecting "distant" terms. These identities are needed in order to obtain the desired directed identities and correspond to dashed arrows in our notation; this means that the connecting terms are not required to satisfy $x \approx t(x, y, x)$. The whole of Proposition 3.2 is devoted to obtain such connections and the main inductive proof is presented in Claim 4.2.

Our methods become slightly easier and our results are somewhat more general in the case of locally finite varieties, or just under the assumption that every algebra in $\mathcal{V}$ generated by 2 elements is finite. In this case the connections tying distant terms are easier to come by and we succeed in dealing with more general configurations associated to paths, as first studied in [8] in a somewhat broader situation. It is an open problem whether these results hold with no finiteness assumption.

2 Preliminaries

We mainly use the notation from [8]. Most of the notions appeared also in [7], sometimes in different terminology. We assume that the reader is familiar with the basic notions of universal algebra, as can be found, e. g., in [14]. Familiarity with [4, 7, 8] would make the paper easier to read. Some useful comments can be found in [12].

Convention 2.1. Throughout this note, we fix a variety $\mathcal{V}$ all whose operations are idempotent. For the sake of brevity, we will simply say that $\mathcal{V}$ is *idempotent*. We work in the free algebra $\mathbf{F}_2(\mathcal{V})$ generated in $\mathcal{V}$ by two elements $\bar{x}$ and $\bar{z}$. Since $\mathcal{V}$ is fixed, we

[1] arXiv:2405.02768v2.

will frequently write $\mathbf{F}_2$ in place of $\mathbf{F}_2(\mathcal{V})$. Elements of F_2 are denoted by $s, s_1, s', r, \dots$ Since $\mathbf{F}_2$ is generated by $\bar{x}$ and $\bar{z}$, to every element $s \in F_2$ there is associated some term $\hat{s}$ depending on the variables x and z such that s is the interpretation of $\hat{s}$ under the assignment $x \mapsto \bar{x}$ and $z \mapsto \bar{z}$. Thus $s = \bar{s}(\bar{x}, \bar{z})$, where $\bar{s}$ is a shorthand for $\hat{s}^{\mathbf{F}_2}$, namely, $\bar{s}(\bar{x}, \bar{z})$ denotes the interpretation of $\hat{s}$ under the above assignment. Notice that $\bar{x}$ is the interpretation of the variable x, hence the notation is consistent. When there is no risk of ambiguity, we will sometimes omit the bars.

The term $\hat{s}$ is not unique; however, since $\mathbf{F}_2$ is free, any other term satisfying the condition is interpreted in the same way inside $\mathcal{V}$.

The Maltsev conditions we consider are defined using ternary terms, but in all proofs we succeeded in working just with binary terms (abstractly, this is due to the fact that the Maltsev conditions we deal with can be expressed using only two variables, hence, say, a variety is congruence distributive if and only if the free algebra $\mathbf{F}_2$ generates a congruence distributive variety. See [4] for further elaboration on this). The main connection among binary and ternary terms is given by the following definition, rephrasing notions from [7, 8].

Definition 2.2. In this note $\approx$ is used in equations with the intended meaning that the equations are always satisfied in $\mathcal{V}$. Also the notions we are going to define depend on $\mathcal{V}$, but we will not explicitly indicate the dependence, since $\mathcal{V}$ will be kept fixed.

We will consider directed graphs whose vertexes are elements of F_2 and whose edges are labeled either as solid or dashed. If $s, r \in F_2$, there is an edge from s to r if and only if there is a ternary $\mathcal{V}$-term t such that the equations $\hat{s}(x, z) \approx t(x, x, z)$ and $t(x, z, z) \approx \hat{r}(x, z)$ are valid in $\mathcal{V}$. We shall denote this as $s \dashrightarrow r$ or $r \dashleftarrow s$. In particular, $s \dashleftarrow r$ means that there is a term t such that $\hat{s}(x, z) \approx t(x, z, z)$ and $t(x, x, z) \approx \hat{r}(x, z)$. Intuitively, an arrow means that the variable x is moved to z in the middle argument of t and in the same direction.

If furthermore t can be chosen in such a way that $x \approx t(x, y, x)$, then the edge from s to r is solid, denoted by $s \to r$ or $r \leftarrow s$. It is convenient to have multiple edges, so that if, say, $s \to r$, then also $s \dashrightarrow r$. As custom, a notation like $s \to s_1 \dashleftarrow s_2 \leftarrow s_3 \dashrightarrow s_4$ means that $s \to s_1$, $s_1 \dashleftarrow s_2$, $s_2 \leftarrow s_3$ and $s_3 \dashrightarrow s_4$ at the same time.

Remark 2.3. As we mentioned, we use a notation quite similar to [8]. In [7], instead, a different notation is used: the relations denoted by F and E in [7, p. 209] correspond to $\dashrightarrow$ and $\to$ in the present notation. In [7] arrows are used to denote transitive closures of the relations F and E. Here we have no use for transitive closure, since we want to deal with the exact length of paths.

Example 2.4. Many Maltsev conditions can be represented by undirected paths from x to z in $\mathbf{F}_2$.

(a) For example, for n even, $\mathcal{V}$ is n-distributive if and only if there are $s_2, s_3, \dots, s_{n-1} \in F_2$ such that $x \to s_2 \leftarrow s_3 \to s_4 \leftarrow s_5 \to \cdots \leftarrow s_{n-1} \to z$. A path representing n-distributivity for n odd is similar, except that we have $\cdots \to s_{n-1} \leftarrow z$ on the right side.

Indeed, the condition is equivalent to the existence of terms $t_1, \dots, t_{n-1}$ such that $x \approx t_i(x, y, x)$, for every $i < n$, $x \approx t_1(x, x, z)$, $t_1(x, z, z) \approx \hat{s}_2(x, z) \approx t_2(x, z, z)$, $t_2(x, x, z) \approx \hat{s}_3(x, z) \approx t_3(x, x, z) \dots$ Notice that if there are terms $t_1, \dots, t_{n-1}$ satisfying the conditions for n-distributivity, then $s_2, s_3, \dots, s_{n-1}$ can be expressed in terms of the t_is.

(b) Similarly, directed distributivity is equivalent to the realizability of $x \to s_2 \to s_3 \to s_4 \to \cdots \to z$.

Paths like $x \to s_2 \to s_3 \to s_4 \to \dots z$ or (undirected) paths like $x \to s_2 \leftarrow s_3 \to s_4 \leftarrow \dots z$ will be called *pattern paths* and a variety $\mathcal{V}$ is said to *realize* the pattern path if $\mathbf{F}_2(\mathcal{V})$ has elements $s_2, s_3, \dots$ such that the relations represented by the path are satisfied. Equivalently, $\mathcal{V}$ has binary terms $\hat{s}_2, \hat{s}_3, \dots, \hat{s}_{n-1}$ and ternary terms $t_1, \dots, t_{n-1}$ such that the equations given by Definition 2.2 hold through $\mathcal{V}$. We will frequently consider additional edges between the vertexes of the above paths.

(c) If we exchange the conditions for i odd and i even in the definition of Jónsson terms, we get a condition which is frequently called the *alvin* condition. See [4, 12] for a discussion. The alvin condition corresponds to the realizability of $x \leftarrow s_2 \to s_3 \leftarrow s_4 \to s_5 \leftarrow s_6 \to s_7 \leftarrow \dots z$.

So far, we have presented conditions involving only solid edges. Any condition represented, as above, by an undirected path from x to z implies congruence distributivity [8, 12]. We now deal with weaker conditions involving dashed edges and which are equivalent to congruence modularity.

(d) If in the alvin condition above we do not ask for the equation $x \approx t_1(x, y, x)$ to be satisfied, we get a sequence of *Gumm terms*. In detail, Gumm terms are terms satisfying the equations in (J3) for i odd, the equations in (J4) for i even, the equations (J2) and (J5) and the equations in (J1) for $1 < i$. The existence of Gumm terms corresponds to the realizability of $x \dashleftarrow s_2 \to s_3 \leftarrow s_4 \to s_5 \leftarrow s_6 \to s_7 \leftarrow \dots z$.

The definition of Gumm terms is not uniform in the literature, see [12, Remark 7.2] for a discussion.

(e) For n even, $n \geqslant 4$, *defective Gumm terms*, introduced in [3] in different terminology, correspond to a path of the form $x \dashleftarrow s_2 \to s_3 \leftarrow s_4 \to s_5 \leftarrow s_6 \to s_7 \leftarrow \cdots \to s_{n-5} \leftarrow s_{n-4} \to s_{n-3} \leftarrow s_{n-2} \to s_{n-1} \dashleftarrow z$. See [12] for further details.

(f) Finally, *directed Gumm terms* [7] correspond to the realizability of $x \to s_2 \to s_3 \to s_4 \to \cdots \to s_{n-3} \to s_{n-2} \to s_{n-1} \dashleftarrow z$.

So far, we have dealt with conditions implying congruence distributivity or at least congruence modularity. It is not the case that every condition involving some path from x to z does imply congruence modularity. In fact, if a dashed right arrow is present, the resulting condition is trivially satisfied by every variety [8]. Intermediate situations might occur.

(g) As a condition which we will use only marginally here, $\mathcal{V}$ is n-permutable if and only if $x \dashleftarrow s_2 \dashleftarrow s_3 \dashleftarrow \cdots \dashleftarrow s_{n-1} \dashleftarrow z$ can be realized. Recall that, for $n \geqslant 4$, n-permutability does not imply congruence modularity.

(h) In examples (d) - (f) above we have taken conditions implying congruence distributivity and we have changed some solid edges to dashed, getting conditions implying congruence modularity. The procedure applies only when left-oriented edges on the two borders are changed. See [12, Section 8].

For example, Polin variety realizes $x \leftarrow s_2 \dashleftarrow s_3 \leftarrow z$ and $x \rightarrow s_2 \dashleftarrow s_3 \rightarrow z$ [12, Remark 10.11], but Polin variety is not congruence modular.

The correspondences described in the present example can be further refined, but we shall not need this here. See [4, 8, 12] for more details and for further Maltsev conditions expressible in the above fashion. Notice that here we have shifted the indices of the terms s_i, in comparison with [8, Section 3.2].

3 A useful proposition

Remark 3.1. The key to our proofs is to nest the terms giving the relevant conditions. While the terms are ternary, it will be almost everywhere sufficient to deal with binary terms. In other words, we need to combine terms associated to elements of F_2. Recalling Convention 2.1, if $s, s_1, s_2 \in F_2$, then $r = \bar{s}(s_1, s_2)$ is the element of F_2 obtained by interpreting the term $\hat{s}$ under the assignment $x \mapsto s_1$, $z \mapsto s_2$. Thus a term $\hat{r}$ corresponding to r is given by $\hat{r}(x, y) = \hat{s}(\hat{s}_1(x, y), \hat{s}_2(x, y))$. Some subtle properties of the above way of generating elements of F_2 are listed in the next proposition. In particular, item (v)(d) below will allow us to reverse some arrows, and item (viii) will give us the possibility of obtaining relations involving new terms not appearing in the assumptions.

For every $s \in F_2$ and $p \geqslant 0$, we define $s^{(p)}$ inductively by $s^{(0)} = s$ and $s^{(p+1)} = \bar{s}(x, s^{(p)})$.

Proposition 3.2. *Assume that $\mathcal{V}$ is a variety all whose operations are idempotent; let $s, s_1, s_1', \dots, r, r_1, \dots \in F_2$ and assume the above notation and definitions. Then the following statements hold.*

- *(i) $s \rightarrow s$, for every s, that is, $\rightarrow$ is reflexive.*
- *(ii) $x \dashrightarrow z$. More generally, $x \dashrightarrow s$ and $s \dashrightarrow z$, for every s.*
- *(iii) The binary relations $\rightarrow$, $\leftarrow$, $\dashrightarrow$ and $\dashleftarrow$ are compatible in $\mathbf{F}_2$.*
- *(iv) More generally, assume that t is an m-ary term, $I \subseteq \{1, \dots, m\}$ and $t(x_1, \dots, x_m) \approx x$ is an identity valid in $\mathcal{V}$ when $x_i = x$, for every $i \in I$. If $s_i \rightarrow s_i'$, for every $i \in I$, and $s_i \dashrightarrow s_i'$ for the remaining indices, $i \leqslant m$, then $t(s_1, \dots, s_m) \rightarrow t(s_1', \dots, s_m')$.*

(v) (a) *If* $s \dashrightarrow r$, $s' \dashrightarrow r'$, $s' \dashrightarrow r''$ *and* $s'' \dashrightarrow r''$, *then* $\bar{s}(s', s'') \dashrightarrow \bar{r}(r', r'')$. *(At first glance, the condition* $s' \dashrightarrow r''$ *might appear spurious, but it is necessary, see Example 3.3(d) below.)*

(b) *If* $s \to r$, $s' \to r'$, $s' \dashrightarrow r''$ *(notice that a dashed arrow is sufficient here) and* $s'' \to r''$, *then* $\bar{s}(s', s'') \to \bar{r}(r', r'')$.

(c) *If* $s \dashleftarrow r$ *(notice the reversed arrow),* $s' \dashrightarrow r'$, $s'' \dashrightarrow r'$ *and* $s'' \dashrightarrow r''$, *then* $\bar{s}(s', s'') \dashrightarrow \bar{r}(r', r'')$.

(d) *If* $s \leftarrow r$, $s' \to r'$, $s'' \dashrightarrow r'$ *and* $s'' \to r''$, *then* $\bar{s}(s', s'') \to \bar{r}(r', r'')$.

(vi) (a) *If* $s \dashrightarrow r$ *and* $s' \dashrightarrow r'$, *then* $\bar{s}(s', z) \dashrightarrow \bar{r}(r', z)$ *and* $\bar{s}(x, s') \dashrightarrow \bar{r}(x, r')$.

(b) *If* $s \to r$ *and* $s' \to r'$, *then* $\bar{s}(s', z) \to \bar{r}(r', z)$ *and* $\bar{s}(x, s') \to \bar{r}(x, r')$.

(c) *In particular, by induction, if* $s \dashrightarrow r$, *then* $s^{(p]} \dashrightarrow r^{(p]}$, *for every* p. *If* $s \to r$, *then* $s^{(p]} \to r^{(p]}$, *for every* p.

(vii) (a) *If* $s \to z$ *and* $s'' \to z$, *then* $\bar{s}(s', s'') \to z$, *for every* $s' \in F_2$.

If $s \leftarrow z$, $s'' \leftarrow z$ *and* $s' \dashrightarrow s''$, *then* $\bar{s}(s', s'') \leftarrow z$.

If $s \dashleftarrow z$, $s'' \dashleftarrow z$ *and* $s' \dashrightarrow s''$, *then* $\bar{s}(s', s'') \dashleftarrow z$.

(b) *If* $x \to s$ *and* $x \to s''$, *then* $x \to \bar{s}(s'', s')$, *for every* $s' \in F_2$.

If $x \leftarrow s$, $x \leftarrow s''$ *and* $s'' \dashrightarrow s'$, *then* $x \leftarrow \bar{s}(s'', s')$.

(viii) *If* $s \dashrightarrow s'$ *and* $s \dashrightarrow s''$, *then* $s \dashrightarrow \bar{r}(s', s'')$, *for every* $r \in F_2$.

If $s' \dashrightarrow s$ *and* $s'' \dashrightarrow s$, *then* $\bar{r}(s', s'') \dashrightarrow s$, *for every* $r \in F_2$.

(ix) *All the statements in (iv), (v) and (vi) hold true if we reverse simultaneously the arrows everywhere.*

Proof. (i) Take $t(x, y, z) = \hat{s}(x, z)$. The edge is solid, since all terms of $\mathcal{V}$ are idempotent, because we assume that all the operations of $\mathcal{V}$ are idempotent.

(ii) To prove the first statement, use the projection onto the second component $t(x, y, z) = y$. To prove $x \dashrightarrow s$, take $t(x, y, z) = \hat{s}(x, y)$, again using idempotence. To prove $s \dashrightarrow z$, take $t(x, y, z) = \hat{s}(y, z)$.

(iii) Let $s_1 \to s'_1$, $s_2 \to s'_2$, $\dots$ be witnessed by $\hat{s}_1(x, z) \approx t_1(x, x, z)$, $t_1(x, z, z) \approx \hat{s}'_1(x, z)$, $\hat{s}_2(x, z) \approx t_2(x, x, z)$, $t_2(x, z, z) \approx \hat{s}'_2(x, z)$, $\dots$ If t is a $\mathcal{V}$-term, then $\bar{t}(s_1, s_2, \dots) \to \bar{t}(s'_1, s'_2, \dots)$ is witnessed by the term $t^{\diamondsuit}(x, y, z) = t(t_1(x, y, z), t_2(x, y, z), \dots)$. As in Remark 3.1, $\bar{t}(s_1, s_2, \dots)$ denotes the interpretation of the term t under the assignment $x_i \mapsto s_i$, where the x_i's are the variables occurring in t. The arrow is solid since t is idempotent.

The case of $\dashrightarrow$ is similar. The relations $\leftarrow$ and $\dashleftarrow$ are the converses of $\to$ and $\dashrightarrow$, hence the conclusion follows from the above arguments.

(iv) Following the proof of (iii), we have $t_i(x, y, x) \approx x$, for every $i \in I$, since the arrow in $s_i \to s'_i$ is solid. Hence $t^\diamondsuit(x, y, x) \approx x$, by the assumption on t.

(v)(a) By the assumption $s \dashrightarrow r$, $\hat{s}(x, z) \approx t(x, x, z)$ and $t(x, z, z) \approx \hat{r}(x, z)$ for some term t. Then $\bar{s}(s', s'') = \bar{t}(s', s', s'') \dashrightarrow \bar{t}(r', r'', r'') = \bar{r}(r', r'')$, by (iii).

(v)(b) is proved in the same way, using (iv).

(v)(c) is similar to (a). Here the assumption is $\hat{s}(x, z) \approx t(x, z, z)$ and $t(x, x, z) \approx \hat{r}(x, z)$ for some term t. Then $\bar{s}(s', s'') = \bar{t}(s', s'', s'') \dashrightarrow \bar{t}(r', r', r'') = \bar{r}(r', r'')$, by (iii).

(v)(d) is proved as (v)(c), using again (iv).

(vi) Clauses (a)(b) are special cases of (v)(a)(b), by (i) and (ii). Then (c) follows by induction.

(vii)(a) The element $\bar{z} \in F_2$ corresponds to the term $\hat{p}_2$, the projection onto the second component. If we write $s \to \bar{z}$ as $s \to p_2$, then $\bar{s}(s', s'') \to \bar{p}_2(s', \bar{z}) = \bar{z}$, by (i), (ii) and (v)(b). To prove the second line, $s \leftarrow \bar{z}$ means $p_2 \to s$, thus $\bar{z} = \bar{p}_2(s', \bar{z}) \to \bar{s}(s', s'')$. Item (vii)(b) is proved in a dual way.

(viii) Since r is idempotent, $s = \bar{r}(s, s) \dashrightarrow \bar{r}(s', s'')$, by (iii). The second statement is proved in a similar way.

(ix) can be proved by repeating the above arguments. However, there is no need of doing this. Just recall that, say, $s \dashrightarrow r$ is the same as $r \dashleftarrow s$. Hence (ix) follows from (iv) - (vi) just by relabeling the elements in the formulas. □

Every argument in the proof of Proposition 3.2 can be translated expressing it in function of the relevant ternary terms. We will exemplify this aspect in a simple case in Example 3.3. A more elaborate situation is described in Example 4.6 in the preprint version of this paper arXiv:2405.02768v2.

Example 3.3. Many items in Proposition 3.2 furnish a compact way for describing widely used techniques. The following examples are basic; more involved applications will be provided in the following sections.

(a) A *Pixley* term is a ternary term t such that $x \approx t(x, z, z) \approx t(x, y, x)$ and $t(x, x, z) \approx z$. This is equivalent to $\bar{x} \leftarrow \bar{z}$, according to Definition 2.2.

As we mentioned in the proof of Proposition 3.2(vii)(a), $\bar{z}$ corresponds to the the projection $\hat{p}_2$ onto the second component; similarly, $\bar{x}$ corresponds to the the projection $\hat{p}_1$ onto the first component, so that $\bar{x} \leftarrow \bar{z}$ can be written as $p_1 \leftarrow p_2$, equivalently, $p_2 \to p_1$.

By Proposition 3.2(i), (v)(d) we get $\bar{x} = \bar{p}_1(\bar{x}, \bar{z}) \to \bar{p}_2(\bar{x}, \bar{z}) = \bar{z}$, since $\bar{z} \to \bar{x}$, hence $\bar{z} \dashrightarrow \bar{x}$ (what is relevant in this example is that the $\bar{z}$ in $\bar{p}_1(\bar{x}, \bar{z})$ is connected by $\dashrightarrow$ to the $\bar{x}$ in $\bar{p}_2(\bar{x}, \bar{z})$).

Thus $\bar{x} \to \bar{z}$, that is, 2-distributivity.

The above argument provides a proof of the fact that a variety with a Pixley term is 2-distributive, that is, has a majority term. Of course, a direct proof using the ternary Pixley term is easy; see the introduction. However, arguments similar to (a) will be very helpful when dealing with more involved situations in which certain conclusions are much more difficult to obtain dealing directly with ternary terms.

(b) On the other hand, a variety with a Pixley term is congruence permutable (= 2-permutable), since $\bar{x} \leftarrow \bar{z}$ implies $\bar{x} \dashleftarrow \bar{z}$. Compare Example 2.4(g).

(c) As well-known, a variety $\mathcal{V}$ has a Pixley term if and only if $\mathcal{V}$ is both congruence permutable and 2-distributive. A proof for necessity has been given above in (a) and (b), using Proposition 3.2.

Conversely, $\mathcal{V}$ is 2-distributive if $\bar{x} \rightarrow \bar{z}$, that is, $p_1 \rightarrow p_2$. $\mathcal{V}$ is congruence permutable if $\bar{x} \dashleftarrow \bar{z}$, that is $\bar{z} \dashrightarrow \bar{x}$. If both properties hold, then $\bar{z} = \bar{p}_1(\bar{z}, \bar{x}) \rightarrow \bar{p}_2(\bar{z}, \bar{x}) = \bar{x}$, by Proposition 3.2(i), (v)(b), taking $s = p_1$, $r = p_2$, $s' = r' = \bar{z}$ and $s'' = r'' = \bar{x}$.

Explicitly, if j is a Jónsson term for 2-distributivity, that is, a majority term, and p is a Maltsev term for congruence permutability, then we need to connect $\bar{z} = \bar{p}_1(\bar{z}, \bar{x}) = \bar{j}(\bar{z}, \bar{z}, \bar{x})$ with $\bar{j}(\bar{z}, \bar{x}, \bar{x}) = \bar{p}_2(\bar{z}, \bar{x}) = \bar{x}$. We use $\bar{z} \dashrightarrow \bar{x}$ which is given by p, thus $j(z, p(x, y, z), x)$ is a Pixley term.

(d) In the above example, $s = p_1 = \bar{x} \rightarrow \bar{z} = p_2 = r$ follows from 2-distributivity and $s' = \bar{z} \rightarrow \bar{z} = r'$, $s'' = \bar{x} \rightarrow \bar{x} = r''$ are from Proposition 3.2(i). Since 2-distributivity does not imply congruence permutability, we actually need $s' \dashrightarrow r''$ in clauses (v)(a) and (v)(b) in Proposition 3.2. In (c) above $s' \dashrightarrow r''$ is $\bar{z} \dashrightarrow \bar{x}$, which needs the additional assumption of congruence permutability.

4 Every n-distributive variety is n-directed distributive

Proposition 4.1. *Suppose that $n \geqslant 2$ and $\mathcal{V}$ is an n-distributive variety, thus $\mathcal{V}$ realizes the pattern path*

$$x = s_1 \rightarrow s_2 \leftarrow s_3 \rightarrow s_4 \leftarrow \dots s_n = z \tag{4.1}$$

with $n - 1$ arrows.

Then we can choose $s_2, s_3, \dots$ in such a way that the above path is realized and, furthermore

(*) $s_i \dashrightarrow s_j$ for every $i \leqslant j \leqslant n$ such that either i is odd, or $i + 2 \leqslant j$.

Proof. It is no loss of generality to consider the terms witnessing n-distributivity as operations of $\mathcal{V}$, and also to assume that $\mathcal{V}$ has no other operation; in particular, we can assume that $\mathcal{V}$ is idempotent.

In view of Proposition 3.2(i)(ii), for $n \leqslant 4$ there is nothing to prove. So let us assume $n > 4$. For even positive $h \leqslant n$, consider the following property.

$(*)_h$ $s_i \dashrightarrow s_j$ for every i, j such that $h \leqslant i \leqslant j \leqslant n$ and either i is odd, or $i + 2 \leqslant j$.

In view of Proposition 3.2(i)(ii), Property $(*)_h$ is satisfied for n odd and $h = n - 1$ (thus h is even) and for n even and $h = n - 2$. We will show the following.

Claim 4.2. *For every even positive $h \leqslant n-3$, if there are $s_1, s_2, \dots s_{n-1}, s_n \in F_2$ realizing the path* (4.1) *and such that* (*)$_{h+2}$ *holds, then there are $s_1^*, s_2^*, \dots s_{n-1}^*, s_n^* \in F_2$ realizing the path* (4.1) *and such that* (*)$_h$ *holds for the s_i^*.*

Assuming we have proved the Claim, a finite induction on decreasing h proves the proposition. The induction terminates at $h = 2$, but then (*) is true in view of Proposition 3.2(ii).

So let us prove the Claim. Assume that (*)$_{h+2}$ holds for certain $s_1, s_2, \dots s_{n-1}, s_n \in F_2$ realizing (4.1). Define

$$\begin{aligned} &s_2^* = s_2, \qquad s_3^* = s_3, \\ &s_{i+2}^* = \bar{s}_{i+2}(s_i^*, z), \text{ for } 2 \leqslant i \leqslant h \text{ and} \\ &s_{i+2}^* = \bar{s}_{i+2}(s_h^*, z), \text{ for } i \geqslant h. \end{aligned} \tag{4.2}$$

Notice that the second and the third lines agree when $i = h$.

We first check that the sequence of the s_i^* realizes the path corresponding to n-distributivity. Indeed, $x \to s_2^* \leftarrow s_3^*$ hold by assumption. Moreover, $s_3^* = s_3 = \bar{s}_3(x, z) \to \bar{s}_4(s_2, z) = \bar{s}_4(s_2^*, z) = s_4^*$, by Proposition 3.2(vi)(b), since $s_3 \to s_4$ and $x \to s_2$ by assumption. Thus $s_3^* \to s_4^*$.

Inductively, we show that if $2 \leqslant i < h$ and, say, i is even and $s_i^* \leftarrow s_{i+1}^*$, then $s_{i+2}^* \leftarrow s_{i+3}^*$. Indeed, $s_{i+2}^* = \bar{s}_{i+2}(s_i^*, z) \leftarrow \bar{s}_{i+3}(s_{i+1}^*, z) = s_{i+3}^*$, by the reversed version (ix) of Proposition 3.2(vi)(b), since $s_{i+2} \leftarrow s_{i+3}$ by assumption and $s_i^* \leftarrow s_{i+1}^*$ by the inductive hypothesis. The case i odd is similar.

The case $i \geqslant h$ is simpler, in that no inductive hypothesis is needed. For $i \geqslant h$ and, say, i even, $s_{i+2}^* = \bar{s}_{i+2}(s_h^*, z) \leftarrow \bar{s}_{i+3}(s_h^*, z) = s_{i+3}^*$, by Proposition 3.2(i), (vi)(b), (ix), since $s_{i+2} \leftarrow s_{i+3}$ (here we use $s_h^* \leftarrow s_h^*$, which holds by Proposition 3.2(i) and we do not need something like $s_i^* \leftarrow s_{i+1}^*$). Again, the case i odd is similar.

Finally, say, for n even, $s_{n-1}^* = \bar{s}_{n-1}(s_h^*, z) \to z$, by the first statement in Proposition 3.2(vii)(a), since $n > 4$ and $s_{n-1} \to z$. If n is odd, use the second statement.

Having proved that the sequence of the s_i^* witnesses n-distributivity, we now show that the sequence of the s_i^* satisfies (*)$_h$, assuming that the sequence of the s_i satisfies (*)$_{h+2}$.

For $j \geqslant i \geqslant h+2$, we have $s_i \dashrightarrow s_j$ by (*)$_{h+2}$, hence $s_i^* = \bar{s}_i(s_h^*, z) \dashrightarrow \bar{s}_j(s_h^*, z) = s_j^*$, by Proposition 3.2(i), (vi)(a).

For $i = h+1$, we already know that $s_i^* \to s_{i-1}^* = s_h^*$, since i is odd. Hence, if $j > i = h+1$, $s_i^* \dashrightarrow \bar{s}_j(s_h^*, z) = s_j^*$, by Proposition 3.2(ii), (viii). Here, since $j > i = h+1$, we apply the third line in (4.2) in the definition of s_j^* (of course, in the case $j = i$ there is nothing to prove, this follows from Proposition 3.2(i)).

If $i = h$ and $j \geqslant i+2$, then $s_i^* = s_h^* \dashrightarrow \bar{s}_j(s_h^*, z) = s_j^*$, again by Proposition 3.2(ii), (viii). □

So far, we have not used the powerful property stated in Proposition 3.2(v)(d). This property will play a key role in the proof of the following theorem.

Theorem 4.3. *For every $n \geqslant 2$, every n-distributive variety is n-directed distributive. Namely, if $\mathcal{V}$ has Jónsson terms $t_1, \dots, t_{n-1}$, then $\mathcal{V}$ has directed Jónsson terms $d_1, \dots, d_{n-1}$.*

Proof. For $n = 2$, there is nothing to prove, so let us assume $n \geqslant 3$.

In any case, $\mathcal{V}$ realizes the pattern path $x \to s_2 \leftarrow s_3 \to s_4 \leftarrow \dots z$, with $n-1$ arrows and, by Proposition 4.1, we may assume that (*) is satisfied.

Define $s_2^* = \bar{s}_2(s_2, s_3)$ and $s_i^* = \bar{s}_i(s_2, s_i)$, for $i \geqslant 3$, thus $s_3^* = \bar{s}_3(s_2, s_3)$.

We have $x \to s_2^*$ by the first statement in Proposition 3.2(vii)(b), using $x \to s_2$ twice.

Moreover, $s_i^* = \bar{s}_i(s_2, s_i) \to \bar{s}_{i+1}(s_2, s_{i+1}) = s_{i+1}^*$, for $i \geqslant 3$, i odd, by the assumptions and Proposition 3.2(i), (v)(b), noticing that $i+1 \geqslant 4$, thus $s_2 \dashrightarrow s_{i+1}$, by (*).

On the other hand, if $i \geqslant 4$ and i is even, then $s_i^* = \bar{s}_i(s_2, s_i) \leftarrow \bar{s}_{i+1}(s_2, s_{i+1}) = s_{i+1}^*$, by the reversed version (ix) of Proposition 3.2(v)(b), since $i \geqslant 4$, thus $s_2 \dashrightarrow s_i$, by (*). Thus $s_3^* \to s_4^* \leftarrow s_5^* \to s_6^* \leftarrow \dots$.

How are connected s_2^* and s_3^*, then? Here Clause (v)(d) in Proposition 3.2 comes to the rescue. We have $s_2^* = \bar{s}_2(s_2, s_3) \to \bar{s}_3(s_2, s_3) = s_3^*$, taking $r = s'' = r'' = s_3$ and $s = r' = s' = s_2$ in Proposition 3.2(v)(d) and using twice $s_3 \to s_2$, both as $s \leftarrow r$ and as $s'' \dashrightarrow r'$.

In the end, we get

$$x \to s_2^* \to s_3^* \to s_4^* \leftarrow s_5^* \to s_6^* \leftarrow \dots \tag{4.3}$$

three right arrows followed by an alternating path.

So far, the arguments prove the theorem in the cases $n = 3$ and $n = 4$, since, say in the latter case, $s_3^* = \bar{s}_3(s_2, s_3) \to z$, by the first statement in Proposition 3.2(vii)(a), applying twice $s_3 \to z$, which holds since $n = 4$. Here we are taking $s = s'' = s_3$.

In order to prove the general case, we need an induction. Before starting the induction, we need to check that the sequence $s_2^*, s_3^*, \dots$ still satisfies (*). Indeed, if $j \geqslant 4$, then $s_2^* = \bar{s}_2(s_2, s_3) \dashrightarrow \bar{s}_j(s_2, s_j) = s_j^*$, by Proposition 3.2(i), (v)(a), since $s_2 \dashrightarrow s_j$ and $s_3 \dashrightarrow s_j$, by (*). If $3 \leqslant i < j$, then $s_i^* = \bar{s}_i(s_2, s_i) \dashrightarrow \bar{s}_j(s_2, s_j) = s_j^*$, by Proposition 3.2(i), (v)(a), since $s_i \dashrightarrow s_j$ and $s_2 \dashrightarrow s_j$, by (*), j being $\geqslant 4$. We have proved that the sequence $s_2^*, s_3^*, \dots$ realizes (4.3) and satisfies (*).

Claim 4.4. *Suppose that h is even, $4 \leqslant h \leqslant n-1$ and $\mathcal{V}$ realizes the pattern path $x \to s_2 \to s_3 \to s_4 \to \dots \to s_h \leftarrow s_{h+1} \to s_{h+2} \leftarrow \dots z$, with $h-1$ right arrows followed by a sequence of alternating arrows. Suppose further that (*) from Proposition 4.1 is satisfied.*

Then $\mathcal{V}$ satisfies the above conditions with $h+2$ in place of h (with just $n-1$ arrows in the exceptional case $h = n-1$).

To prove the Claim, define

$$\begin{aligned} s_i^* &= \bar{s}_i(s_i, s_{h+1}), \text{ for } i \leqslant h, \text{ and} \\ s_i^* &= \bar{s}_i(s_h, s_i), \text{ for } i \geqslant h+1. \end{aligned} \tag{4.4}$$

For $i < h$, we have $s_i^* = \bar{s}_i(s_i, s_{h+1}) \to \bar{s}_{i+1}(s_{i+1}, s_{h+1}) = s_{i+1}^*$ by Proposition 3.2(i), (v)(b), since $s_i \to s_{i+1}$, by the assumptions. We have also used $s_i \dashrightarrow s_{h+1}$, which holds by (*), since $i < h$, hence $i + 2 \leqslant h + 1$.

For $i = h$, we employ the method used above when dealing with s_2 and s_3. Namely, $s_h^* = \bar{s}_h(s_h, s_{h+1}) \to \bar{s}_{h+1}(s_h, s_{h+1}) = s_{h+1}^*$, by Proposition 3.2(i), (v)(d), using twice $s_{h+1} \to s_h$.

For $i \geqslant h + 1$, i odd, as usual by now, $s_i^* = \bar{s}_i(s_h, s_i) \to \bar{s}_{i+1}(s_h, s_{i+1}) = s_{i+1}^*$, by Proposition 3.2(i), (v)(b), using twice $s_i \to s_{i+1}$, and since $s_h \dashrightarrow s_{i+1}$, by (*), noticing that $i+1 \geqslant h+2$, since $i \geqslant h+1$. The case $i \geqslant h+1$, i even is similar: $s_i^* = \bar{s}_i(s_h, s_i) \leftarrow \bar{s}_{i+1}(s_h, s_{i+1}) = s_{i+1}^*$, since $s_i \leftarrow s_{i+1}$. Notice that if i is even, then $i \geqslant h + 2$, since h is even, hence (*) actually gives $s_h \dashrightarrow s_i$.

So far, we have showed that the sequence of the s_i^* realizes the pattern path $x \to s_2 \to s_3 \to \dots \to s_h \to s_{h+1} \to s_{h+2} \leftarrow s_{h+3} \to s_{h+4} \leftarrow \dots z$. It remains to show that the sequence of the s_i^* also satisfies (*).

This is standard by now when $i < j \leqslant h$, since then $i + 2 \leqslant h + 1$ hence we can apply (*) holding for the sequence of the s_i, namely, $s_i \dashrightarrow s_{h+1}$ (the case $i = j$ is from Proposition 3.2(i)). Similarly, if $h + 1 \leqslant i < j$, then we always have $s_h \dashrightarrow s_j$. The cases when $i \leqslant h < j$ present no particular difficulty, once we check that (*) for the s_i can be applied. For example, $s_{h-1}^* = \bar{s}_{h-1}(s_{h-1}, s_{h+1}) \dashrightarrow \bar{s}_{h+1}(s_h, s_{h+1}) = s_{h+1}^*$, by the usual Proposition 3.2(i), (v)(a), since $s_{h-1} \dashrightarrow s_{h+1}$ and $s_{h-1} \dashrightarrow s_h$, by (*) for the s_i, $h - 1$ being odd. Similarly, in $s_{h-1}^* = \bar{s}_{h-1}(s_{h-1}, s_{h+1}) \dashrightarrow \bar{s}_{h+2}(s_h, s_{h+2}) = s_{h+2}^*$, besides $s_{h-1} \dashrightarrow s_h$, we use $s_{h+1} \dashrightarrow s_{h+2}$, since $h + 1$ is odd and $s_{h-1} \dashrightarrow s_{h+2}$, holding by (*). We have already proved $s_h \to s_{h+1}$. In $s_h^* = \bar{s}_h(s_h, s_{h+1}) \dashrightarrow \bar{s}_{h+2}(s_h, s_{h+2}) = s_{h+2}^*$ we use again $s_{h+1} \dashrightarrow s_{h+2}$. In all the remaining cases the significant components are sufficiently "far away" so that (*) for the sequence of the s_i can be always applied with no need of special care.

Having proved Claim 4.4, the theorem follows from the arguments at the beginning of the proof. There we proved the assumptions of the Claim for the case $h = 4$, thus a finite induction shows that we can have $h \geqslant n$, that is, a sequence of directed terms, by Example 2.4(b). □

Recall the definitions of alvin, Gumm and directed Gumm terms from Example 2.4.

Theorem 4.5. *(1) For every $n \geqslant 2$, every variety with n alvin terms has n-directed Jónsson terms.*

(2) For every $n \geqslant 2$, every variety with n Gumm terms has n-directed Gumm terms.

Proof. (1) can be proved by arguments similar to Theorem 4.3. Otherwise, with no need of repeating the arguments, if the alvin condition is witnessed by $x \leftarrow r_2 \to r_3 \leftarrow r_4 \to \dots z$ relabel the elements as $s_3 = r_2$, $s_4 = r_3$, …and take $s_2 = s_1 = x$. The sequence $x \to s_2 \leftarrow s_3 \to s_4 \leftarrow s_5 \to \dots z$ witnesses $n + 1$-distributivity. Applying the proof of

Theorem 4.3, we get a sequence $x \rightarrow s_2^\diamond \rightarrow s_3^\diamond \rightarrow s_4^\diamond \rightarrow \dots z$ witnessing $n+1$-directed distributivity.

Since we have taken $s_2 = x$, going through the proof of Theorem 4.3, one sees that still $s_2^\diamond = x$. Thus the sequence $x = s_1^\diamond = s_2^\diamond \rightarrow s_3^\diamond \rightarrow s_4^\diamond \rightarrow \dots z$ witnesses n-directed distributivity.

(2) Exchanging at the same time the order of terms and of variables, the existence of Gumm terms corresponds to the realizability of $x \rightarrow s_2 \leftarrow s_3 \rightarrow s_4 \leftarrow s_5 \rightarrow s_6 \leftarrow \dots \rightarrow s_{n-3} \leftarrow s_{n-2} \rightarrow s_{n-1} \dashleftarrow z$, for n odd and of $x \leftarrow s_2 \rightarrow s_3 \leftarrow s_4 \rightarrow s_5 \leftarrow s_6 \rightarrow s_7 \leftarrow \dots \rightarrow s_{n-3} \leftarrow s_{n-2} \rightarrow s_{n-1} \dashleftarrow z$, for n even.

For n odd, repeat the proof of Theorem 4.3, stopping the induction at $h = n-1$ (if we reverse the dashed arrow in $s_{n-1} \dashleftarrow z$, a trivial condition arises).

For n even, use the arguments in (1). In both cases one needs the third statement in Proposition 3.2(vii)(a). $\square$

5 Adding edges to undirected paths in locally finite varieties

We now deal with arbitrary undirected paths from x to z, namely, we do not assume that the directions of the arrows alternate. Under a finiteness assumption, we show that we can always add dashed right arrows between pairs of vertexes in the path. On the contrary, left arrows cannot be generally added, since congruence distributivity does not imply n-permutability, for some n. Compare Example 2.4(g).

Assumption 5.1. In detail, we fix some variety $\mathcal{V}$ and some $n \geqslant 1$. It is no loss of generality to assume that $\mathcal{V}$ is idempotent, arguing as at the beginning of the proof of Proposition 4.1, when necessary. We deal with a sequence $x = s_1, s_2, \dots, s_n = z$ such that, for every $i < n$, either $s_i \rightarrow s_{i+1}$, or $s_i \leftarrow s_{i+1}$, or $s_i \dashleftarrow s_{i+1}$. The case $s_i \dashrightarrow s_{i+1}$ need not be considered, since it always corresponds to a trivial condition; see [8].

To establish some notation, we will consider some fixed function f from $\{1, \dots, n-1\}$ to $\{\rightarrow, \leftarrow, \dashleftarrow\}$ and we will write $s_i \stackrel{i}{\rightsquigarrow} s_{i+1}$ to mean $s_i \ f(i) \ s_{i+1}$. The *pattern path associated to* f is the path $x = s_1 \stackrel{1}{\rightsquigarrow} s_2 \stackrel{2}{\rightsquigarrow} s_3 \stackrel{3}{\rightsquigarrow} \dots s_n = z$. Thus a sequence $s_1, s_2, \dots$ realizes the pattern path associated to f if $s_i \stackrel{i}{\rightsquigarrow} s_{i+1}$ holds for every $i < n$. Notice that we will always assume $x = s_1$ and $s_n = z$.

Our aim is to show that, under the above assumptions, we can further have $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant n$. We need a finiteness assumption. We say that some variety $\mathcal{V}$ is 2-*locally finite* if the free algebra $\mathbf{F}_2$ in $\mathcal{V}$ generated by 2 elements is finite, equivalently, if every algebra in $\mathcal{V}$ generated by 2 elements is finite.

Lemma 5.2. *Under the assumptions in 5.1, fix some $k \leqslant n$.*

(i) *Define* $s_i^{(1,k)} = \bar{s}_i(x, s_k)$, *for* $i \leqslant k$, *and* $s_i^{(1,k)} = \bar{s}_i(x, s_i)$, *for* $i \geqslant k$. *Note that the definitions coincide when* $i = k$. *For* $p \geqslant 1$, *define inductively* $s_i^{(p+1,k)} = \bar{s}_i(x, s_k^{(p,k)})$, *for* $i \leqslant k$ *and* $s_i^{(p+1,k)} = \bar{s}_i(x, s_i^{(p,k)})$, *for* $i \geqslant k$.

Then $x = s_1^{(p,k)}$, $s_n^{(p,k)} = z$ *and* $s_i^{(p,k)} \overset{i}{\rightsquigarrow} s_{i+1}^{(p,k)}$, *for every* $i < n$ *and every* $p \geqslant 1$.

If $1 \leqslant i \leqslant j \leqslant k$ *and* $s_i \dashrightarrow s_j$, *then* $s_i^{(p,k)} \dashrightarrow s_j^{(p,k)}$.

(ii) *Suppose further that* $\mathcal{V}$ *is* 2*-locally finite. Then there is some* $p \geqslant 1$ *such that the sequence* $(s_i^{(p,k)})_{i \leqslant n}$ *still satisfies Assumption 5.1 and furthermore, for every* $i \leqslant k$, $s_i^{(p,k)} \dashrightarrow s_k^{(p,k)}$.

Proof. (i) is immediate from Proposition 3.2(i), (vi)(a)(b), (ix), by an induction, using the assumptions. For example, if $s_n^{(p,k)} = z$, then $s_n^{(p+1,k)} = \bar{s}_n(x, s_n^{(p,k)}) = \bar{s}_n(x, z) = z$, since $s_n = z$, that is, $\hat{s}_n$ is the second projection. As another example, if $k \leqslant i < n$ and $s_i^{(p,k)} \leftarrow s_{i+1}^{(p,k)}$, then $s_i^{(p+1,k)} = \bar{s}_i(x, s_i^{(p,k)}) \leftarrow \bar{s}_{i+1}(x, s_{i+1}^{(p,k)}) = s_{i+1}^{(p+1,k)}$ by the reversed version (ix) of Proposition 3.2(v)(b).

(ii) Since F_2 is finite, there are $p > p' > 0$ such that $s_k^{(p,k)} = s_k^{(p',k)}$. If $i \leqslant k$, then $s_i^{(p,k)} = \bar{s}_i(x, \bar{s}_k(x, \dots \bar{s}_k(x, s_k^{(p',k)}) \dots))$, with $p - p'$ open parenthesis and $p - p'$ closed parenthesis on the base line. Then $s_i^{(p,k)} = \bar{s}_i(x, \bar{s}_k(x, \dots \bar{s}_k(x, s_k^{(p',k)}) \dots)) \dashrightarrow s_k^{(p',k)} = s_k^{(p,k)}$, by Proposition 3.2(i)(ii) and iterating the second statement in (viii). $\square$

Proposition 5.3. *Suppose that* $n \geqslant 2$ *and* f *is a function from* $\{1, \dots, n-1\}$ *to* $\{\rightarrow, \leftarrow, \dashleftarrow\}$. *If* $\mathcal{V}$ *is* 2*-locally finite and* $\mathcal{V}$ *realizes the pattern path associated to* f, *then the path can be realized in* $\mathcal{V}$ *by* $s_1, s_2, \dots \in F_2$ *in such a way that* $s_i \dashrightarrow s_j$, *for every* $i \leqslant j \leqslant n$.

Proof. By a finite induction on $k \geqslant 1$, we prove that, for every $k \leqslant n$, there is a sequence such that $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant k$. The case $k = n$ gives the proposition.

The base case $k = 1$ is Proposition 3.2(i).

If $k > 1$ and the statement holds for $i \leqslant j \leqslant k-1$ for some sequence, then the sequence constructed in Lemma 5.2(ii) satisfies $s_i \dashrightarrow s_k$, for every $i \leqslant k$. By 5.2(i) and the inductive assumption, the new sequence also satisfies $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant k-1$. This completes the induction and thus the proof of the proposition. $\square$

In the next theorem, for 2-locally finite varieties, we generalize Theorems 4.3 and 4.5, to the effect that if some variety $\mathcal{V}$ realizes some pattern path, then $\mathcal{V}$ realizes a path in which any number of solid left arrows of our choice are changed into solid right arrows.

Theorem 5.4. *Suppose that* $n \geqslant 2$ *and* f, g *are functions from* $\{1, \dots, n-1\}$ *to* $\{\rightarrow, \leftarrow, \dashleftarrow\}$ *such that, for every* $i \leqslant n$, *if* $f(i) \neq \leftarrow$, *then* $g(i) = f(i)$ *(in other words,* f *and* g *possibly differ only on those* i *such that* $f(i) = \leftarrow$*).*

(1) If $\mathcal{V}$ realizes the pattern path associated to f by a sequence $s_1, s_2, \dots$ such that $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant n$, then $\mathcal{V}$ realizes the pattern path associated to g.

(2) If $\mathcal{V}$ is 2-locally finite and $\mathcal{V}$ realizes the pattern path associated to f, then $\mathcal{V}$ realizes the pattern path associated to g.

Proof. (1) The proof goes as in the proof of Theorem 4.3 with no essential modification.

In detail, if $f(h) = \leftarrow$ and $s_1, s_2, \dots$ is a sequence realizing the path associated to f, define another sequence $s_1^*, s_2^*, \dots$ by (4.4). Then the sequence of the s_i^* realizes the path associated to the function f' such that $f'(h) = \rightarrow$ and f' coincides with f on all the i different from h. Moreover, all the relations of the form $s_i \dashrightarrow s_j$, for $i \leqslant j \leqslant n$ are preserved. The proof is exactly the same as in Claim 4.4. Note that here we have $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant n$, hence we need not to deal with the parity of elements (recall that in (*) in Proposition 4.1 we do not necessarily have $s_i \dashrightarrow s_{i+1}$, for i even).

To prove (1) iterate the above procedure a sufficient number of times.

(2) is immediate from (1) and Proposition 5.3. □

Definition 5.5. [12] For $n \geqslant 3$, a variety $\mathcal{V}$ is *n-directed with alvin heads* if $\mathcal{V}$ realizes the path $x = s_1 \leftarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow \cdots \rightarrow s_{n-3} \rightarrow s_{n-2} \rightarrow s_{n-1} \leftarrow s_n = z$.

For $n \geqslant 3$, a variety $\mathcal{V}$ is *n-two-headed directed Gumm* if $\mathcal{V}$ realizes the path $x = s_1 \dashleftarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow \cdots \rightarrow s_{n-3} \rightarrow s_{n-2} \rightarrow s_{n-1} \dashleftarrow s_n = z$.

The next corollary is immediate from Theorem 5.4(2).

Corollary 5.6. *Suppose that $\mathcal{V}$ is a 2-locally finite variety.*

If $n \geqslant 3$ and $\mathcal{V}$ is n-directed with alvin heads, then $\mathcal{V}$ is n-directed distributive.

More generally, if $\mathcal{V}$ realizes some undirected path as in Assumption 5.1 with $n-1$ edges and with all arrows as solid, then $\mathcal{V}$ is n-directed distributive.

If $n \geqslant 4$, n even and $\mathcal{V}$ is n-defective Gumm (that is, $\mathcal{V}$ realizes the path from Example 2.4(e)), then $\mathcal{V}$ is n-two-headed directed Gumm.

Problems 5.7. (a) Can we remove the assumption that $\mathcal{V}$ is 2-locally finite in Theorem 5.4(2) and Corollary 5.6?

Of course, the problem has affirmative answer if the assumption of 2-local finiteness can be removed from Proposition 5.3.

Remark: the point is *not* that in Section 4 we only proved and used (*) from Proposition 4.1, rather than $s_i \dashrightarrow s_j$, for every $i \leqslant j \leqslant n$. The point *is* that we needed the arrows in the path to alternate between left and right, in order to carry over the proof of Proposition 4.1.

(b) Can we simplify the proofs of the main Theorems 4.3 and 4.5? In particular, is there some hidden combinatorial principle underlying the compositions we have performed?

Were this the case, it would be probably very useful in similar contexts, for example when dealing with SD($\vee$) terms, or when comparing the number of Day and Gumm terms in a congruence modular variety. On the other hand, the results we have proved are quite unexpected, hence it would be really surprising if a significantly simpler proof could be devised.

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