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POWERS AND LIMITATIONS OF URQUHART-STYLE SEMANTICS I: BASIC SUBSTRUCTURAL LOGICS

A b s t r a c t. This paper addresses three kinds of binary operational semantics, called here *Urquhart-style* semantics, for basic substructural logics. First, we discuss the most basic substructural logic **GL** introduced by Galatos and Ono and its expansions with structural axioms and their algebraic semantics. Next, we provide one kind of Urquhart-style semantics, whose frames form the same structures as algebraic semantics, for those substructural logics and consider powers and limitations of this kind of semantics in substructural logic. We then introduce another kind of Urquhart-style semantics, whose canonical frames are based on prime theories, for **DL**, the **GL** with distributivity, and some of its non-associative expansions and extend it to the semantics with star operations for negations. Similarly, we consider powers and limitations of these two kinds of semantics in substructural logic.

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1 Introduction

Using binary accessibility relations, for modal and intuitionistic logics a relational semantics called *Kripke semantics* was first introduced by Kripke [26, 27, 28]. Since then, lots of semantics called *Kripke-style semantics* have been introduced as its generalizations. (One common feature of these semantics is that they all have forcing relation $\Vdash$ for evaluation.) One interesting generalization is Urquhart's operational semantics (briefly Urquhart semantics) for relevant implication (see [38, 39, 40]). This semantics, called '(join) semi-lattice semantics,' has the following standard clauses for extensional conjunction, disjunction, and implication: For arbitrary sentences φ, ψ and any $x \in A$, a carrier set,

(1) $x \Vdash \varphi \wedge \psi$ if and only if (iff) $x \Vdash \varphi$ and $x \Vdash \psi$;

(2) $x \Vdash \varphi \vee \psi$ iff $x \Vdash \varphi$ or $x \Vdash \psi$; and

($\rightarrow_U$) $x \Vdash \varphi \rightarrow \psi$ iff $y \Vdash \varphi$ entails $x * y \Vdash \psi$, for every $y \in A$.

However, many well-known substructural logics can not have semantics with these three conditions since one is unable to prove completeness for those logics using such semantics. The reasons are as follows: First, while the conditions (1) and (2) force the distributivity law, substructural logics with extensional conjunction and disjunction, in general, need not prove it. Namely, such semantics cannot be provided for non-distributive substructural logics. Second, such semantics cannot be established for distributive substructural logics in general. Because there are *distributive* substructural logics for which one cannot provide such semantics. The most well-known example is the system **R** (Relevance logic), which is one of the most famous relevance logics denoted by **T** (Ticket entailment logic), **R** and **E** (Entailment logic). While those three conditions force the sentence

$$(\alpha)\ ((\varphi \rightarrow (\psi \vee \chi)) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$$

to be true in **R**, this sentence is not a theorem of **R** (see [16, 40]).[1] Because of this, Restall ([33] presented two ways to deal with the problem in substructural logic: One is to provide an evaluation different from (2) and to keep ($\rightarrow_U$) for implication, like Fine [17, 18]. The other is to keep (2) and instead to provide an evaluation different from ($\rightarrow_U$), like Routley and Meyer [34, 35, 36].

Urquhart semantics is regarded as an *operational* and *relational* semantics (see [49]). This semantics is operational in the sense that it provides model structures with binary

[1]Note that Urquhart already knew this fact when he [40] first introduced the operational semantics.

operation $*$ in place of binary relation R for accessibility.[2] This semantics is relational in the sense that it has forcing relations for evaluations like Kripke semantics. Urquhart first introduced ($\rightarrow_U$) as the evaluation clause for implication. Since then, many similar semantics having ($\rightarrow_U$) have been introduced for substructural logics. As far as the author knows, there are at least two routes in such introduction.

One is to have a clause for disjunction different from (2) as Restall presented as the first way. For instance, using ($\rightarrow_U$) but with a clause for disjunction different from (2), Humberstone [25] established an operational semantics for $\mathbf{R}^+$, the positive part of $\mathbf{R}$; Došen [9, 10] provided groupoid frames based on uni-residuated lattice-ordered groupoids, briefly *urlogs*, for other substructural logics; and Ono [32] considered similar frames for modal and substructural logics. The other is to have (2) and instead to accept formulas such as (α) as provable formulas. This provides logics generated by the semilattice semantics, i.e., Urquhart Semantics. For example, Urquhart [41] himself introduced a sequent system for the **R** proving such formulas, called **UR** by Standefer [37], and proved completeness; Charlwood [4, 5] presented a natural deduction system for **UR**; in particular, Standefer [37] presented an axiomatic system for **UR** and provided an overview of recent work on operational models building on Urquhart semantics. Moreover, Montagna and Ono [29], Montagna and Sacchetti [30, 31], and Yang [43, 48, 50] provided linearly ordered such frames for fuzzy logics such as **MTL** (Monoidal t-norm logic) and **UL** (Uninorm logic). (Note that the formulas such as (α) are provable in these fuzzy logics.)

Yang [50] called semantics with ($\rightarrow_U$) *Urquhart-style* semantics (US semantics briefly) in honor of the inventor of ($\rightarrow_U$) Urquhart. Following him, we henceforth call such semantics US semantics. Here we note that *powers* and *limitations* of these US semantics in providing completeness results for *basic* substructural logics[3] have not yet been fully investigated and so they have not yet been fully elucidated. For instance, semantics with all of (1), (2), and ($\rightarrow_U$) are not working as semantics to provide completeness results for distributive basic substructural logics in general, whereas such semantics are still working as semantics to provide completeness results for basic substructural *fuzzy* logics as linearly ordered substructural logics. Moreover, they can still work as semantics to establish completeness results for some non-associative distributive basic substructural *non-fuzzy* logics. But these facts have not yet been exactly addressed. In order to overcome such deficiencies, we introduce three kinds of US semantics, based on the Urquhart evaluation of implication ($\rightarrow_U$), for basic substructural logics and discuss their powers and limitations.

For this, in Section 2, we first discuss the most basic substructural logic **GL** and its

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[2] In Kripke semantics, the clause for implication is defined using the relation R below:

($\rightarrow_{R_K}$) $x \Vdash \varphi \rightarrow \psi$ iff xRy and $y \Vdash \varphi$ entail $y \Vdash \psi$, for each $y \in A$.

[3] Basic substructural logics denote substructural logics with some or all of the structural axioms associativity, exchange, contraction, expansion, left weakening and right weakening, see [21, 49]. Other substructural logics can be regarded as expansions of those logics.

expansions with structural axioms and their algebraic semantics.[4] This is a preliminary work for our investigation of three kinds of US semantics for basic substructural logics.

In Section 3, we address one kind of US semantics in the first way, i.e., semantics having a clause for disjunction different from (2) and deal with powers and limitations of such semantics. To be more precise, Section 3.1 introduces one sort of US semantics, called here *nuclear* US semantics, for basic substructural logics introduced in Section 2. This kind of semantics can be also called *algebraic* Urquhart-style semantics (briefly AUS semantics) because frames for such logics form the same structures as algebraic semantics.[5] The nuclear US semantics is *as powerful as algebraic semantics* in the sense that algebraically complete basic substructural logics are complete on nuclear US semantics. However, this reduction cannot be applied to semantics irreducible to algebraic semantics such as Routley–Meyer semantics for **R** in [16]. We consider such powers and limitations in Section 3.2.

In Section 4, we address two kinds of US semantics in the second way, i.e., semantics having all of the conditions (1), (2), and ($\rightarrow_U$) and deal with powers and limitations of such semantics. To be more concrete, Section 4.1 introduces another US semantics having all the conditions for some *distributive* basic substructural logics. We call this kind of semantics *Urquhart–Fine-style* semantics since it is based on the Urquhart semantics with Fine-style interpretation of the intensional conjunction (or fusion). This semantics can be also called *prime* US semantics since it is working for prime theories (with parameters). Section 4.2 extends this semantics to the semantics with star operations for negations for those logics with the axioms for *de Morgan laws*. Similarly we introduce *star-based* Urquhart–Fine-style semantics. These two sorts of semantics are *less powerful* than AUS semantics in that they can be applied to *restricted* distributive basic substructural logics. However, these semantics instead use all the standard clauses (1), (2), and ($\rightarrow_U$) and so the models are easier to work with. Thus, they are more powerful than AUS semantics in that most people working for semantics of a formal system would be familiar with these standard clauses and thus such semantics are more intuitive to them. In Section 4.3, we more exactly deal with such powers and limitations of these two sorts of semantics.

We finally note that basic substructural logics, in general, do not prove formulas such as (α). Our investigation is to provide US semantics for such logics. This means that while Urquhart and his supporters have investigated logics built on Urquhart semantics, this paper conversely addresses US semantics for substructural logics rejecting formulas

[4]We first used the term “structural rules” in place of “structural axioms” since they are all usually called structural rules although they can be all presented as axioms. One reviewer pointed out that **GL** has a structural rule related to exchange. I agree his opinion and so I changed it into “structural axioms.”

[5]The notion ‘algebraic Urquhart-style semantics’ first called ‘algebraic Kripke-style semantics’ by Yang [44] so as to express US semantics having the same structures as algebraic semantics. Here we instead call such semantics AUS semantics since various sorts of relational semantics are called Kripke-style semantics and we just want to denote one sort among them.

such as (α).

2 Preliminaries: logics and algebraic semantics

The term *(pointed) residuated lattice-ordered unital groupoids (briefly, (p)rlu-groupoid)-based logics* denote substructural logic systems having semantics based on $(p)rlu$-groupoids, where a groupoid with unit and its residua interpret the intensional conjunction and implication connectives '&,' '$\rightarrow$,' and '$\rightsquigarrow$.' The system **GL** of Galatos and Ono [21] is the weakest logic in this framework. One can base this logic and its expansions on a propositional countable language having Fm (a set of sentences), inductively constituted from VAR (a set of propositional variables); connectives $\&, \vee, \wedge, \rightarrow, \rightsquigarrow$; constants $\overline{1}, (\overline{0},)$; and defined connectives (df1) $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$; (df2) $\varphi \leftrightsquigarrow \psi := (\varphi \rightsquigarrow \psi) \wedge (\psi \rightsquigarrow \varphi)$; ((df3) $\sim \varphi := \varphi \rightarrow \overline{0}$; (df4) $-\varphi := \varphi \rightsquigarrow \overline{0}$.)

Notice that $\overline{0}$, df3, and df4 are the additional constant and the definitions of two negation connectives for substructural logic systems based on $prlu$-groupoids. A consequence relation $\vdash$ is henceforth provided using axiom systems.

Definition 2.1. ([6, 7, 21]) The axioms and rules below are for ***GL***:

$\varphi \rightarrow (\varphi \vee \psi)$, $\psi \rightarrow (\varphi \vee \psi)$	($\vee$-introduction, $\vee$-I)
$((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)$	($\vee$-elimination, $\vee$-E)
$((\varphi \rightsquigarrow \chi) \wedge (\psi \rightsquigarrow \chi)) \rightarrow ((\varphi \vee \psi) \rightsquigarrow \chi)$	($\vee$-elimination, $\vee$-$E_{\rightsquigarrow}$)
$((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \wedge \chi))$	($\wedge$-introduction, $\wedge$-I)
$(\varphi \wedge \psi) \rightarrow \varphi$, $(\varphi \wedge \psi) \rightarrow \psi$	($\wedge$-elimination, $\wedge$-E)
$\varphi \rightarrow (\overline{1} \rightarrow \varphi)$	(Push)
$\overline{1} \rightarrow (\varphi \rightarrow \varphi)$	(R')
$\overline{1}$	($\overline{1}$)
$\varphi \rightarrow ((\varphi \rightsquigarrow \psi) \rightarrow \psi)$	(assertion$_{\rightsquigarrow}$, $ASl_{\rightsquigarrow}$)
$\varphi \rightarrow (\psi \rightarrow (\psi \& \varphi))$	(&-adjunction,,-Adj)
$\varphi, \psi \vdash \varphi \wedge \psi$	(adjuction, adj)
$\varphi \rightarrow \psi$, $\varphi \vdash \psi$	(modus ponens, mp)
$\psi \rightarrow \chi \vdash (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)$	(prefixing, pf)
$\varphi \rightarrow \psi \vdash (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$	(suffixing, sf)
$\varphi \vdash (\varphi \rightarrow \psi) \rightarrow \psi$	(ASl)
$\psi \rightarrow (\varphi \rightarrow \chi) \vdash (\varphi \& \psi) \rightarrow \chi$	(residuation, res)
$\varphi \rightarrow (\psi \rightarrow \chi) \vdash \psi \rightarrow (\varphi \rightsquigarrow \chi)$	(permutation$_{\rightsquigarrow}$, $per_{\rightsquigarrow}$)
$\varphi \rightsquigarrow \psi \vdash \varphi \rightarrow \psi$	(symmetry, $symm$)

We call a logic $\mathbf{L}'$ an axiomatic expansion (briefly expansion) of a logic $\mathbf{L}$ if it is obtained from $\mathbf{L}$ by adding either new constants or connectives and their corresponding axioms. If these logics have the same language, then we call $\mathbf{L}'$ an extension of $\mathbf{L}$.

Definition 2.2. (*i*) (Non-distributive logics) Consider the following structural axioms:

e	$\varphi\&\psi \to \psi\&\varphi$	exchange
a	$(\varphi\&\psi)\&\chi \leftrightarrow \varphi\&(\psi\&\chi)$	associativity
c	$\varphi \to (\varphi\&\varphi)$	contraction
p	$(\varphi\&\varphi) \to \varphi$	expansion
i	$\varphi \to (\psi \to \varphi)$	left weakening
o	$\overline{0} \to \varphi$	right weakening

For any $\alpha \subseteq \{e, a, c, p, i, o\}$, **GL** is expanded by the basic structural axioms in α. GL_α is said to be a *non-distributive* basic substructural logic since it does not require the axiom for distributivity.

(*ii*) (Distributive logics) **DL** is **GL** plus:

- $(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$ (distributivity, D).

Similarly **DL** is expanded by the basic structural axioms in α, i.e., DL_α is a GL_α with (D). By $\mathrm{DL}_{\alpha\backslash a^i_c}$, we denote a $\mathrm{GL}_{\alpha\backslash\{a,i\}}$ with (D) or a $\mathrm{GL}_{\alpha\backslash\{a,c\}}$ with (D).[6] The systems denoted by DL_α and $\mathrm{DL}_{\alpha\backslash a^i_c}$ are said to be *distributive* basic substructural logics expanding ***DL*** since they require the axiom for distributivity.

(*iii*) (De Morgan distributive logics) dmDL_α is a DL_α with $\overline{0}$, df3, df4, and the axioms:

- $\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi)$ (strong de Morgan$\mathrm{I}_\sim$, $sdmI_\sim$),
- $-(\varphi \wedge \psi) \to (-\varphi \vee -\psi)$ ($sdmI_-$),
- $(\sim\varphi \wedge \sim\psi) \to \sim(\varphi \vee \psi)$ ($sdmII_\sim$),
- $(-\varphi \wedge -\psi) \to -(\varphi \vee \psi)$ ($sdmII_-$).

By $\mathrm{dmDL}_{\alpha\backslash a^i_c}$, we denote $\mathrm{DL}_{\alpha\backslash a^i_c}$ with $\overline{0}$, df3, df4, ($sdmI_\sim$), ($sdmI_-$), ($sdmII_\sim$), and ($sdmII_-$). The systems denoted by dmDL_α and $\mathrm{dmDL}_{\alpha\backslash a^i_c}$ are said to be *de Morgan distributive* basic substructural logics expanding ***DL*** since they require the axioms for de Morgan negation properties.

For convenience, we denote the sets of substructural logics defined above as follows.

Definition 2.3. (*i*) GLs $= \{GL_\alpha : \alpha \subseteq \{e, a, c, p, i, o\}\}$.

(*ii*) DLs $= \{DL_\alpha : \alpha \subseteq \{e, a, c, p, i, o\}\}$; $\mathrm{DLs}^- = \{DL_{\alpha\backslash a^i_c} : \alpha \subseteq \{p, i, o, e\}$ or $\alpha \subseteq \{p, c, o, e\}\}$.

[6] As the results in Section 4 will show, $\mathrm{DL}_{a,i}$ and $\mathrm{DL}_{a,c}$ both do not prove sentences such as (α), whereas their related semantics satisfy such sentences. So here we drop such distributive basic substructural logics.

(*iii*) dmDLs $= \{dmDL_\alpha : \alpha \subseteq \{e, a, c, p, i, o\}\}$; dmDLs$^- = \{dmDL_{\alpha\backslash a^i_c} : \alpha \subseteq \{p, i, o, e\}$ or $\alpha \subseteq \{p, c, o, e\}\}$.

(*iv*) Ls = GLs ∪ DLs ∪ dmDLs.

A logic is called *finitary* if all its deduction rules are finite. Since L ∈ Ls has finite deduction rules, it is finitary.

One can easily prove the following proposition.

Proposition 2.4. (*i*) *(cf, [6, 8]) Let* $\varphi_{\overline{1}}$ *be* $\varphi \wedge \overline{1}$. **GL** *proves:*

(1) $(\varphi\&(\varphi \to \psi)) \to \psi$

(2) $((\varphi \rightsquigarrow \psi)\&\varphi) \to \psi$

(3) $\psi \to \chi \vdash (\varphi \rightsquigarrow \psi) \to (\varphi \rightsquigarrow \chi)$ *($Pf_\rightsquigarrow$)*

(4) $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$ *(transitivity, T)*

(5) $\varphi \vdash \varphi_{\overline{1}}$ *(adj_u)*

(6) $\varphi \to \psi \vdash (\chi\&\varphi) \to (\chi\&\psi)$ *(&-monotonicity,,-mon1)*

(7) $\varphi \to \psi \vdash (\varphi\&\chi) \to (\psi\&\chi)$ *(&-monotonicity,,-mon2)*

(8) $\psi \to (\varphi \rightsquigarrow \chi) \vdash \varphi \to (\psi \to \chi)$ *($per_\rightsquigarrow 2$)*

(9) $(\varphi\&\psi) \to \chi \vdash \psi \to (\varphi \to \chi)$ *(res2)*

(10) $\varphi \to \psi \vdash \varphi \rightsquigarrow \psi$ *(symm2)*

(*ii*) **GL** *with* $\overline{0}$*, df3, and df4 proves:*

(1) $\varphi \to -\sim\varphi$ *(double negation introduction$_{-\sim}$, $DNI_{-\sim}$)*

(2) $\varphi \to \sim -\varphi$ *($DNI_{\sim -}$)*

(3) $(\sim\varphi \vee \sim\psi) \to \sim(\varphi \wedge \psi)$ *(weak de MorganI$_\sim$, wdmI$_\sim$)*

(4) $(-\varphi \vee -\psi) \to -(\varphi \wedge \psi)$ *(wdmI$_-$)*

(5) $-(\varphi \vee \psi) \to (-\varphi \wedge -\psi)$ *(wdmII$_-$)*

(6) $\sim(\varphi \vee \psi) \to (\sim\varphi \wedge \sim\psi)$ *(wdmII$_\sim$)*

(7) $\varphi \to \psi \vdash \sim\psi \to \sim\varphi$ *(contraposition$_\sim$, $CP_\sim$)*

(8) $\varphi \to \psi \vdash -\psi \to -\varphi$ *(CP_-)*

(*iii*) **DL**, *i.e., the* **GL** *with (D), proves:*

(1) $(\varphi \wedge (\psi \vee \chi)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$

(2) $(\varphi \wedge (\psi \vee \chi)) \leftrightsquigarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$

(*iv*) *Let* **dmDL** *be the* **DL** *with* $\overline{0}$*, df3, df4, (sdmI$_\sim$), (sdmI$_-$), (sdmII$_\sim$), and (sdmII$_-$).* **dmDL** *proves:*

(1) $-(\varphi \wedge \psi) \leftrightarrow (-\varphi \vee -\psi)$ *(dmI$_\sim$)*

(2) $-(\varphi \vee \psi) \leftrightarrow (-\varphi \wedge -\psi)$ *(dmI$_-$)*

(3) $-(\varphi \wedge \psi) \leftrightsquigarrow (-\varphi \vee -\psi)$

(4) $-(\varphi \vee \psi) \leftrightsquigarrow (-\varphi \wedge -\psi)$

(5) $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$ (dmII$_\sim$)
(6) $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$ (dmII$_-$)
(7) $\sim(\varphi \wedge \psi) \leftrightsquigarrow (\sim\varphi \vee \sim\psi)$
(8) $\sim(\varphi \vee \psi) \leftrightsquigarrow (\sim\varphi \wedge \sim\psi)$

Remark 2.5. *Proposition 2.4 (i) to (iv) show the following.*

1. *(i) (4), (6), and (7) ensure that **GL** satisfies transitivity and monotonicity.*
2. *(ii) ensures that **GL** with $\overline{0}$, df3, and df4 satisfies contraposition, double negation introduction, and weak forms of de Morgan laws.*
3. *(iii) ensures that **DL** satisfies distributivity.*
4. *(iv) ensures that **dmDL** satisfies de Morgan laws.*

A *theory* of L is a set of sentences such that $T \vdash_L \varphi$ entails $\varphi \in T$. In a theory T on L $\in$ Ls, a *proof* is defined as a sequence of sentences σ, where each element is either an axiom of L, a member of T, or follows from preceding elements of σ using a rule of L. $T \vdash_L \varphi$ means that there exists a proof of φ in T on L.[7]

For convenience, as propositional connectives and as algebraic operators we ambiguously use '$\vee$,' '$\wedge$,' '$\sim$,' and '$-$'.

Now we introduce suitable algebraic structures for substructural logics.

Definition 2.6. An *rlu-groupoid* is an algebra $(A, *, 1, \wedge, \vee, \backslash, /)$, where $(A, *, 1)$ is a groupoid with unit; $(A, \wedge, \vee)$ is a lattice; and $b \leq a\backslash c$ iff $a * b \leq c$ iff $a \leq c/b$, for all $a, b, c \in A$ (residuation). A *prlu*-groupoid is an rlu-groupoid with an arbitrary element 0.

Since the system **GL** is characterized by the set of *rlu-groupoids*, rlu-groupoids are henceforth called *GL-algebras.*

Definition 2.7. (*i*) (GL$_\alpha$-algebras) Suitable algebraic (in)equations for the structural axioms introduced in Definition 2.2 (*ii*) are defined as follows:

- $a * b \leq b * a$, for all $a, b \in A$ ($e^{\mathcal{A}}$)
- $(a * b) * c = a * (b * c)$, for all $a, b, c \in A$ ($a^{\mathcal{A}}$)
- $a * a \leq a$, for all $a \in A$ ($p^{\mathcal{A}}$)
- $a \leq a * a$, for all $a \in A$ ($c^{\mathcal{A}}$)
- $a \leq 1$, for all $a \in A$ ($i^{\mathcal{A}}$)
- $0 \leq a$, for all $a \in A$ ($o^{\mathcal{A}}$).

[7]This implies that T is closed under the rules of L and contains theorems of L.

Thus, for any $\alpha \subseteq \{e^{\mathcal{A}}, a^{\mathcal{A}}, p^{\mathcal{A}}, c^{\mathcal{A}}, i^{\mathcal{A}}, o^{\mathcal{A}}\}$, we define *$GL_\alpha$-algebras* as GL-algebras with their corresponding (in)equations. If $\alpha = \emptyset$, then GL_α-algebras are just GL-algebras.

(ii) (DL_α-algebras) A *DL_α-algebra* is a distributive lattice-ordered GL_α-algebra. In particular, a *$DL_{\alpha\backslash a^i_c}$-algebra* is a DL_α-algebra dropping either $(a^{\mathcal{A}})$ and $(i^{\mathcal{A}})$ or $(a^{\mathcal{A}})$ and $(c^{\mathcal{A}})$.

(iii) (dmDL_α-algebras) Let the negation operations $\sim$ and $-$ be defined as follows: for all $a \in A$, (df3$^{\mathcal{A}}$) $\sim a := a\backslash 0$ and (df4$^{\mathcal{A}}$) $-a := 0/a$. A *$dmDL_\alpha$-algebra* is a pointed DL_α-algebra satisfying (df3$^{\mathcal{A}}$), (df4$^{\mathcal{A}}$), and de Morgan properties[8], i.e., $(dmI^{\mathcal{A}}_{\sim})$ $\sim (a \wedge b) = (\sim a \vee \sim b)$, $(dmI^{\mathcal{A}}_{-})$ $-(a \wedge b) = (-a \vee -b)$, $(dmII^{\mathcal{A}}_{\sim})$ $\sim (a \vee b) = (\sim a \wedge \sim b)$, and $(dmII^{\mathcal{A}}_{-})$ $-(a \vee b) = (-a \wedge -b)$. In particular, a *$dmDL_{\alpha\backslash a^i_c}$-algebra* is a dmDL_α-algebra dropping either $(a^{\mathcal{A}})$ and $(i^{\mathcal{A}})$ or $(a^{\mathcal{A}})$ and $(c^{\mathcal{A}})$.

(iv) By *L-algebras*, we say all the algebras introduced in *(i)* to *(iii)*.

For an L-algebra $\mathcal{A}$, an *$\mathcal{A}$-evaluation* is defined as a map $v : Fm \to \mathcal{A}$ such that $v : Fm \to \mathcal{A}$ satisfying: $v(\#(\varphi_1, \ldots, \varphi_n)) = \#^{\mathcal{A}}(v(\varphi_1), \ldots, v(\varphi_n))$, where $\# \in \{\&, \vee, \wedge, \to, \rightsquigarrow, \overline{1}, \overline{0}\}$ and $\#^{\mathcal{A}} \in \{*, \vee, \wedge, \backslash, /, 1, 0\}$. A sentence φ is said to be an $\mathcal{A}$-tautology in case $1 \leq v(\varphi)$ for all $\mathcal{A}$-evaluation v. An $\mathcal{A}$-evaluation v is said to be an *$\mathcal{A}$-model* of a theory T in case $1 \leq v(\varphi)$ for all $\varphi \in T$. *Mod(T, $\mathcal{A}$)* denotes the set of all $\mathcal{A}$-models of T. A sentence φ is said to be a *semantic consequence* of T on a class of L-algebras $\mathcal{K}$, denoted by $T \models_{\mathcal{K}} \varphi$, in case $Mod(T \cup \{\varphi\}, \mathcal{A}) = Mod(T, \mathcal{A})$ for each $\mathcal{A} \in \mathcal{K}$. $\mathcal{A}$ is said to be an ***L**-algebra* in case $\mathcal{A}$ is a semantic consequence of T on $\{\mathcal{A}\}$ whenever φ is L-provable in any T. *MOD(L)* denotes the set of **L**-algebras. For simplicity, instead of $T \models_{MOD(L)} \varphi$, we write $T \models_L \varphi$.

Theorem 2.8. (*Strong completeness*) *For a theory T on $L \in Ls$ and a sentence φ, it holds that $T \vdash_L \varphi$ iff $T \models_L \varphi$.*

Proof. The claim is obtained as a corollary of Theorem 2.1.25 in [8]. □

3 Semantics I: AUS semantics

In this section, one kind of US semantics, called AUS semantics, is introduced for Ls, i.e., all the systems introduced in Section 2 (see Definition 2.3 (*iv*)), and its powers and limitations are discussed.

[8]In order to emphasize that dmDL_α-algebras satisfy de Morgan laws, we introduce de Morgan properties as equations satisfying de Morgan laws.

3.1 Nuclear US semantics for Ls

AUS semantics as Urquhart-style semantics being equivalent to algebraic semantics can be introduced for Ls. To verify this, we introduce one sort of such semantics called here *nuclear US semantics*, which is based on Urquhart's operational evaluation of implication[9] and nuclear completion. For this, we first introduce several Kripke frames, some of which are already introduced in [44, 45, 49].

Definition 3.1. (i) (Kripke frames [44]) A structure $\mathcal{F} = (F, 1, \leq)$ is said to be a *Kripke frame* if $\leq$ is a partial order on the carrier set F and 1 is a special element in F. We call the elements of $\mathcal{F}$ *states of information*.

(ii) (Operational Kripke frames [45]) A Kripke frame $\mathcal{F} = (F, 1, \leq, *)$ is said to be an *operational Kripke frame* (briefly OK frame) if $(F, 1, *)$ forms a groupoid with unit.

(iii) (Residuated operational Kripke frames [49]) A *residuated* OK frame is an OK frame, where the sets $\{c : a * c \leq b\}$ and $\{c : c * a \leq b\}$ have suprema, denoted by $a\backslash b$ and b/a, respectively, for every $a, b \in F$.

(iv) (GL frames [49]) A *GL frame* is a residuated OK frame, where $\leq$ is lattice-ordered on F.

(v) (L frames) A GL frame is said to be *pointed* if it has an element 0. For any $\alpha \subseteq \{e^{\mathcal{A}}, a^{\mathcal{A}}, p^{\mathcal{A}}, c^{\mathcal{A}}, i^{\mathcal{A}}, o^{\mathcal{A}}\}$, a (pointed) GL frame is said to be (pointed) *GL_α frame* if it has the α additionally. A GL_α frame is said to be a DL_α frame if $\leq$ is distributive lattice-ordered, and a pointed DL_α frame is said to be a dmDL_α frame if it satisfies (df3$^{\mathcal{A}}$), (df4$^{\mathcal{A}}$), and de Morgan laws. All these frames are said to be *L frames*.

Remark 3.2. *We recall the facts associated with Definition 3.1 mentioned in [49].*

1. *The definition* (i) *is for the intuitionistic logic* ***H***.
2. (ii) *shows that frames take groupoid operations in place of binary relations for accessibility.*
3. (iii) *is important to give AUS semantics for Ls because the suprema provide left and right divisions, which are operations corresponding to two implication connectives.*
4. (iv) *shows that the operations* meet *and* join *are defined as* inf *and sup, respectively, on a lattice.*

[9] More exactly, we consider a generalization of Urquhart's evaluation of implication because, while Urquhart considered only one implication in [40], we introduce two implications here.

5. (v) *shows that the additional structural properties of* α *for L frames are provided by groupoid operations.*

For an *evaluation* on an L frame, we need to introduce closure operators.

Definition 3.3. (Closure operators) Given a partially ordered set F, a function $cl : F \to F$ is said to be a *closure operator* on F if it satisfies: for any $a, b \in F$, $a \leq cl(a)$ (increasing); $clcl(a) = cl(a)$ (idempotent); if $a \leq b$, then $cl(a) \leq cl(b)$ (monotone). For an element $a \in F$, if $a = cl(a)$, a is called *closed*. F_{cl} denotes the class of all closed elements of F, called *closure set*.

Let $\mathcal{F}$ be a partially ordered groupoid (po-groupoid briefly). A *nucleus* on $\mathcal{F}$ is a function $h : F \to F$ such that h is a closure operator on $(F, \leq)$ satisfying $h(a) * h(b) \leq h(a * b)$, where $*$ is a groupoid operator on F. Residuated OK frames are po-groupoids. By virtue of nuclear completions, it is verified that one can embed L frames into *complete* L frames.

Proposition 3.4. *Every L frame is embeddable into a* complete *L frame.*

Proof. For GL_α and DL_α frames, see Proposition 1 in [49] and Proposition 4 in [44], respectively. Here we consider $dmDL_\alpha$ frames. Let $\mathcal{F}$ be a residuated lattice. First we recall the definition of its embeddable residuated lattice $\mathcal{F}^+$:

1. For any $F \subseteq \mathcal{F}$, $N(F)$ denotes the intersection of all sets H satisfying: (I) $F \subseteq H$, (II) H is closed downwards, and (III) for any $G \subseteq H$, there exists $sup(G)$ in $\mathcal{F}$ entails $sup(G) \in H$. One can ensure that N is a closure operator. The domain of $\mathcal{F}^+$ is $\{F : F \subseteq \mathcal{F}$ such that $N(F) = F\}$, the N-closed subsets of $\mathcal{A}$.

2. The definitions of the operations of $\mathcal{F}^+$ are given as follows: $F \vee G = N(F \cup G)$; $F \wedge G = F \cap G$; and $F \circ G = N(F * G)$ such that $F * G = \{a * b : a \in F$ and $b \in G\}$, where $*$ is the groupoid operator of $\mathcal{F}$. In addition, we have a residuated pair $(\backslash, /)$ as follows: $F \backslash G = \{c \in \mathcal{F} : \forall a \in F,\ a * c \in G\}$ and $G/F = \{c \in \mathcal{F} : \forall a \in F,\ c * a \in G\}$. The definition ensures that N is a nucleus on $(\mathcal{F}^+, \subseteq)$ since for $F, G \in \mathcal{F}^+$, $N(F) \circ N(G) = F \circ G \subseteq N(F \circ G)$.

3. $1^+ = \{c \in \mathcal{F} : c \leq 1\}$ and $0^+ = \{c \in \mathcal{F} : c \leq 0\}$ are the constants in $\mathcal{F}^+$.

For the distributivity in $\mathcal{F}^+$, see Proposition 4 in [44]. For the de Morgan laws in $\mathcal{F}^+$, we further note that two negations are defined as follows:

4. $\sim F = F \backslash 0^+$ and $-F = 0^+/F$.

One has to prove: $\sim(F \wedge G) = (\sim F \vee \sim G)$, $\sim(F \vee G) = (\sim F \wedge \sim G)$, $-(F \wedge G) = (-F \vee -G)$, and $-(F \vee G) = (-F \wedge -G)$. For the first case, we verify that $\sim(F \cap G) = N(\sim F \cup \sim G)$. Since for $x \in F$ and $y \in G$, $\sim(x \wedge y) = \sim x \vee \sim y$ and thus $\sim(F \cap G) = \sim F \cup \sim G$, one can easily prove its left-to-right direction. For its reverse direction, we note that Lemma 3.33 in [20] ensures that $F\backslash G$ and G/F are in $\mathcal{F}^+$ for any $F, G \in \mathcal{F}^+$ since N is a nucleus on $\mathcal{F}^+$. This means that $F\backslash G = N(F\backslash G)$ and $G/F = N(G/F)$. Then, since $F \cap G \subseteq F, G$ and thus $\sim F, \sim G \subseteq \sim(F \cap G)$, we can ensure that $\sim F \cup \sim G \subseteq \sim(F \cap G)$. Then, since $\sim(F \cap G) \in \mathcal{F}^+$ and thus $N(\sim F \cup \sim G) \subseteq N(\sim(F \cap G)) = \sim(F \cap G)$, we can obtain that $N(\sim F \cup \sim G) \subseteq \sim(F \cap G)$. The proof for the other cases is analogous.

Therefore, if $\mathcal{F}$ is an L frame, then so is $\mathcal{F}^+$. □

Given a po-groupoid $\mathcal{F}$ and a nucleus N on $\mathcal{P}(\mathcal{F})$, if all N-closed sets are closed downward, N is said to be a *downward* nucleus. For any set F, any downward nucleus N on $\mathcal{P}(F)$, and any evaluation v from sentences to closed subsets of F, an *evaluation* on an L frame is given as a forcing $\Vdash$ between the states of information and the propositional variables, constants, and any sentences satisfying the below conditions. For all propositional variables p,

(p-closure) $N(v(p)) = v(p)$;
(AHC) $b \leq a$ and $a \Vdash p$ entail $b \Vdash p$;
($\Vdash$) $a \Vdash p$ iff $a \in v(p)$,

for constants $\overline{1}, \overline{0}$,

(1) $a \Vdash \overline{1}$ iff $a \in N(v(\overline{1}))$;
((0) $a \Vdash \overline{0}$ iff $a \in N(v(\overline{0}))$, if frames are pointed,) and

for any sentences,

(&) $a \Vdash \varphi \& \psi$ iff $a \in N(v(\varphi) * v(\psi))$;
($\vee$) $a \Vdash \varphi \vee \psi$ iff $a \in N(v(\varphi) \cup v(\psi))$;
($\wedge$) $a \Vdash \varphi \wedge \psi$ iff $a \Vdash \varphi$ and $a \Vdash \psi$;
($\rightarrow$) $a \Vdash \varphi \rightarrow \psi$ iff for every $b \in F$, $b \Vdash \varphi$ entails $b * a \Vdash \psi$;
($\rightsquigarrow$) $a \Vdash \varphi \rightsquigarrow \psi$ iff for every $b \in F$, $b \Vdash \varphi$ entails $a * b \Vdash \psi$.

A pair $(\mathcal{F}, \Vdash)$, where $\mathcal{F}$ is an L frame and $\Vdash$ is an evaluation on $\mathcal{F}$, is said to be an *L model*. For an L model $(\mathcal{F}, \Vdash)$, a state of information $a \in \mathcal{F}$ and a sentence φ, *a forces* φ means that $a \Vdash \varphi$. φ is said to be *true* in $(\mathcal{F}, \Vdash)$ in case $1 \Vdash \varphi$; *valid* in the frame $\mathcal{F}$ in case φ is true in $(\mathcal{F}, \Vdash)$ for any evaluation $\Vdash$ on $\mathcal{F}$. If all axioms of L are valid in $\mathcal{F}$, an L frame $\mathcal{F}$ is said to be an **L** frame; if $\mathcal{F}$ is an **L** frame, an L model $(\mathcal{F}, \Vdash)$ is said to be an ***L** model.*

Lemma 3.5. *(i) (HL, Hereditary lemma) Given an L frame $\mathcal{F}$, all states of information $a, b \in \mathcal{F}$, and a sentence φ, $a \Vdash \varphi$ and $b \leq a$ entail $b \Vdash \varphi$.*

(ii) $v(\varphi)$ is closed for any L model and for any sentence φ.

Proof. See Lemma 1 in [49] for the proof for (i) and (ii). □

Given a downward nucleus N on $\mathcal{P}(F)$ and closed sets G, H ($\subseteq F$), we define $G \cup_N H := N(G \cup H)$, $G *_N H := N(G * H)$, and $X \backslash G$ and G/X as in Proposition 3.4. The following proposition shows an important connection between algebraic semantics and nuclear OK semantics for Ls.

Proposition 3.6. *Let $\mathcal{F}$ be $(F, 1, (0,) \leq, *)$.*

(i) $\mathcal{F}$ is an L frame in case it is the reduct of an L-algebra $\mathcal{A}$.

(ii) The structure $\mathcal{P}(\mathcal{F})_N = (\mathcal{P}(F)_N, *_N, \cup_N, \cap, \backslash, /, N(\{1\}), (N(\{0\})))$ forms an L-algebra in case $\mathcal{F}$ is an L frame.

(iii) $(\mathcal{F}, \Vdash)$ is an L model and $a \Vdash \varphi$ iff $a \in v(\varphi)$ for each sentence φ and for each $a \in \mathcal{A}$, in case $\mathcal{F}$ is the reduct of an L-algebra $\mathcal{A}$ and v is an evaluation in $\mathcal{P}(\mathcal{A})_N$.

Proof. See Proposition 2 in [49] for the proof for (i), (ii), and (iii). □

Now the soundness and completeness for Ls are provided as follows.

Lemma 3.7. ([*49*]) *$1 \Vdash \varphi \to \psi$ iff for any $a \in \mathcal{F}$, $a \Vdash \varphi$ entails $a \Vdash \psi$.*

Proposition 3.8. *(Soundness) φ is valid in each L frame in case $\vdash_L \varphi$.*

Proof. The validity of the axioms $(sdmI_{\sim})$, $(sdmI_{-})$, $(sdmII_{\sim})$, and $(sdmII_{-})$ is proved here as examples. For $(sdmI_{\sim})$, one needs to verify $1 \Vdash\, \sim (\varphi \wedge \psi) \to (\sim \varphi \vee \sim \psi)$. By Lemma 3.7, for any state of information $a \in F$ one may instead assume that $a \Vdash\, \sim (\varphi \wedge \psi)$ and prove that $a \Vdash\, \sim \varphi \vee \sim \psi$. By the condition $(\vee)$ and Proposition 3.6 (iii), we assume that $a \in v(\sim (\varphi \wedge \psi))$ and show that $a \in N(v(\sim \varphi) \cup v(\sim \psi))$. Note that $a \in v(\sim (\varphi \wedge \psi))$ iff $a \in v((\varphi \wedge \psi) \to \overline{0})$ iff $a \in v(\varphi \wedge \psi)\backslash 0^{+}$ iff $a \in\, \sim (v(\varphi) \cap v(\psi))$. Then, since $\sim (v(\varphi) \cap v(\psi)) = \sim v(\varphi) \cup \sim v(\psi) \subseteq N(\sim v(\varphi) \cup \sim v(\psi))$, one can obtain that $a \in N(v(\sim \varphi) \cup v(\sim \psi))$. The proof for the other strong forms of de Morgan laws is analogous. □

Theorem 3.9. (*Strong completeness*) *Let T be a theory on L, φ a sentence, and $\mathcal{L}$ a set of all L frames. $T \vdash_L \varphi$ in case $T \models_{\mathcal{L}} \varphi$.*

Proof. Suppose contrapositively that T is a theory on L and φ is a sentence such that $T \nvdash_L \varphi$. Theorem 2.8 assures that one can construct an **L**-algebra $\mathcal{A}$ and an evaluation v on $\mathcal{A}$ such that $1 \leq v(\chi)$ for all $\chi \in T$ and $v(\varphi) < 1$. Then, the complete embeddability of Theorem 6.29 in [20] further ensures that $N(\{1\}) \subseteq N(\{v(\psi)\})$ and $N(\{1\}) \nsubseteq N(v(\varphi))$ in $\mathcal{P}(\mathcal{A})_N$. Thus, for an evaluation v' in $\mathcal{P}(\mathcal{A})_N$, one has that for any $\chi \in T$, $1 \in v'(\chi)$ and $1 \notin v'(\varphi)$ by Proposition 3.6; therefore, $T \not\models_{\mathcal{L}} \varphi$. □

Remark 3.10. *Since every L frame forms its corresponding L-algebras, the semantics constructed by these frames are AUS semantics.*

3.2 Powers and limitations

The AUS semantics introduced in Section 3.1 is the *most powerful* of the known US semantics so far in the sense that it covers all the basic substructural systems introduced in Definition 2.3. As is shown above, this semantics is as powerful as algebraic semantics in the sense that algebraically complete basic substructural logics are complete on this semantics and vice versa. Associated with it, we note that such semantics have been introduced for some of them (but not all of them). For instance, Humberstone [25] introduced such semantics for $\mathbf{R}^+$; Došen [9, 10] introduced (semi-lattice-ordered) groupoid frames for logics based on urlogs and their extensions with structural axioms; Venema [42] generalized Došen's work to substructural logics with two implications $\rightarrow, \rightsquigarrow$ and two constants $\top, \overline{1}$; and Yang [44, 49] introduced AUS semantics for $\mathbf{R}$ and its neighbors, and GL_α.

Especially the nuclear US semantics for Ls addressed in Section 3.1 is valuable in the following two senses: First, this semantics is *fully* operational in the sense that it provides operational evaluations for disjunction $\vee$ and intensional conjunction & as well as two implications $\rightarrow, \rightsquigarrow$ (see $(\rightarrow)$, $(\rightsquigarrow)$, $(\vee)$ and $(\&)$ above). However, the semantics introduced by Humberstone [25], Došen [10] and Venema [42] are not in the sense that they do not provide operational evaluations for $\vee$ and &. The following first two are the evaluations by Humberstone [25] the last two are by Došen [10] and Venema [42].

$(\vee_H)$ $a \Vdash \varphi \vee \psi$ iff there are $b, c \in A$ such that $a = b + c$, $b \Vdash \varphi$ and $c \Vdash \psi$.

$(\&_H)$ $a \Vdash \varphi \& \psi$ iff there are $b, c, d \in A$ such that $d + a = b * c$, $b \Vdash \varphi$ and $c \Vdash \psi$.

$(\vee_D)$ $a \Vdash \varphi \vee \psi$ iff there are $b, c \in A$ such that either $b \Vdash \varphi$, $c \Vdash \psi$ and $b \cap c \leq a$, or $a \Vdash \varphi$, or $a \Vdash \psi$.

$(\&_F)$ $a \Vdash \varphi \& \psi$ iff there are $b, c \in A$ such that $b \Vdash \varphi$, $c \Vdash \psi$, and $b * c \leq a$.[10]

[10]This operational-relational evaluation was first introduced by Fine [17]. So we used the index "F" in honor of the inventor, Fine.

This implies that the nuclear US semantics faithfully follows Urquhart's idea of semantics based on operations.

Second, regarding the first, as Standefer mentioned on operational semantics in [37], the nuclear US semantics *philosophically* provides a natural interpretation of connectives in terms of information.[11] (Note that to avoid ontological commitment many philosophical logicians prefer the term "states of information" or "situations" to the term "possible worlds".) Let the elements of the domain be states of information. A state of information a verifies an implication, denoted by $\rightarrow$, whenever any state of information b verifying the antecedent results in a state of information combining b with a in order, denoted by $b * a$, verifying the consequent; A state of information a verifies another implication, denoted by $\rightsquigarrow$, whenever any state of information b verifying the antecedent results in a state of information combining a with b in order, denoted by $a * b$, verifying the consequent;[12] and such relations of verification can be naturally extended to conjunction, intensional conjunction, and disjunction. This means that it provides a more intuitive way to understand connections between states of information forcing a sentence in a logic. In particular, it provides interpretations of states of information verifying intensional conjunction and disjunction based on nuclear completions.

In addition, regarding the second, on disjunction and intensional conjunction the nuclear US semantics provides simpler interpretations than other US semantics. As one can see, ($\vee$) and (&) are simpler than ($\vee_H$), ($\&_H$), ($\vee_D$) and ($\&_F$) since they just requires the nuclear operator N. In fact, to provide semantics based on nuclear completions is not my own idea. Such semantics has a long history. For instance, semantics based on nuclear completions such as Beth, Dragalin, phase, and quantale semantics were introduced for intuitionistic and linear logics [1, 2, 11, 12, 22, 53]. In particular, for non-distributive substructural logics, Restall [33] considered semantics with the evaluations (&) and ($\vee$), each of which is based on nuclear operation N, in Section 3.1. However, his semantics has Routley–Meyer-style evaluations for implications below.[13]

($\rightarrow_R$) $x \Vdash \varphi \rightarrow \psi$ iff for all $y, z \in A$, $Rxyz$ and $y \Vdash \varphi$ entail $z \Vdash \psi$,

($\rightsquigarrow_R$) $x \Vdash \varphi \rightsquigarrow \psi$ iff for all $y, z \in A$, $Ryxz$ and $y \Vdash \varphi$ entail $z \Vdash \psi$.

[11] As one reviewer pointed out, this interpretation originated with Urquhart [38, 39, 40], and it was emphasized by Humberstone [25].

[12] These two interpretations of implications show that states of information obtained by intensional conjunction need not commutative. For instance, the state of information that I have a breakfast and go to school is different from the state of information that I go to school and have a breakfast since here 'and' means 'and then.'

[13] In his semantics, one is not easy to catch the intuitive meanings of $Rxyz$ and $Ryxz$ in the conditions. The most natural interpretations of R are that $Rxyz := x * y = z$ and $Rxyz := x * y \leq z$, which are operational and operational-relational ones, respectively. The first one is related to Urquhart's evaluation of implication and the second to Fine's evaluation of implication. For more details on these interpretations, see [13, 14, 15, 49].

This shows that the nuclear US semantics is the first introduction of US semantics based on nuclear completions for all the basic substructural logics in Definition 2.3.[14]

However, AUS semantics has an obvious limitation. Because it does not have its own semantics distinguished from algebraic semantics and so is not very interesting in novelty. As its natural consequences, we can say the following two limitations of nuclear US semantics. First, the reduction of AUS semantics to algebraic semantics cannot be applied to frames irreducible to algebraic structures. For instance, the nuclear semantics cannot be applied to the Routley–Meyer semantics for the relevance logic **R** introduced in [16] since the interpretations of R based on fusion operation such as $Rxyz := x * y = z$ and $Rxyz := x * y \leq z$ cannot be applied to the R in [16]: The semantics for **R** has the postulate (p) $Rxxx$. According to these definitions, one has $x * x = x$ and $x * x \leq x$, and so $(p^{\mathcal{A}})$. However, the algebraic structures for **R** do not require $(p^{\mathcal{A}})$. Namely, it causes a problem of overgeneration. This shows that the postulates for Routley–Meyer semantics for **R** interpreted by $Rxyz := x * y = z$ and $Rxyz := x * y \leq z$ do not provide related algebraic structures and so do not frame structures, see [49].

Second, this semantics is very limited in providing set-theoretic completeness for basic substructural logics. For set-theoretic completeness, we in general need canonical evaluation. However, as Yang mentioned in [51], while the *standard* canonical evaluation can be defined as

$$(CE_S)\ x \Vdash_{can} \varphi \text{ iff } \varphi \in x,$$

it does not work for basic substructural logics in general. Because of this, he instead introduced basic substructural logics with the nucleus connective N, called *substructural nuclear image-based logics*, and defined the canonical evaluation as follows

$$(CE_N)\ x \Vdash_{can} \varphi \text{ iff } N\varphi \in x.$$

Then he provided a set-theoretic completeness for them.[15] This implies that one has some difficulties in establishing set-theoretic completeness for the basic substructural logics in Definition 2.3. Namely, nuclear US semantics does not provide set-theoretic completeness results for the basic substructural logics. These two are the limitations of AUS semantics we can say in this paper.

[14]Recently, Galatos and Jipsen [19] introduced residuated frames providing relational semantics based on nuclear relations for basic substructural logics. This semantics is interesting in that the residuated frames are based on ternary relations, whereas their corresponding semantics are based on algebras, i.e., *rlu*-groupoids. However, it is slightly different from traditional relational semantics in the sense that, while evaluations in relational semantics are in general provided by forcing relations, evaluations in their semantics are not.

Hartonas [23, 24] also introduced relational semantics based on closure operators for propositional lattice logics as non-distributive propositional logics. However, the semantics in [23] deals with implications using ternary relations and the semantics in [24] does not deal with them.

[15]See [51] for more details.

4 Semantics II, III: (Star-based) Urquhart–Fine-style semantics

As mentioned in Section 1, Urquhart semantics does not work for distributive substructural logics in general. But the reason of its failure is not in the distributivity itself but in some properties, in particular associativity, of intensional conjunction &. We verify this fact by introducing the Urquhart semantics with a variant of Fine's evaluation of & for DLs$^-$ in Section 4.1. US semantics for some distributive substructural logics may have star operations for negations introduced in Routley–Meyer semantics for relevance logics. However, this fact has not yet been elucidated exactly. We investigate this by introducing similar semantics with two operations for negations for dmDLs$^-$ in Section 4.2. In Section 4.3, we finally consider powers and limitations of these two kinds of semantics.

4.1 Prime US semantics for DLs$^-$

One can deal with Urquhart–Fine-style semantics for DLs$^-$ introduced in Definition 2.3 (ii). To verify this, we introduce one sort of such semantics called here *prime US semantics*, which has an operational and relational interpretation of &, and prime theories based on parameterized disjunctions in place of closed theories.[16] We first introduce Urquhart-Fine frames (briefly *UF frames*) as GL frames on distributive lattices in honor of Urquhart and Fine, the inventors of basic idea of the prime US semantics.

Definition 4.1. (i) (UF$_\alpha$ frames) Let *UF frame* be an GL frame, where $\leq$ forms a distributive lattice order. For any $\alpha \subseteq \{e^{\mathcal{A}}, a^{\mathcal{A}}, p^{\mathcal{A}}, c^{\mathcal{A}}, i^{\mathcal{A}}, o^{\mathcal{A}}\}$, a (pointed) UF frame is said to be a (pointed) *UF$_\alpha$ frame* if it has the α additionally.

(ii) (UF$_{\alpha\backslash a^i_c}$ frames) By *UF$_{\alpha\backslash a^i_c}$ frame*, we denote a UF$_\alpha$ frame, where $\alpha \subseteq \{e^{\mathcal{A}}, p^{\mathcal{A}}, c^{\mathcal{A}}, o^{\mathcal{A}}\}$ or $\alpha \subseteq \{e^{\mathcal{A}}, p^{\mathcal{A}}, i^{\mathcal{A}}, o^{\mathcal{A}}\}$.

For any set F, an evaluation v is a map from propositional variables to elements of F. This evaluation is extended to an *evaluation* on a UF$_{\alpha\backslash a^i_c}$ frame, which is given as a forcing $\Vdash$ between the states of information and the propositional variables, constants, and any sentences satisfying: For all propositional variables p,

($\Vdash$) $a \Vdash p$ iff $a \leq v(p)$;
(AHC) $a \Vdash p$ and $b \leq a$ entail $b \Vdash p$, and

for constants $\overline{1}, \overline{0}$

(1) $a \Vdash \overline{1}$ iff $a \leq 1$;
((0) $a \Vdash \overline{0}$ iff $a \leq 0$, if frames are pointed;) and

[16] Since DL$_\alpha$ has a more general form of deduction theorem containing parameters like GL$_\alpha$, it also requires prime theories containing parameters. For more general deduction theorems, see [7, 8, 47].

for any sentences,[17]

(&)	$a \Vdash \varphi \& \psi$ iff there exist $b, c \in F$ such that $a \le b * c$, $b \Vdash \varphi$, and $c \Vdash \psi$;	
($\wedge$)	$a \Vdash \varphi \wedge \psi$ iff $a \Vdash \varphi$ and $a \Vdash \psi$;	
($\vee$)	$a \Vdash \varphi \vee \psi$ iff $a \Vdash \varphi$ or $a \Vdash \psi$;	
($\to$)	$a \Vdash \varphi \to \psi$ iff for every $b \in F$, $b \Vdash \varphi$ entails $b * a \Vdash \psi$;	
($\rightsquigarrow$)	$a \Vdash \varphi \rightsquigarrow \psi$ iff for every $b \in F$, $b \Vdash \varphi$ entails $a * b \Vdash \psi$.	

$\mathrm{UF}_{\alpha\backslash a^i_c}$ models, validity, and $\mathbf{U}_{\alpha\backslash a^i_c}$ frames and models are defined as in Section 3.1. First, the following lemmas, which can be easily proved, are introduced.

Lemma 4.2. (*HL*) *For a* $UF_{\alpha\backslash a^i_c}$ *frame* $\mathcal{F}$, *for all states of information* $a, b \in \mathcal{F}$, *and for any sentence* φ, $b \le a$ *and* $a \Vdash \varphi$ *entail* $b \Vdash \varphi$.

Lemma 4.3. $1 \Vdash \varphi \to \psi$ *iff for any* $a \in F$, $a \Vdash \varphi$ *entails* $a \Vdash \psi$.

Proposition 4.4. *Let* $\mathcal{F} = (F, 1, (0,) \le, *)$ *be a* $UF_{\alpha\backslash a^i_c}$ *frame and* v *be an evaluation in* $\mathcal{F}$. *Then* $(\mathcal{F}, \Vdash)$ *is a* $UF_{\alpha\backslash a^i_c}$ *model and for every sentence* φ *and for every* $a \in \mathcal{A}$, *one has:* $a \Vdash \varphi$ *iff* $a \le v(\varphi)$.

Proof. The induction steps for $\varphi = \psi \vee \chi$, $\varphi = \psi \& \chi$, $\varphi = \psi \to \chi$, and $\varphi = \psi \rightsquigarrow \chi$ need to be considered. The interesting case is $\varphi = \psi \& \chi$ because the proof for the other ones are already well known. Let φ be $\psi \& \chi$. The condition (&) assures that $a \Vdash \psi \& \chi$ iff one can construct $b, c \in F$ such that $b \Vdash \psi$, $c \Vdash \chi$, and $a \le b * c$, so by the induction hypothesis, iff one can construct $b, c \in F$ such that $b \le v(\psi)$, $c \le v(\chi)$ and $a \le b * c$. Therefore, one obtains $a \le b * c \le v(\psi) * v(\chi) = v(\psi \& \chi)$. For the other direction, let $a \le v(\psi) * v(\chi) = v(\psi \& \chi)$. Take $b = v(\psi)$ and $c = v(\chi)$. One obtains $b \Vdash \psi$, $c \Vdash \chi$ and $a \le b * c$; hence $a \Vdash \psi \& \chi$. □

Proposition 4.5. (*Soundness*) φ *is valid in every* $UF_{\alpha\backslash a^i_c}$ *frame in case* $\vdash_{DL_{\alpha\backslash a^i_c}} \varphi$.

Proof. As examples, the validity of (e), (p), (c), (i), and (o) is proved here.
(e): One has to verify $1 \Vdash (\varphi \& \psi) \to (\psi \& \varphi)$. Lemma 4.3 ensures that for every $a \in F$, we may assume $a \Vdash \varphi \& \psi$ and prove $a \Vdash \psi \& \varphi$. Let $a \Vdash \varphi \& \psi$. The condition (&), monotonicity, and Lemma 4.3 ensure that there exist $b, c \in F$ such that $a \le b * c \le v(\varphi) * v(\psi)$. By ($e^{\mathcal{A}}$), one has $v(\varphi) * v(\psi) \le v(\psi) * v(\varphi)$. Thus, by Proposition 4.4, one obtains that $a \Vdash \psi \& \varphi$.
(c): As above, for every $a \in F$, we assume $a \Vdash \varphi$ and prove $a \Vdash \varphi \& \varphi$. Let $a \Vdash \varphi$. Then, one has $a \le v(\varphi)$ by Proposition 4.4 and so $a \le a * a \le v(\varphi) * v(\varphi)$ by the monotonicity

[17]Fine defined a ternary relation *Rabc* of semantics for relevance logics as $a * b \le c$ in [17] and stated that one can use *Rabc* as a "relativized inclusion (written by $b \le_a c$) with the sense that c is as strong as b relative to a." Note that '$\le$' in (&) is order reversed (compare with $\le$ in Fine's definition ($\&_F$)).

and $(c^{\mathcal{A}})$. Therefore, one has $a \Vdash \varphi \& \varphi$ by the condition (&) and Proposition 4.4.
(p): For every $a \in F$, assume $a \Vdash \varphi \& \varphi$. We need to prove $a \Vdash \varphi$. The condition (&) and Lemma 4.3 ensure that one can construct $b \in F$ such that $a \leq b * b \leq v(\varphi) * v(\varphi)$. By $(p^{\mathcal{A}})$, one has $v(\varphi) * v(\varphi) \leq v(\varphi)$. Thus, one obtains $a \Vdash \varphi$ by Proposition 4.4.
(i): The proof is analogous to that of (p).
(o): For every $a \in F$, let $a \Vdash \overline{0}$. We prove $a \Vdash \varphi$. Using Proposition 4.4 and $(o^{\mathcal{A}})$, one obtains $a \Vdash \varphi$. □

This proposition ensures that $\mathrm{UF}_{\alpha \backslash a^i_c}$ frames are $\mathbf{UF}_{\alpha \backslash a^i_c}$ frames.

Now, by a definition of canonical $\mathrm{UF}_{\alpha \backslash a^i_c}$ frames, *set-theoretic* completeness of DLs^- is provided. We fix $\mathrm{DL}_{\alpha \backslash a^i_c}$ as a non-associative distributive basic substructural logic dropping left weakening or contraction.

Let a $DL_{\alpha \backslash a^i_c}$*-theory* be a theory on $\mathrm{DL}_{\alpha \backslash a^i_c}$ and $\nabla(p, q, \vec{r})$ be a set of sentences with two propositional variables p, q and a sequence (possibly either empty, or finite, or countable infinite) of other variables $\vec{r}$ called parameters. We define $\varphi \nabla \psi$ as follows:

$$\varphi \nabla \psi := \bigcup \{\nabla(\varphi, \psi, \vec{\chi} : \vec{\chi} \in Fm\}.$$

We write $\varphi \vee \psi$ in place of $\varphi \nabla \psi$ if there are no parameters in the set $\nabla(p, q, \vec{r})$. If $T \vdash \varphi$ or $T \vdash \psi$ for a theory T and for any pair φ, ψ of sentences such that $T \vdash \varphi \nabla \psi$, T is called ∇-*prime*; T is just called *prime* if $\nabla = \vee$. A logic L has the *prime extension property*, for brevity PEP, with respect to ∇ if for each theory T and sentence φ such that $T \nvdash_L \varphi$, there is a ∇-prime theory T' such that $T \subseteq T'$ and $T' \nvdash_L \varphi$. ∇ is called a *p-disjunction* in L if it satisfies:

(PD, p-protodisjunction) $\varphi \vdash_L \varphi \nabla \psi$ and $\psi \vdash_L \varphi \nabla \psi$, and

(PCP, proof by cases property) $T, \varphi \vdash_L \chi$ and $T, \psi \vdash_L \chi$ entail $T, \varphi \nabla \psi \vdash_L \chi$.

L is called *p-disjunctional* if it has ∇ which is a p-disjunction in L.

First, the following is a fact following from Theorem 2.7.23 and Remark 2.7.24 in [8].

Fact 4.6. *Let L be a finitary logic, T a theory, and φ, ψ and χ sentences. Then the following are equivalent.*

(i) ∇ is a p-disjunction in L.

(ii) L satisfies the PEP with respect to ∇.

Moreover we note the following fact.

Fact 4.7. (*Theorem 2.7.20,* [8]) *For a finitary weakly implicative logic L*[18]*, the following are equivalent.*

[18] A logic L is called a *weakly implicative logic* if the following are elements of L: (R) $\vdash_L \varphi \to \varphi$; (MP) $\varphi \to \psi, \varphi \vdash_L \psi$; (T) $\varphi \to \psi, \psi \to \chi \vdash_L \varphi \to \chi$; and ($sCng^i_{\#}$, symmetrized congruence) $\varphi \leftrightarrow \psi \vdash_L \#^n(\vec{\chi}, \varphi_i) \to \#^n(\vec{\chi}, \psi_i)$ for each n-nary connective $\#$, a part of $\mathcal{L}$, and each $i \leq n$. For more details, see [8].

(i) L is p-disjunctional.

(ii) The lattice Th(L), the set of all theories of L, is distributive.

Then since $\mathrm{DL}_{\alpha\backslash a^i_c}$ is a finitary weakly implicative distributive logic and so $\mathrm{Th}(\mathrm{DL}_{\alpha\backslash a^i_c})$ is distributive, we have the following as a corollary.

Corollary 4.8. *$DL_{\alpha\backslash a^i_c}$ satisfies the PEP with respect to ∇.*

Let T be a ∇-prime $\mathrm{DL}_{\alpha\backslash a^i_c}$-theory. The *canonical $UF_{\alpha\backslash a^i_c}$ frame* determined by T is defined as a structure $\mathcal{F} = (F_{can}, 1_{can}, (0_{can},) \leq_{can}, *_{can})$, where F_{can} is the set of ∇-prime $\mathrm{DL}_{\alpha\backslash a^i_c}$-theories extending T, 1_{can} is the least ∇-prime theory extending T with $\{\bar{1}\}$, (0_{can} is the least ∇-prime theory extending T with $\{\bar{0}\}$,) $\leq_{can}$ is $\supseteq$ restricted to F_{can}, and $*_{can}$ is defined as follows: $a *_{can} b := \{\varphi\&\psi :$ for some $\varphi \in a$, $\psi \in b\}$, where $*_{can}$ satisfies groupoid properties corresponding to $\mathrm{UF}_{\alpha\backslash a^i_c}$ frames on $(F_{can}, 1_{can}, \leq_{can})$. A canonical $\mathrm{UF}_{\alpha\backslash a^i_c}$ frame is partially ordered since the partial ordering of the canonical $\mathrm{UF}_{\alpha\backslash a^i_c}$ frame depends on $\leq_{can}$ restricted on F_{can}.

Let v_{can} be a canonical evaluation function from sentences to the least sets of sentences, i.e, $v_{can}(\varphi) = \{\varphi\}$. A canonical evaluation is defined:

$$(\square)\ a \Vdash_{can} \varphi \text{ iff } \varphi \in a.$$

Lemma 4.9. *$1_{can} \Vdash_{can} \varphi \to \psi$ only if for every $a \in F_{can}$, $a \Vdash_{can} \varphi$ entails $a \Vdash_{can} \psi$.*

Proof. By ($\square$), one can instead prove that $\varphi \to \psi \in 1_{can}$ only if for every $a \in F_{can}$, $\varphi \in a$ entails $\psi \in a$. Suppose that $\varphi \to \psi \in 1_{can}$ and $\varphi \in a$. We need to prove that $\psi \in a$. One has $\varphi\&(\varphi \to \psi) \in a *_{can} 1_{can} = a$ by the definition of $*_{can}$. Using Proposition 2.4 (i) (1), one further has $(\varphi\&(\varphi \to \psi)) \to \psi \in 1_{can}$ and so $(\varphi\&(\varphi \to \psi)) \to \psi \in a$; hence $\psi \in a$ by (mp). $\square$

Lemma 4.10. *The forcing relation $\Vdash_{can}$ canonically defined is an evaluation.*

Proof. Consider first the conditions for any propositional variables p.

For ($\Vdash$), one has to verify:

$$a \Vdash_{can} p \text{ iff } a \leq_{can} v_{can}(p).$$

By ($\square$), we instead prove that $p \in a$ iff $a \supseteq v_{can}(p)$. Since $v_{can}(p) = \{p\}$, the claim directly follows.

For (AHC), one has to verify:

$$b \leq_{can} a \text{ and } a \Vdash_{can} p \text{ entail } b \Vdash_{can} p.$$

By (□), we instead assume that $b \supseteq a$ and $p \in a$ and prove that $p \in b$. Since $b \supseteq a$, clearly $p \in b$.

Consider next the conditions for constants $\overline{1}$, $\overline{0}$.

For (1), one has to verify:

$$a \Vdash_{can} \overline{1} \text{ iff } a \leq_{can} 1_{can}.$$

By (□), we prove that $\overline{1} \in a$ iff $1_{can} \subseteq a$. This follows from the definition of 1_{can}.

For (0), one has to verify:

$$a \Vdash_{can} \overline{0} \text{ iff } a \leq_{can} 0_{can}.$$

The proof is similar to that of (1).

Finally consider the conditions for any sentences φ, ψ.

For (&), one has to verify:

$$a \Vdash_{can} \varphi \& \psi \text{ iff there exist } b, c \in F_{can} \text{ such that } a \leq_{can} b *_{can} c,\ b \Vdash_{can} \varphi, \text{ and } c \Vdash_{can} \psi.$$

By (□), we instead prove that $\varphi\&\psi \in a$ iff there exist $b, c \in F_{can}$ such that $b *_{can} c \subseteq a$, $\varphi \in b$, and $\psi \in c$. $(\Longrightarrow)$ Let $\varphi\&\psi \in a$. Take b as the least ∇-prime theory extending 1_{can} with $\{\varphi\}$ and c as the least ∇-prime theory extending 1_{can} with $\{\psi\}$. Then, $b *_{can} c$ is the least ∇-prime theory including $\varphi\&\psi$; therefore, $b *_{can} c \subseteq a$. $(\Longleftarrow)$ Let there be $b, c \in F_{can}$ such that $b *_{can} c \subseteq a$, $\varphi \in b$, and $\psi \in c$. The definition of $*_{can}$ assures $\varphi\&\psi \in b *_{can} c$; therefore, $\varphi\&\psi \in a$ since $b *_{can} c \subseteq a$.

For (∨), one has to verify:

$$a \Vdash_{can} \varphi \vee \psi \text{ iff } a \Vdash_{can} \varphi \text{ or } a \Vdash_{can} \psi.$$

We verify that $\varphi \vee \psi \in a$ iff $\varphi \in a$ or $\psi \in a$. $(\Longrightarrow)$ From the fact that a is ∇-prime and so is prime in case $\nabla = \vee$, it follows. $(\Longleftarrow)$ It is obtained using the axioms $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$ and the rule (*mp*).

For (∧), one has to verify:

$$a \Vdash_{can} \varphi \wedge \psi \text{ iff } a \Vdash_{can} \varphi \text{ and } a \Vdash_{can} \psi.$$

We verify that $\varphi \wedge \psi \in a$ iff $\varphi \in a$ and $\psi \in a$. $(\Longrightarrow)$ It is obtained using the axioms $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$ and the rule (*mp*). $(\Longleftarrow)$ It follows from the rule (*adj*).

For (→), one has to verify:

$$a \Vdash_{can} \varphi \to \psi \text{ iff for every } b \in F_{can},\ b \Vdash_{can} \varphi \text{ entails } b *_{can} a \Vdash_{can} \psi.$$

We prove that $\varphi \to \psi \in a$ iff for every $b \in F_{can}$, $\varphi \in b$ entails $\psi \in b *_{can} a$. $(\Longrightarrow)$ Suppose that $\varphi \to \psi \in a$ and $\varphi \in b$. We prove that $\psi \in b *_{can} a$. By the definition of $*_{can}$, one can have $\varphi\&(\varphi \to \psi) \in b *_{can} a$. Moreover, by Proposition 2.4 (*i*) (1) and Lemma 4.9, one further has that $\psi \in b *_{can} a$. $(\Longleftarrow)$ Suppose contrapositively that $\varphi \to \psi \notin a$. One has to prove that there is a ∇-prime theory b such that $\varphi \in b$ but $\psi \notin b *_{can} a$. Take b' as the least $\mathrm{DL}_{\alpha\backslash a^i_c}$-theory, which extends 1_{can} with $\{\varphi\}$ and satisfies $b' *_{can} a = \{\beta :$ there exists $a \in a$ and $T \vdash (\varphi \& a) \to \beta\}$. Clearly, $\varphi \in b'$ and $\psi \notin b' *_{can} a$. (Otherwise, for some $\alpha \in a$, $T \vdash (\varphi \& \alpha) \to \psi$ and so $T \vdash \alpha \to (\varphi \to \psi)$; therefore, $\varphi \to \psi \in a$, a contradiction.) Moreover, by Corollary 4.8, the PEP with respect to ∇ assures that one is capable of obtaining a ∇-prime theory b such that $b' \subseteq b$ and $b *_{can} a = \{\beta :$ there exists $\alpha \in a$ and $T \vdash (\varphi \& \alpha) \to \beta\}$. Hence, one further has $\varphi \in b$ and $\psi \notin b *_{can} a$.

For ($\rightsquigarrow$), one has to verify:

$$a \Vdash_{can} \varphi \rightsquigarrow \psi \text{ iff for every } b \in F,\ b \Vdash_{can} \varphi \text{ entails } a *_{can} b \Vdash_{can} \psi.$$

The proof is similar to that of ($\to$). □

Then, we can show the strong completeness of $\mathrm{DL}_{\alpha\backslash a^i_c}$, using Lemma 4.10 and the PEP with respect to ∇.

Theorem 4.11. (*Strong completeness*) *Let T be a $DL_{\alpha\backslash a^i_c}$-theory, φ a sentence, and $\mathcal{UF}_{\alpha\backslash a^i_c}$ a set of all $UF_{\alpha\backslash a^i_c}$ frames. $T \vdash_{DL_{\alpha\backslash a^i_c}} \varphi$ in case $T \models_{\mathcal{UF}_{\alpha\backslash a^i_c}} \varphi$.*

It is well known that Urquhart semantics in [40] does not work for the system **R** because it validates sentences such as (α) in Section 1, which is not a theorem in **R** (see [16, 33]). In Urquhart semantics, the groupoid operation $*$ requires identity, commutativity, associativity, and idempotence. In fact, by taking conditions for $*$ weaker than or a little different from the $*$ in Urquhart semantics, we can give such a result. Here we finally verify this fact.

Example 4.12. (1) A $\mathrm{UF}_{\{a,c\}}$ model $(\mathcal{F}, \Vdash)$ validates (α) $((\varphi \to (\psi \vee \chi)) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$, i.e., $1 \Vdash ((\varphi \to (\psi \vee \chi)) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$.

(2) A $\mathrm{UF}_{\{a,i\}}$ model $(\mathcal{F}, \Vdash)$ validates (β) $(\varphi \to (\psi \vee \chi)) \to ((\psi \to \chi) \rightsquigarrow (\varphi \to \chi))$.

Proof. (1): The condition ($\wedge$) and Lemma 4.3 assure that one may instead suppose that $a \Vdash \varphi \to (\psi \vee \chi)$ and $a \Vdash \psi \to \chi$ and prove that $a \Vdash \varphi \to \chi$. Let $a \Vdash \varphi \to (\psi \vee \chi)$ and $a \Vdash \psi \to \chi$. By the condition ($\to$), we additionally suppose that $b \Vdash \varphi$ and verify that $b * a \Vdash \chi$. Using the suppositions and ($\to$), one is capable of obtaining $b * a \Vdash \psi \vee \chi$ and so either $b * a \Vdash \psi$ or $b * a \Vdash \chi$ by the condition ($\vee$). We need to consider the case $b * a \Vdash \psi$. Using ($\to$), one can have that $(b * a) * a \Vdash \chi$ since $a \Vdash \psi \to \chi$. Then, by ($a^{\mathcal{A}}$), one moreover has $b * (a * a) \Vdash \chi$. Therefore, using Lemma 4.2, one obtains that $b * a \Vdash \chi$ since $a \leq a * a$ by ($c^{\mathcal{A}}$).

(2): Similarly one may instead suppose that $a \Vdash \varphi \to (\psi \vee \chi)$ and prove that $a \Vdash (\psi \to \chi) \rightsquigarrow (\varphi \to \chi)$. Let $a \Vdash \varphi \to (\psi \vee \chi)$. By the condition ($\rightsquigarrow$), we additionally suppose that $b \Vdash \psi \to \chi$ and verify that $a * b \Vdash \varphi \to \chi$. To verify this, we also suppose that $c \Vdash \varphi$ and prove that $c * (a * b) \Vdash \chi$. Using the suppositions and ($\to$), one is capable of obtaining $c * a \Vdash \psi \vee \chi$ and so either $c * a \Vdash \psi$ or $c * a \Vdash \chi$ by ($\vee$). First, consider the case $c*a \Vdash \psi$. Since $b \Vdash \psi \to \chi$, using ($\to$) one has that $(c*a)*b \Vdash \chi$ and so $c*(a*b) \Vdash \chi$ by the condition ($a^{\mathcal{A}}$). Second, consider the case $c * a \Vdash \chi$. Since $1 \Vdash \chi \to \chi$, using ($\to$), one has that $(c * a) * 1 \Vdash \chi$. Then, since $b \leq 1$ by ($i^{\mathcal{A}}$), using Lemma 4.2 one moreover get $(c * a) * b \Vdash \chi$; therefore, $c * (a * b) \Vdash \chi$. □

(1) shows that UF models with associativity and contraction as weaker conditions for $*$ validate (α) and (2) shows that UF models with associativity and left weakening as different conditions for $*$ validate (β). Note that the distributive basic logic systems $\mathrm{DL}_{\{a,c\}}$ and $\mathrm{DL}_{\{a,i\}}$ do not prove (α) and (β), respectively, in Example 4.12. Therefore, $\mathrm{UF}_{\{a,c\}}$ and $\mathrm{UF}_{\{a,i\}}$ models do not work for their corresponding logics. This means that UF_{α} models, where $\{a, c\} \subseteq \alpha \subseteq \{e, a, c, p, i, o\}$ or $\{a, i\} \subseteq \alpha \subseteq \{e, a, c, p, i, o\}$, do not work for their corresponding DL_{α} logics.

Open Problem: A $\mathrm{UF}_{\{a\}}$ model can be introduced similarly. We do not know whether this model can work for the logic $\mathrm{DL}_{\{a\}}$. This remains an open problem.

4.2 Star-based prime US semantics for dmDLs^{-}

We can consider star-based Urquhart–Fine-style semantics for dmDLs^{-} introduced in Definition 2.3 (*iii*). As one sort of this semantics, we introduce star-based prime US semantics for dmDLs^{-}. For this, we first introduce pointed $\mathrm{UF}_{\alpha\backslash a^i_c}$ frames with star operations for negations as $\mathrm{dmUF}_{\alpha\backslash a^i_c}$ frames.[19]

Definition 4.13. ($\mathrm{dmUF}_{\alpha\backslash a^i_c}$ frames) A $\mathrm{UF}_{\alpha\backslash a^i_c}$ frame with 0 is said to be a *pointed* $\mathrm{UF}_{\alpha\backslash a^i_c}$ frame. A *dmUF*$_{\alpha\backslash a^i_c}$ *frame* is a pointed $\mathrm{UF}_{\alpha\backslash a^i_c}$ frame with two unary operations $\times$ and $+$.

An *evaluation* on a $\mathrm{dmUF}_{\alpha\backslash a^i_c}$ frame is given as in Section 4.1 but with the additional conditions below. For every sentence φ,

($\sim$) $a \Vdash \sim \varphi$ iff $a^{\times} \nVdash \varphi$;
($-$) $a \Vdash -\varphi$ iff $a^{+} \nVdash \varphi$.

Note that the conditions for ($\sim$) and ($-$) are new ones for the negations $\sim$ and $-$, respectively, satisfying de Morgan laws. Moreover, $\mathrm{dmUF}_{\alpha\backslash a^i_c}$ models, validity, and $\mathbf{dmU}_{\alpha\backslash a^i_c}$ frames and models are defined as in Section 4.1.

Now we prove soundness for dmDLs^{-}.

Proposition 4.14. (*Soundness*) *φ is valid in each dmUF$_{\alpha\backslash a^i_c}$ frame in case* $\vdash_{dmDL_{\alpha\backslash a^i_c}}$ φ.

[19]In order to express two star operations for two negations, we introduce the notations $\times$ and $+$ in place of the notation $*$ used in relevance logic because this notation was already used to express groupoid operations.

Proof. We must consider the axioms (sdmI$_\sim$), (sdmI$_-$), (sdmII$_\sim$), and (sdmII$_-$). As its example, we verify the validity of (sdmI$_\sim$). To verify $1 \Vdash \sim (\varphi \wedge \psi) \to (\sim \varphi \vee \sim \psi)$, for every $a \in F$, one can suppose $a \Vdash \sim (\varphi \wedge \psi)$ and prove $a \Vdash \sim \varphi \vee \sim \psi$. The condition $(\sim)$ ensures that $a \Vdash \sim (\varphi \wedge \psi)$ iff $a^\times \nVdash \varphi \wedge \psi$ and thus iff $a^\times \nVdash \varphi$ or $a^\times \nVdash \psi$ iff $a \Vdash \sim \varphi$ or $a \Vdash \sim \psi$; therefore, $a \Vdash \sim \varphi \vee \sim \psi$ by $(\vee)$. The proof for the other axioms is analogous. □

For completeness results for dmDLs$^-$, let T be a ∇-prime dmDL$_{\alpha\backslash a_c^i}$-theory. The *canonical dmUF$_{\alpha\backslash a_c^i}$ frame* determined by T is defined as a structure: $\mathcal{F} = (F_{can}, 1_{can}, 0_{can}, \leq_{can}, *_{can}, {}^\times_{can}, {}^+_{can})$, where $(F_{can}, 1_{can}, 0_{can}, \leq_{can}, *_{can})$ are defined as in Section 4.1 and the canonical operations ${}^\times_{can}, {}^+_{can}$ are defined as follows:

$$({}^\times_{can}) \sim \varphi \in a \text{ iff } \varphi \notin a^\times; \text{ and}$$
$$({}^+_{can}) -\varphi \in a \text{ iff } \varphi \notin a^+.$$

As above, clearly a canonical dmUF$_{\alpha\backslash a_c^i}$ frame is partially ordered.

Next, as (□) in Section 4.1, we define a canonical evaluation.

Lemma 4.15. *The forcing relation $\Vdash_{can}$ canonically defined is an evaluation.*

Proof. It suffices to consider the conditions $(\sim)$ and $(-)$.

For $(\sim)$, one has to verify:

$$a \Vdash_{can} \sim \varphi \text{ iff } a^\times \nVdash_{can} \varphi.$$

By (□), one may instead verify that $\sim \varphi \in a$ iff $\varphi \notin a^\times$. The claim follows from the definition $({}^\times_{can})$.

The proof for $(-)$ is analogous to that of $(\sim)$. □

This lemma ensures that the $(\mathcal{F}, \Vdash_{can})$ canonically defined is a dmUF$_{\alpha\backslash a_c^i}$ model. Thus, we can prove the strong completeness of dmDL$_{\alpha\backslash a_c^i}$, using Lemma 4.15 and the PEP with respect to ∇.

Theorem 4.16. (*Strong completeness*) *Let T be a dmDL$_{\alpha\backslash a_c^i}$-theory, φ a sentence, and dm$\mathcal{UF}_{\alpha\backslash a_c^i}$ a set of all dmUF$_{\alpha\backslash a_c^i}$ frames. $T \vdash_{dmDL_{\alpha\backslash a_c^i}} \varphi$ in case $T \models_{dm\mathcal{UF}_{\alpha\backslash a_c^i}} \varphi$.*

Note that for a negation to be a de Morgan negation it requires classical contraposition and double negation elimination as well as contraposition, double negation introduction, and de Morgan laws (see [16]). By adding their corresponding axioms and postulates, we can extend dmDL$_{\alpha\backslash a_c^i}$ logics and dmUF$_{\alpha\backslash a_c^i}$ frames to logics and frames with de Morgan negations. This means that we can introduce star-based semantics with negations *weaker* than de Morgan negations. The same can be done in Routley-Meyer models as well.

Remark 4.17.

1. *We may consider* $dmDL_{\alpha\backslash a_c^i}$ *logics dropping the constant* $\overline{0}$*, df3, and df4. For their basic logic, we can introduce the negations* $\sim$ *and* $-$ *as primitive connectives and add* $(CP_{\sim})$*,* (CP_{-}) *in Proposition 2.4* (ii)*, and* (sdmI$_{\sim}$)*,* (sdmI$_{-}$)*,* (sdmII$_{\sim}$)*, and* (sdmII$_{-}$) *as the additional axioms to* ***DL****, the* ***GL*** *with (D). Let us call this system* ***wdmDL****. Then, in order to provide UF frames for* ***wdmDL****, called henceforth* wdmUF frames *similarly, we need to consider* $\sim$ *and* $-$ *as primitive negation operators and* $(CP^{\mathcal{A}})$ $a \leq b$ *entails* $\sim b \leq \sim a$ *and* $-b \leq -a$ *and de Morgan laws as the additional conditions for wdmUF frames for* ***wdmDL****, the basic wdmDL*$_{\alpha\backslash a_c^i}$ *logic.*[20]

2. *Note that for the logic* ***wdmDL*** *and its corresponding wdmUF frames in* 1*, we need not add* $(DNI_{-\sim})$ *and* $(DNI_{\sim-})$ *and their corresponding frame properties because in order to validate these two double negation introductions we must have the additional conditions* $(-\sim)$ $a^{+\times} \leq a$ *and* $(\sim -)$ $a^{\times +} \leq a$*, respectively.*[21] *This means that for* $(DNI_{-\sim})$ *and* $(DNI_{\sim-})$*, we need the conditions* $(-\sim)$ *and* $(\sim -)$*, respectively.*[22]

3. *1 and 2 show that the logical principles contrapositions and strong de Morgan laws are* minimal *conditions for a distributive logic with negations to have star-based semantics.*

4.3 Powers and limitations

Prime US semantics is *less powerful* than AUS semantics in that it cannot cover non-distributive systems introduced in Section 2. However, prime US semantics is still *very powerful* in the sense that it can cover a lot of distributive logics, even though not all of them. To verify this, we considered the Urquhart–Fine-style semantics for DLs$^-$. Especially this semantics addressed in Section 4.1 is valuable in the following several senses: First, the Urquhart–Fine-style semantics is also *very powerful* in an another sense that frames for DLs$^-$ need not have the same structures as algebraic semantics for DLs$^-$. As is shown in Section 4.1, this semantics only requires operational(-relational) properties of the groupoid operator $*$ for intensional conjunction and implications. Note that it

[20]The star operations $\times$, $+$ validate $(CP_{\sim})$ and (CP_{-}), respectively. For instance, for $(CP_{\sim})$, one can assume that $1 \Vdash \varphi \to \psi$ and verify that $1 \Vdash \sim \psi \to \sim \varphi$. To prove this, for any state of information a, one may further assume that $a \Vdash \sim \psi$ and verify that $a \Vdash \sim \varphi$. Then, since $b \Vdash \sim \varphi$ iff $b^{\times} \nVdash \varphi$ by $(\times)$, we instead assume that $a^{\times} \Vdash \varphi$ and prove that $a^{\times} \Vdash \psi$. Since $a^{\times} = a^{\times} * 1 \Vdash \varphi \& (\varphi \to \psi)$, we have that $a^{\times} \Vdash \psi$ as above. Therefore, we need $(CP_{\sim})$ and (CP_{-}) as axioms.

[21]For instance, consider $(DNI_{-\sim})$. To validate this, one can assume that $a \Vdash \varphi$ and verify that $a \Vdash - \sim \varphi$. Then, using $(-\sim)$, we have that $a^{+\times} \Vdash \varphi$. Thus, since $a^{+\times} \Vdash \varphi$ iff $a^{+} \nVdash \sim \varphi$ iff $a \Vdash - \sim \varphi$ by $(-)$ and $(\sim)$, we further obtain $a \Vdash - \sim \varphi$.

[22]Note that dmUF frames for **dmDL** do not require these conditions because $(DNI_{-\sim})$ and $(DNI_{\sim-})$ can be validated without introducing the conditions $(-\sim)$ and $(\sim -)$, respectively.

does not require any operational evaluations for extensional conjunction and disjunction. Therefore, we can think of this semantics as *minimal* US semantics.

Second, regarding the first, the Urquhart–Fine-style semantics is independent of algebraic semantics and moreover provides its own set-theoretic models for DLs^-. Note that many logicians who work on semantics of formal logic systems are familiar with prime theories rather than closed theories. They are also interested in having set-theoretic models because of their familiarity with sets. By use of set-theoretic models instead of algebraic models, one could have a more intuitive grasp on the structures of the logic. Therefore, the Urquhart–Fine-style semantics would be *very useful* to such people because of its set-theoretic consideration.

Third, like the nuclear US semantics this semantics *philosophically* provides a natural interpretation of connectives in terms of information too. The only difference between them is in disjunction and intensional conjunction. The nuclear US semantics requires operational interpretations based on nuclear completions on those connectives, whereas the Urquhart–Fine-style semantics requires an operational-relational interpretation on intensional conjunction and does not require any operational interpretation on disjunction, i.e., it just requires primeness on it. In this sense, this semantics is simpler than the nuclear US semantics and so philosophers and logicians may more easily understand their intuitive meanings as combinations of information.

However, it has a clear limit because it does not work for basic substructural logics in general, even distributive such logics in general. Related to this, we can say the following two limitations of Urquhart–Fine-style semantics. First, as a natural consequence, it covers less logics than the nuclear US semantics. Note that Kripke semantics has been widely used for the propositional intuitionistic logic **H** since it is easier to handle **H** than other semantics such as Beth, Dragalin, and topological semantics. But it covers less logics than those semantics (see [3]). Similarly, Urquhart–Fine-style semantics is easier to handle DLs^- than nuclear US semantics but the former semantics covers less logics than the latter semantics.

Second, it is not fully operational. As is well-known, the Urquhart semantics for the relevant implication is fully operational. Similarly, the nuclear US semantics is also fully operational as seen in Section 3. However, the evaluation of the intensional conjunction (&) is operational and relational. In this sense, Urquhart–Fine-style semantics does neither fully support nor fully follow Urquhart's operational idea. In the context of operational semantics, we need to introduce the Urquhart semantics with an operation evaluation of intensional conjunction.[23]

Interestingly, these two limitations instead provide a motivation for new researches. Related to this, we note that Urquhart motivated logicians to introduce logic systems based on semilattice semantics. As mentioned in Section 1, such logic systems have been

[23] This is an interesting future work.

introduced. The logics such as **UR**, **URW**, **UT** and **UTW**, which are called operational logics by Standefer [37], are good examples. Similarly we may investigate logic systems based on Urquhart–Fine-style semantics as logics proving sentences such as (α).

Star-based Urquhart–Fine-style semantics is a particular kind of Urquhart–Fine-style semantics applied to dmDLs$^-$. Thus this semantics has powers and limitations similar to Urquhart–Fine-style semantics when we compare it with nuclear US semantics. In particular, as the 3 in Remark 4.17 shows, star-based Urquhart–Fine-style semantics verifies what are the *minimal* conditions for a distributive logic with negations to have star-based semantics. This is one important contribution of this star-based Urquhart–Fine-style semantics. Moreover, this semantics provides operational evaluations to negations and so follows the idea of operational semantics with respect to negations. Finally, we note that star-based Urquhart–Fine-style semantics is *less powerful* than Urquhart–Fine-style semantics in general in the sense that it can only cover dmDLs$^-$. Instead, star-based Urquhart–Fine-style semantics is *more useful* than Urquhart–Fine-style semantics to people who are interested in semantics with star operations for negations. These are the powers and limitations of star-based Urquhart–Fine-style semantics we can say in this paper.

5 Concluding remarks

We investigated three kinds of US semantics for basic substructural logics and their distributive extensions. We in particular considered powers and limitations of these three kinds of semantics.

As the class of logics satisfying reflexivity, transitivity, modus ponens, and tonicity, Yang and Dunn [52] studied implicational tonoid logics. Then, since all the logic systems introduced in Definition 2.3 satisfy those properties, they are also concrete implicational tonoid logics. We will consider this more exactly in a subsequent paper.

Note that fuzzy extensions of basic substructural logics have been introduced as basic fuzzy logics (see [6, 7, 8, 46, 47]). We can introduce US semantics for these extensions. But this paper has not enough space to do it. We also anticipate another paper to investigate powers and limitations of US semantics for basic substructural fuzzy logics.

6 Declarations

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