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Huishan WU

EFFECTIVE ASPECTS OF SEMIPERFECT RINGS

A b s t r a c t. This paper studies effective aspects of semiperfect rings from the standpoint of reverse mathematics. Based on first-order Jacobson radicals of rings, we define a ring R with the Jacobson radical $Jac(R)$ to be semiperfect if the quotient ring $R/Jac(R)$ is semisimple, and idempotents of the quotient ring can be lifted to R. Using elementary matrix operations in linear algebra, we show that RCA_0 proves a characterization of semiperfect rings in terms of idempotents of rings. Semiperfect rings are generalizations of semisimple rings and local rings, and semiperfect rings R with $R/Jac(R)$ simple are isomorphic to matrix rings over local rings. Based on the effective characterization of semiperfect rings via idempotents, we prove the structure theorem of semiperfect rings R with $R/Jac(R)$ simple in RCA_0. Left perfect rings or right perfect rings are always semiperfect. Finally, we provide a proof for the structure theorem of one-sided perfect rings R with $R/Jac(R)$ simple in WKL_0.

1 Introduction

Reverse mathematics is a program initiated by Friedman [8, 9] in 1970s. To study a theorem in reverse mathematics, we often first formalize the theorem by using second-

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order language L_2 and then look for a weakest subsystem τ of second-order arithmetic Z_2 such that τ proves the theorem. Early empirical work of the area reveals that theorems in various disciplines of mathematics are often equivalent to one of the big five subsystems of Z_2 over a properly chosen base system, where the five systems are listed as RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1-\mathrm{CA}_0$. RCA_0 serves as the base system. It contains basic axioms about operations and orders on natural numbers (see Definition I.2.4(i), [15]), induction axioms restricted to Σ^0_1 formulas (i.e., Σ^0_1 induction) and axioms asserting the existence of sets that can be defined by both Σ^0_1 and Π^0_1 formulas (i.e., Δ^0_1 comprehension). For systematic developments of countable mathematics within the big five systems, refer to Simpson's monograph [15]. For a general introduction to reverse mathematics, see a recent book of Stillwell [19]. For new developments of reverse mathematics in recent decades, refer to Dzhafarov and Mummert's new book [7].

Reverse mathematics is closely connected with computable mathematics, where computable mathematics studies algorithmic content of mathematics from the perspective of computability theory [12, 16, 17]. The base system RCA_0 has a minimal ω-model whose second-order part contains the set of computable sets (see page 65, [15]), so we often view theorems as effective if they are provable in RCA_0 and noneffective otherwise. Take algebraic closures for example [10, 13]. The statement "every countable field has an algebraic closure" is provable in RCA_0, this implies that the effective version of the statement holds. That is, every computable field has a computable algebraic closure. In addition, the statement "every countable field has a unique algebraic closure" is equivalent to WKL_0 over RCA_0, this implies that the effective version of the statement fails. That is, two computable algebraic closures of a computable field may not be computably unique in the sense that they are not isomorphic via a computable isomorphism. For more examples in countable algebra that illustrate the connections between reverse mathematics and computable mathematics, we refer to articles such as Solomon [18], Downey, Lempp, and Mileti [5, 6], Conidis [3, 4].

Modules over rings are generalizations of algebraic structures (e.g., groups, vector spaces) that are extensively studied in reverse mathematics. Yamazaki first studied modules over general rings in the context of reverse mathematics in [24], he also initiated the study of homological algebra in reverse mathematics in [25]. Following the work of Yamazaki, we are interested in studying rings and modules from the perspective of reverse mathematics. Semiperfect rings form an important class of rings, and they are generalizations of various kinds of rings like one-sided artinian rings, semisimple rings and local rings. In this paper, we study effective aspects of semiperfect rings. For the background of algebra studied here, please refer to Lam's book [11]. In the following, we only consider rings with identity but not necessarily commutative.

To define semiperfect rings, we need the notion of Jacobson radicals of rings, which is often defined in a second-order way as the intersection of all maximal left ideals of rings. As in [20], we define Jacobson radicals in a first-order way.

Definition 1.1. (RCA_0) The Jacobson radical of a ring R is defined as the Π^0_2 set $Jac(R) = \{x \in R : \forall y \in R \exists z \in R[z(1_R - yx) = 1_R]\}$.

The Π^0_2 set $Jac(R)$ is actually a two-sided ideal of R. The classical proof of the fact depends on second-order characterizations of $Jac(R)$. Since for general rings, $Jac(R) = \bigcap\{\mathfrak{M} : \mathfrak{M}$ is a maximal left ideal of $R\}$ is equivalent to ACA_0 over RCA_0 (see Theorem 6.19 in Sato's thesis [14]), the classical proof works in ACA_0. We have provided a direct proof for the fact in RCA_0.

Proposition 1.2. *[20] Let R be a ring such that the Jacobson radical $Jac(R)$ exists. Then* RCA_0 *proves that $Jac(R)$ is a two-sided ideal of R.*

For a ring $R \subseteq \mathbb{N}$, one can form the quotient ring $R/Jac(R)$, whose elements are $\leq_{\mathbb{N}}$-least representatives under the equivalence relation: $x \sim y \Leftrightarrow x - y \in Jac(R)$. As usual, we write $\overline{R} := R/Jac(R) = \{\overline{r} : r \in R\}$, where $\overline{r}$ is the least representative of $r \in R$ under the equivalence relation above. For any $r, s \in R$, we have $\overline{r} = \overline{s} \Leftrightarrow r - s \in Jac(R)$.

To define semiperfect rings, we also need the notion of semisimple rings. Semisimple rings have various classical characterizations (refer to Chapter 1 of [11]). Sato first studied semisimple rings in the context of reverse mathematics based on ring-theoretic definitions at the end of his thesis [14], where semisimple rings are defined as finite direct products of simple artinian rings. Recently, we have studied semisimple rings based on module-theoretic definitions [21, 22], where semisimple rings are defined as rings whose regular modules are semisimple modules. Semisimple modules possess known characterizations which are only equivalent over ACA_0. When studying semisimple rings in a module-theoretic way, different characterizations of semisimple modules result in different definitions of semisimple rings.

Definition 1.3. (RCA_0) A nonzero left R-module M is simple if the Π^0_2 condition $\forall x \in M \setminus \{0_M\} \forall y \in M \exists r \in R[y = rx]$ holds.

Since we are interested in structures of rings, we view semisimple modules as those that can be decomposed as direct sums of simple submodules. Then semisimple rings are defined as follows.

A left R-module is an abelian group with a left scalar multiplication such that the usual module axioms hold. We often use ${}_RM$ to denote a left R-module. Each ring R possesses a natural left R-module structure, namely, the left regular module ${}_RR$, with the addition and left scalar multiplication defined by the addition and multiplication of the ring itself.

Definition 1.4. (RCA_0) A ring R is left semisimple if the left regular module ${}_RR$ is a direct sum of simple submodules.

An element $e \in R$ is called an *idempotent* if $e = e^2$.

Definition 1.5. (RCA_0) A ring R is semiperfect if $Jac(R)$ exists, and the quotient ring $\overline{R} = R/Jac(R)$ meets the following two conditions:

(1) $\overline{R}$ is a left semisimple ring;

(2) idempotents of $\overline{R}$ can be lifted to R; that is, if $\overline{r}$ is an idempotent of $\overline{R}$, then there is an idempotent $e \in R$ such that $\overline{r} = \overline{e}$.

Based on previous work on Jacobson radicals of rings and semisimple rings, we study effective aspects of semiperfect rings. We first consider a basic characterization of semiperfect rings in terms of idempotents of rings.

Theorem 1.6. *[11] The following are equivalent for a ring R.*

(1) R is semiperfect.

(2) $1_R = e_1 + \cdots + e_n$ for some pairwise orthogonal idempotents $e_i (1 \leq i \leq n)$ with each Re_i strongly indecomposable.

The classical proof of Theorem 1.6 appeared in [11] uses various arithmetic sets, and thus works in ACA_0; especially, the proof depends on Nakayama's Lemma, whose classical proof often requires ACA_0. In Section 3, using elementary matrix operations of linear algebra (see e.g., Lemma 3.3 below), we develop an effective version of Theorem 1.6 in RCA_0.

We next consider the structure theorem of a subclass of semiperfect rings.

Theorem 1.7. *[11] The following are equivalent for a ring R.*

(1) R is semiperfect with $R/Jac(R)$ simple.

(2) $R \cong M_n(S)$ for some $n \geq 1$ and some local ring S, where $M_n(S)$ is the $n \times n$ matrix ring over S.

In Section 4, based on the characterization of semiperfect rings in terms of idempotents above, we provide an effective proof for Theorem 1.7 in RCA_0.

Semiperfect rings are generalizations of one-sided perfect rings. As an application of Theorem 1.7, we continue to study the structure theorem of one-sided perfect rings R with $R/Jac(R)$ simple.

Theorem 1.8. *[11] The following are equivalent for a ring R.*

(1) R is left (resp., right) perfect with $R/Jac(R)$ simple.

(2) $R \cong M_n(S)$ for some $n \geq 1$ and some local ring S whose maximal ideal is left (resp., right) T-nilpotent.

The classical proof of Theorem 1.8 appeared in [11] relies on an equivalent characterization of left (resp., right) T-nilpotent ideals, proofs involving such arguments often require ACA_0. In Section 5, based on Bounded König's Lemma, we develop a proof for Theorem 1.8 in WKL_0. We would like to know whether RCA_0 proves Theorem 1.8 or not.

The rest of the sections are organized as follows. In Section 2, we provide necessary properties of semisimple rings and local rings. In Section 3, we study the characterization of semiperfect rings in terms of idempotents of rings. In Sections 4 and 5, we study structure theorems of a subclass of semiperfect rings and one-sided perfect rings, respectively.

2 Semisimple rings and local rings

Semiperfect rings are generalizations of semisimple rings and local rings. In this section, we provide necessary properties of semisimple rings and local rings.

2.1 Semisimple rings

Semisimple rings have a nice characterization in terms of idempotents of rings. Recall that two idempotents $e, f \in R$ are *orthogonal* if $ef = fe = 0_R$.

Proposition 2.1. *The following are equivalent over* RCA_0 *for a ring* R.

(1) R *is a left semisimple ring.*

(2) $1_R = e_1 + \cdots + e_n$ *for some pairwise orthogonal idempotents* $e_i (1 \leq i \leq n)$ *of* R *with each* Re_i *a simple left* R*-module; in this case,* ${}_RR = Re_1 \oplus \cdots \oplus Re_n$.

Proof. That (2) $\Rightarrow$ (1) is clear. We provide a proof for (1) $\Rightarrow$ (2). If R is left semisimple, by definition, the left regular module can be written as ${}_RR = S_1 \oplus \cdots \oplus S_n$ for some simple submodules $S_1, \ldots, S_n$. Then $1_R = x_1 + \cdots + x_n$ for some $x_i \in S_i (1 \leq i \leq n)$. For any $y \in S_i$, we have $y = y1_R = y(x_1 + \cdots + x_n) = yx_1 + \cdots + yx_n$. Then

$$y - yx_i = \sum_{j \neq i} yx_j = 0_R,$$

so $y = yx_i$ and $yx_j = 0_R$ for $j \neq i$. This implies that $S_i = Rx_i$ and that $x_1, \ldots, x_n$ are pairwise orthogonal idempotents of R. (2) holds. □

For an idempotent $e \in R$, we point out that $Re = \{re : r \in R\}$ exists in RCA_0 because for any $x \in R$, $x \in Re \Leftrightarrow x = xe$. Similarly, $eRe = \{ere : r \in R\}$ also exists in RCA_0.

To prove the effective characterization of semiperfect rings in Section 3, we will use the following technical lemma on semisimple rings.

Lemma 2.2. (RCA_0) *Let R be a semisimple ring such that $R = Re_1 \oplus \cdots \oplus Re_n$ for some pairwise orthogonal idempotents $e_1, \cdots, e_n$ with Re_i a simple left R-module for all $1 \le i \le n$. For any left ideal I of R, there is a set $A \subseteq \{1, \cdots, n\}$ such that $R = I \oplus \bigoplus_{i \in A} Re_i$ as left R-modules.*

Proof. For any subset $\{i_1, \cdots, i_k\}$ of $\{1, \cdots, n\}$, $I \cap \bigoplus_{j=1}^{k} Re_{i_j} = \{0_R\}$ if and only if the Π^0_1 condition

$$\forall r \in I\ \forall r_1, \cdots, r_k \in R[r = r_1 e_{i_1} + \cdots + r_k e_{i_k} \to r = 0_R]$$

holds. Form a set $X := \{\{i_1, \cdots, i_k\} \subseteq \{1, \cdots, n\} : I \cap \bigoplus_{j=1}^{k} Re_{i_j} = \{0_R\}\}$. Then X exists by bounded Π^0_1 comprehension.

Build a maximal set A in X as follows.

(1) If for all $1 \le i \le n$, $\{i\} \notin X$, i.e., $I \cap Re_i \neq \{0_R\}$, then there is an element $s \in R$ such that $se_i \in I \setminus \{0_R\}$. Since Re_i is simple, we have that $Re_i = Rse_i \subseteq I$ and $e_i \in I$. Then $R = Re_1 \oplus \cdots \oplus Re_n = I$, take $A = \emptyset$.

(2) If there is an $1 \le i \le n$ such that $\{i\} \in X$. We can define a maximal set $A \in X$ extending $\{i\}$ as follows:

(2.1) If $\{i\}$ is maximal, i.e., for any $j \in \{1, \cdots, n\} \setminus \{i\}$, $\{i, j\} \notin X$, then let $A = \{i\}$.

(2.2) If (2.1) fails, then there is a $j \neq i$ with $\{i, j\} \in X$. There are two subcases:

- if $\{i, j\}$ is maximal, then let $A = \{i, j\}$;
- otherwise, there is a $k \in \{1, \cdots, n\} \setminus \{i, j\}$ with $\{i, j, k\} \in X$. Continue the process finitely many steps, we obtain the set $A \in X$ such that for any $j \in \{1, \cdots, n\} \setminus A$, we have $A \cup \{j\} \notin X$.

We claim that the maximal set A in X is the desired set. For any e_j with $j \notin A$, by $A \cup \{j\} \notin X$, there is a nonzero $r \in I$ such that $r = se_j + \sum_{i \in A} r_i e_i$ for some $s, r_i \in R$. By $A \in X$, we see that $se_j \neq 0_R$. Again, we have $Re_j = Rse_j$ and $e_j = tse_j$ for some $t \in R$ because Re_j is simple. Now

$$e_j = tse_j = t(r - \sum_{i \in A} r_i e_i) = tr - \sum_{i \in A} tr_i e_i \in I \oplus \bigoplus_{i \in A} Re_i.$$

Therefore, $R = Re_1 \oplus \cdots \oplus Re_n \subseteq I \oplus \bigoplus_{i \in A} Re_i \subseteq R$ and $R = I \oplus \bigoplus_{i \in A} Re_i$. □

Matrix rings over division rings are typical examples of left semisimple rings. The structure theorem of semisimple rings (i.e., Wedderburn-Artin Theorem on page 33, [11]) says that left semisimple rings are isomorphic to finite direct products of matrix rings over division rings. With the help of the theory of idempotents of rings, we have obtained a proof for the theorem in RCA_0.

Theorem 2.3. *[22]*(RCA_0) *A left semisimple ring is isomorphic to a finite direct product of matrix rings over division rings.*

A ring is *right semisimple* if the right regular module R_R is a finite direct sum of right simple modules. By symmetry, we have a right version of Theorem 2.3 above. Therefore, RCA_0 proves that a ring is left semisimple iff it is right semisimple. In the following, we only deal with left semisimple rings.

2.2 Local rings

An element r of a ring R is *left invertible* if $sr = 1_R$ for some $s \in R$. Let $U_l(R) = \{r \in R : \exists s \in R[sr = 1_R]\}$.

Definition 2.4. [20](RCA_0) A ring R is left local if $U_l(R)$ exists, and the set of non left invertible elements $R \setminus U_l(R)$ is closed under the addition of R.

One can also define right local rings and local rings via right invertible elements and invertible elements, respectively. That is, R is *right local* if $U_r(R) = \{r \in R : \exists s \in R[rs = 1_R]\}$ exists, and the set of non right invertible elements $R \setminus U_r(R)$ is closed under the addition of R; R is *local* if $U(R) = \{r \in R : \exists s \in R[sr = rs = 1_R]\}$ exists, and the set of non invertible elements $R \setminus U(R)$ is closed under the addition of R. By Corollary 3.4 in [20], RCA_0 can prove that a ring is left local iff it is right local iff it is local. In the following, we only focus on left local rings.

Lemma 2.5. (RCA_0) *The following are equivalent for a ring R.*

(1) R is left local.

(2) $Jac(R)$ exists, and $Jac(R) = R \setminus U_l(R)$.

(3) $Jac(R)$ exists, and the quotient ring $\overline{R} := R/Jac(R)$ is a division ring.

Proof.
(1) $\Rightarrow$ (2). Let R be a left local ring. That is, $U_l(R)$ exists, and $R \setminus U_l(R)$ is closed under addition. $Jac(R) = \{x \in R : \forall y \in R \exists z \in R[z(1_R - yx) = 1_R]\}$.

- If $x \in U_l(R)$, let $yx = 1_R$, then for any $z \in R$, we have $z(1_R - yx) = 0_R \neq 1_R$, by definition, $x \notin Jac(R)$.

- If $x \notin Jac(R)$, there is a $y \in R$ such that for any $z \in R$, we have $z(1_R - yx) \neq 1_R$. That is, $1_R - yx \notin U_l(R)$. Suppose otherwise that $x \notin U_l(R)$. Then $yx \notin U_l(R)$, we see that $1_R = yx + (1_R - yx) \notin U_l(R)$ because $R \setminus U_l(R)$ is closed under addition, this derives a contradiction. So we have $x \in U_l(R)$.

We have shown that $U_l(R) = R \setminus Jac(R)$. (2) holds.

(2) $\Rightarrow$ (1) is clear because $Jac(R)$ is a two-sided ideal of R by Proposition 1.2.

(3) $\Rightarrow$ (2). Since $U_l(R) \subseteq R \setminus Jac(R)$, it suffices to prove that $R \setminus Jac(R) \subseteq U_l(R)$. If $x \notin Jac(R)$, then $\overline{x} \in \overline{R}$ is nonzero, and thus invertible in the division ring $\overline{R}$. Let $\overline{y}$ be an inverse of $\overline{x}$ in $\overline{R}$, i.e., $1_R - yx \in Jac(R)$. Then, by the definition of $Jac(R)$, there is an element $z \in R$ such that $zyx = z[1_R - (1_R - yx)] = 1_R$, so $x \in U_l(R)$. (2) holds.

(2) $\Rightarrow$ (3). Let $\overline{x} \in \overline{R}$ be nonzero, i.e., $x \notin Jac(R)$, we need to show that $\overline{x}$ is invertible in the quotient ring $\overline{R}$. First, by $Jac(R) = R \setminus U_l(R)$, $x \in U_l(R)$, let $yx = 1_R$ for some $y \in R$. Second, by Proposition 1.2, $Jac(R)$ is a two-sided ideal of R, we have $y \notin Jac(R)$. Again, this means that $y \in U_l(R)$ and $zy = 1_R$ for some $z \in R$. Now we have $z = z(yx) = (zy)x = x$, and $yx = xy = 1_R$. So $\overline{y}$ is an inverse of $\overline{x}$ in $\overline{R}$. $\overline{R}$ is a division ring. □

For a general ring R, $Jac(R) = \bigcap\{\mathfrak{M} : \mathfrak{M}$ is a maximal left ideal of $R\}$ is equivalent to ACA_0 over RCA_0 (see Theorem 6.19, [14]). For left local rings, by Lemma 2.5, we see that the equality holds in RCA_0.

Proposition 2.6. (RCA_0) *For a left local ring R, $Jac(R) = R \setminus U_l(R)$ is the unique maximal left ideal of R.*

In classical algebra, local rings are used to define strongly indecomposable modules, where a left R-module M is *strongly indecomposable* if the endomorphism ring $End({}_RM)$ is local. Although the endomorphism ring of general modules is a third-order object, for a cyclic R-module Re with e an idempotent of R (such modules are called *principal modules*), the endomorphism ring can be defined as follows.

Definition 2.7. (RCA_0) Let e be an idempotent of a ring R. The endomorphism ring of the left R-module Re is (encoded by) the ring $eRe = \{ere : r \in R\}$ with identity e.

It is not hard to see the idea behind Definition 2.7. In fact, if $\varphi : Re \to Re$ is a left R-endomorphism, the desired code is just the image of φ at e, i.e., $(e)\varphi = (ee)\varphi = e(e)\varphi \in eRe$; conversely, each element $ere \in eRe$ determines a left R-endomorphism on Re which sends e to ere.

Definition 2.8. (RCA_0) A left R-module Re with e an idempotent of R is strongly indecomposable if eRe is a local ring.

Schur's Lemma says the endomorphism ring $End({}_RM)$ of a simple R-module M is a division ring. When restricted to principal modules, based on Definition 2.7, we can prove the lemma as follows.

Proposition 2.9. (RCA_0) *Schur's Lemma: If Re is a simple R-module, then eRe is a division ring with identity e.*

Proof. Let $r \in R$. Since Re is simple, if $ere \neq 0_R$, then $Re = Rere$, and $e = sere$ for some $s \in R$. Now $e = ee = e(sere) = (ese)(ere)$. On the other hand, $ese \neq 0_R$ implies that $Re = Rese$ and $(ete)(ese) = e$ for some $t \in R$. As usual, one can check that $ere = ete$, and thus, ere is invertible in the ring eRe. □

Proposition 2.10. (RCA_0) *For a ring R and an idempotent $e \in R$, the following are equivalent.*

(1) Re is an indecomposable left R-module.

(2) e is not a sum of two nonzero orthogonal idempotents of R.

Proof. (1) $\Rightarrow$ (2). If (2) fails, let $e = f + g$ for some nonzero orthogonal idempotents of R. Then $fe = f(f+g) = f^2 + fg = f$, $ge = g(f+g) = gf + g^2 = g$, and Re can be decomposed as $Re = Rfe \oplus Rge$. (1) fails.

(2) $\Rightarrow$ (1). If (1) fails, that is, $Re = M_1 \oplus M_2$ for two nonzero submodules $M_1, M_2 \subseteq Re$. Then $e = x_1 + x_2$ for some nonzero $x_1 \in M_1, x_2 \in M_2$, and $x_1 = x_1 e = x_1 x_1 + x_1 x_2$ implies that x_1 is an idempotent of R and $x_1 x_2 = 0_R$. Similarly, x_2 is an idempotent of R with $x_2 x_1 = 0_R$. (2) fails. □

Simple modules Re are strongly indecomposable because eRe is a division ring, which is always local. On the other hand, using Proposition 2.10, one can check that strongly indecomposable modules Re are always indecomposable. Proposition 2.11 shows that the three notions are the same for semisimple rings.

Proposition 2.11. (RCA_0) *For a semisimple ring R and a nonzero idempotent $e \in R$, the following conditions are equivalent.*

(1) Re is a simple R-module.

(2) Re is a strongly indecomposable R-module.

(3) Re is an indecomposable R-module.

Proof.

(1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are true for general rings.

For (3) $\Rightarrow$ (1). Let Re be an indecomposable R-module. As R is left semisimple, there are mutually orthogonal idempotents $e_i (1 \leq i \leq n)$ with Re_i a simple R-module such that $1_R = e_1 + \cdots + e_n$. By Lemma 2.2, there is a set $A \subseteq \{1, \cdots, n\}$ such that $R = Re_1 \oplus \cdots \oplus Re_n = Re \oplus \bigoplus_{i \in A} Re_i$. Then $Re \cong \bigoplus_{i \notin A} Re_i$. Since Re is indecomposable, there is exactly one number $1 \leq i \leq n$ such that $i \notin A$; thus $Re \cong Re_i$ and Re is a simple left R-module. □

3 Semiperfect rings

In this section, we develop an effective proof for the characterization of semiperfect rings in terms of idempotents of rings in Theorem 1.6 above.

Definition 3.1. (Yamazaki, [25]) (RCA_0) A left R-module P is projective if it is a direct summand of a free left R-module.

Free modules are isomorphic to direct sums of copies of regular modules, refer to Proposition 1.6, [23]. For an idempotent e of a ring R, Re is projective because it is a direct summand of the left regular module ${}_RR$.

Proposition 3.2. *(Yamazaki, [25]) For a left R-module P, the following conditions are equivalent over* RCA_0.

(1) P is projective.

(2) For any R-modules M, N, for any surjective R-homomorphism $\varepsilon : M \to N$ and R-homomorphism $\varphi : P \to N$, there is an R-homomorphism $\psi : P \to M$ such that $\psi\varepsilon$[1]$= \varphi$.

For a left R-module M and a two-sided ideal I of R, IM is the submodule of M generated by $\{rx : r \in I, x \in M\}$. That is,

$$IM = \{r_1x_1 + \cdots + r_nx_n : n \geq 1, r_1, \ldots, r_n \in I, x_1, \ldots, x_n \in M\}.$$

Lemma 19.27 in Lam's book [11] is crucial to prove various properties of semiperfect rings. The classical proof of it uses various arithmetic sets, and thus requires ACA_0. We now develop an effective proof for Lam's Lemma 19.27 in Lemma 3.3 below.

Lemma 3.3. *Let R be a ring with the Jacobson radical $J := Jac(R)$, and P, Q be finitely generated projective left R-modules with generating sets $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$, respectively. Assume that JP, JQ exist. Then the following conditions are equivalent over* RCA_0.

(1) $P \cong Q$ as left R-modules.

(2) $P/JP \cong Q/JQ$ as left $\overline{R} := R/J$-modules.

[1]Following the convention in algebra text [11], we write left R-module homomorphisms on right, that is, $\varphi : M \to N; x \mapsto (x)\varphi$, and write the composition of two left R-homomorphisms from left to right, that is, for two R-homomorphisms $\varphi : M \to N; x \mapsto (x)\varphi$ and $\psi : N \to K; x \mapsto (x)\psi$, the composition is $\varphi\psi : M \to K; x \mapsto ((x)\varphi)\psi$.

Proof. For the quotient modules, we write $P/JP = \{\overline{x} : x \in P\}$, where $\overline{x}$ is the $\leq_{\mathbb{N}}$-least representative of $x \in P$ under the equivalence relation $x \sim y \Leftrightarrow x - y \in JP$ on P; similarly, write $Q/JQ = \{\overline{y} : y \in Q\}$.

(1) $\Rightarrow$ (2). Let $\varphi : P \to Q$ be a left R-module isomorphism. Define $\overline{\varphi} : P/JP \to Q/JQ$ by sending $\overline{x}$ to $\overline{(x)\varphi}$ for all $x \in P$. It is a direct check that $\overline{\varphi}$ is an $\overline{R}$-module isomorphism.

(2) $\Rightarrow$ (1). Assume that $\psi : P/JP \to Q/JQ$ is an $\overline{R}$-module isomorphism. Consider the natural surjective R-homomorphisms $\pi_P : P \to P/JP$; $x \mapsto \overline{x}$ and $\pi_Q : Q \to Q/JQ$; $y \mapsto \overline{y}$. Since P is a projective R-module, there is an R-module homomorphism $\varphi : P \to Q$ such that $\varphi\pi_Q = \pi_P\psi$, i.e., the following diagram commutes:

$$\begin{array}{ccccc} P & \xrightarrow{\pi_P} & P/JP & & \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} & & \\ Q & \xrightarrow{\pi_Q} & Q/JQ & \longrightarrow & 0 \end{array}$$

We first show that $\varphi : P \to Q$ is onto, that is, $im(\varphi) = \{y \in Q : \exists x \in P[y = (x)\varphi]\} = Q$. By assumption, Q is generated by $\{y_1, \cdots, y_m\}$. Since $\varphi\pi_Q = \pi_P\psi$ is a surjective R-module homomorphism, for each $y_j \in Q$ with $1 \leq j \leq m$, there is a $z_j \in P$ such that $\overline{y_j} = (z_j)\varphi\pi_Q = \overline{(z_j)\varphi}$, i.e., $y_j - (z_j)\varphi \in JQ$. Then there are $a_{j1}, \cdots, a_{jm} \in J$ such that

$$y_j - (z_j)\varphi = a_{j1}y_1 + \cdots + a_{jm}y_m$$

for $1 \leq j \leq m$, and we have the following matrix equations:

$$\begin{pmatrix} 1_R - a_{11} & -a_{12} & \cdots & -a_{1m} \\ -a_{21} & 1_R - a_{22} & \cdots & -a_{2m} \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \cdots & 1_R - a_{mm} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} (z_1)\varphi \\ (z_2)\varphi \\ \vdots \\ (z_m)\varphi \end{pmatrix}$$

Let B be the square matrix in the equality above. We claim that B is left invertible in the matrix ring $M_m(R)$. As in linear algebra (see Theorem 1.2.16, [2]), it suffices to show that the row echelon form of B is the identity matrix I_m in the matrix ring $M_m(R)$.

For all $1 \leq i \leq m$, as $a_{ii} \in J = Jac(R)$, the i-th diagonal entry $1_R - a_{ii}$ of B is left invertible in R; so there is a $u_i \in R$ such that $u_i(1_R - a_{ii}) = 1_R$. Compute the row echelon form of B as follows:

- *Step 1.* Left multiply the first row of B by $a_{21}u_1$, and then add the resulting first row to the second row of B, obtaining a matrix with the $(2,1)$-entry

  $$a_{21}u_1(1_R - a_{11}) + (-a_{21}) = a_{21} - a_{21} = 0_R,$$

  the $(2,2)$-entry $a_{21}u_1(-a_{12}) + (1_R - a_{22}) = 1_R - (a_{22} + a_{21}u_1a_{12}) \in 1_R + J$ and the $(2,j)$-entry $a_{21}u_1(-a_{1j}) + (-a_{2j}) \in J$ for all $3 \leq j \leq m$.

Continue this process by adding left multiples of the first row to the remaining rows one by one, we obtain a matrix B_1 whose $(i,1)$-entries with $i \neq 1$ are zero, (i,i)-entries are of the form $1_R - x$ with $x \in J$ and thus left invertible, and (i,j)-entries with $j \geq 2$ and $i \neq j$ are inside J.

- *Step* i $(2 \leq i \leq m)$. Suppose that we have obtained B_{i-1} at Step $i-1$. Add left multiple of the i-th row of B_{i-1} to other rows one by one, obtaining a matrix B_i whose nondiagonal entries with columns $j \leq i$ are zero, diagonal entries belong to $1_R + J$ and thus left invertible, and other nondiagonal entries belong to J. In particular, B_m is a matrix with diagonal entries left invertible and other entries zero.

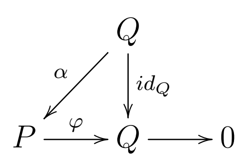
- *Step* $m+1$. Left multiply the rows of B_m one by one to obtain the identity matrix I_m.

Now the row echelon form of B is the identity matrix I_m, and B is left invertible. Let $CB = I_m$ for some $C \in M_m(R)$. We have

$$(y_1, \cdots, y_m)^T = CB(y_1, \cdots, y_m)^T = C((z_1)\varphi, \cdots, (z_m)\varphi)^T,$$

and $y_j (1 \leq j \leq m)$ is an R-linear sum of elements in $\{(z_1)\varphi, \cdots, (z_m)\varphi\}$. So $y_j \in im(\varphi)$ for all $1 \leq j \leq m$. Since $\{y_1, \ldots, y_m\}$ generates Q as left R-modules, $Q = im(\varphi)$ and $\varphi : P \to Q$ is onto.

We next show that $\varphi : P \to Q$ is one-to-one, that is, $ker(\varphi) = \{x \in P : (x)\varphi = 0_Q\} = \{0_P\}$. By the projectivity of Q, for the identity homomorphism $id_Q : Q \to Q$ and the surjective homomorphism $\varphi : P \to Q$, there is a homomorphism $\alpha : Q \to P$ such that $id_Q = \alpha\varphi$.



First of all, $im(\alpha)$ exists in RCA_0. Indeed, let $x \in P$, $(x)\varphi = (((x)\varphi)\alpha)\varphi$ implies that $x - ((x)\varphi)\alpha \in ker(\varphi)$. If $x \in im(\alpha)$, then $x - ((x)\varphi)\alpha = (y)\alpha \in ker(\varphi)$ for some $y \in Q$, and $y = ((y)\alpha)\varphi = 0_Q$. So $x = ((x)\varphi)\alpha$. Conversely, if $x = ((x)\varphi)\alpha$, then $x \in im(\alpha)$. Hence,

$$x \in im(\alpha) \Leftrightarrow x = ((x)\varphi)\alpha.$$

$im(\alpha)$ exists by Σ^0_0 comprehension. Furthermore, we see that $P = ker(\varphi) \oplus im(\alpha)$ because $ker(\varphi) \cap im(\alpha) = \{0_P\}$, and for any $x \in P$, $x = (x - ((x)\varphi)\alpha) + ((x)\varphi)\alpha$.

Claim. $ker(\varphi) = Jker(\varphi) = JP \cap ker(\varphi)$.

Proof of the claim. First, $Jker(\varphi) = JP \cap ker(\varphi)$. Clearly, $Jker(\varphi) \subseteq JP \cap ker(\varphi)$. Now let $x \in JP \cap ker(\varphi)$. Since $P = ker(\varphi) \oplus im(\alpha)$ is generated by $\{x_1, \ldots, x_n\}$, let

$x_i = y'_i + z'_i$ be the unique decomposition with $y'_i \in ker(\varphi)$ and $z'_i \in im(\alpha)$. Then there are $r_1, \ldots, r_n \in J$ such that

$$x = r_1x_1 + \cdots + r_nx_n = (r_1y'_1 + \cdots + r_ny'_n) + (r_1z'_1 + \cdots + r_nz'_n) \in JP \cap ker(\varphi).$$

Let $z = r_1z'_1 + \cdots + r_nz'_n \in im(\alpha)$. Then $(z)\varphi = (x)\varphi = 0_Q$, and $z \in ker(\varphi) \cap im(\alpha)$. So $z = 0_P$ and $x = r_1y'_1 + \cdots + r_ny'_n \in Jker(\varphi)$. This shows that $JP \cap ker(\varphi) \subseteq Jker(\varphi)$. Hence, $Jker(\varphi) = JP \cap ker(\varphi)$.

Second, $ker(\varphi) = Jker(\varphi)$. It suffices to show that $ker(\varphi) \subseteq Jker(\varphi)$. Let $x \in ker(\varphi)$. Then $(\overline{x})\psi = (x)\pi_P\psi = (x)\varphi\pi_Q = \overline{(x)\varphi}$ is zero in Q/JQ. Since $\psi : P/JP \to Q/JQ$ is an $\overline{R}$-isomorphism, $\overline{x}$ is zero in P/JP, and $x \in JP$. Thus, we have $x \in JP \cap ker(\varphi) = Jker(\varphi)$.

This ends the proof of the claim.

We are ready to show that $ker(\varphi) = \{0_P\}$. P is generated by $\{x_1, \cdots, x_n\}$. Again, for each $1 \leq i \leq n$, let $x_i = y'_i + z'_i$ with $y'_i \in ker(\varphi), z'_i \in im(\alpha)$. Then $ker(\varphi)$ is generated by $\{y'_1, \cdots, y'_n\}$, and $y'_i \in ker(\varphi) = Jker(\varphi)$ implies that there are $b_{i1}, \cdots, b_{in} \in J = Jac(R)$ such that $y'_i = b_{i1}y'_1 + b_{i2}y'_2 + \cdots + b_{in}y'_n$ for all $1 \leq i \leq n$. That is,

$$\begin{pmatrix} 1_R - b_{11} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & 1_R - b_{22} & \cdots & -b_{2n} \\ \vdots & \vdots & & \vdots \\ -b_{n1} & -b_{n2} & \cdots & 1_R - b_{nn} \end{pmatrix} \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{pmatrix} = \begin{pmatrix} 0_P \\ 0_P \\ \vdots \\ 0_P \end{pmatrix}$$

As before, since $b_{ij} \in J = Jac(R)$ for all $1 \leq i, j \leq n$, the square matrix on the equation above is left invertible and thus $y'_i = 0_P$ for all $1 \leq i \leq n$. Then $ker(\varphi) = \{0_P\}$ and $\varphi : P \to Q$ is an R-module isomorphism. □

We remark that the classical proof of Lemma 19.27 in [11] uses Nakayama's Lemma. Here, we avoid the use of Nakayama's Lemma that appeared in the classical proofs like $im(\varphi) + JQ = Q \Rightarrow im(\varphi) = Q$, which requires the existence of both $im(\varphi)$ and the sum $im(\varphi) + JQ$, and thus ACA_0.

For a proof of Proposition 3.4 below, refer to Theorem 21.10, [11].

Proposition 3.4. (RCA_0) *Let R be a ring with the Jacobson radical $J := Jac(R)$. For an idempotent $e \in R$, $Jac(eRe) = J \cap eRe = eJe = \{ere : r \in J\}$.*

For a semiperfect ring R, using Lemma 3.3 above, we show that indecomposable modules Re are always strongly indecomposable.

Proposition 3.5. (RCA_0) *Let R be a semiperfect ring with $J := Jac(R)$ and e an idempotent of R. The following conditions are equivalent.*

(1) Re is an indecomposable R-module.

(2) Re is a strongly indecomposable R-module.

Proof. We only need to show (1) $\Rightarrow$ (2). Let Re be an indecomposable R-module. Consider the least representative $\overline{e} \in \overline{R} := R/J$. We first show that $\overline{R}\overline{e}$ is an indecomposable $\overline{R}$-module. Suppose otherwise that $\overline{R}\overline{e}$ is decomposable. Then $\overline{e} = x + y$ with x, y nonzero orthogonal idempotents of $\overline{R}$. Since idempotents of $\overline{R}$ can be lifted to R, there are idempotents f, g of R such that $x = \overline{f}, y = \overline{g}$, and $fg, gf \in J$. Claim that there is an idempotent $h \in R$ with $\overline{h} = \overline{g} = y$ and $fh = hf = 0_R$. The classical proof of the claim works in RCA_0 (see Proposition 21.22, [11]), we omit the details here.

Let $e' = f + h$. e' is an idempotent of R, and $\overline{e} = x + y = \overline{f} + \overline{h} = \overline{f+h} = \overline{e'}$. Then we have the following isomorphisms of $\overline{R}$-modules:

$$Re/Je \cong \overline{R}\overline{e} = \overline{R}\ \overline{e'} \cong Re'/Je'.$$

Since Re and Re' are projective R-modules, by Lemma 3.3, $Re \cong Re' = R(f + h) = Rf \oplus Rh$ as R-modules, and Re is decomposable, a contradiction. So $\overline{R}\overline{e}$ is indecomposable as an $\overline{R}$-module.

As $\overline{R}$ is a semisimple ring, by Proposition 2.11, $\overline{R}\overline{e}$ is a simple $\overline{R}$-module. Then $eRe/Jac(eRe) = eRe/eJac(R)e \cong \overline{e}\overline{R}\overline{e}$ is a division ring. By Lemma 2.5, $Jac(eRe) = eRe \setminus U_l(eRe)$, the set of non left invertible elements of eRe. This shows that eRe is a local ring. By definition, Re is strongly indecomposable. ☐

Proposition 3.5 above will be used to prove the characterization of semiperfect rings in terms of idempotents. Following the classical proof of the characterization of semiperfect rings, we also need (1) of Ex 21.16 in [11], for completeness, we provide a proof for the exercise in the following proposition.

Proposition 3.6. *Let e_1, e_2 be two idempotents of a ring R and let $f_1 = 1_R - e_1, f_2 = 1_R - e_2$. The following conditions are equivalent over* RCA_0.

(1) $Re_1 \cong Re_2$ and $Rf_1 \cong Rf_2$ as left R-modules.

(2) $e_1 = u^{-1}e_2u$ for some unit $u \in R$.

Proof. (1) $\Rightarrow$ (2). Let $\varphi : Re_1 \to Re_2;\ e_1 \mapsto ae_2$ and $\psi : Rf_1 \to Rf_2;\ f_1 \mapsto cf_2$ be left R-module isomorphisms with inverse $\varphi^{-1} : Re_2 \to Re_1;\ e_2 \mapsto be_1$ and $\psi^{-1} : Rf_2 \to Rf_1;\ f_2 \mapsto df_1$, respectively. Then we have

$$\begin{aligned}
&ae_2 = (e_1)\varphi = (e_1e_1)\varphi = e_1(e_1)\varphi = e_1ae_2,\\
&be_1 = (e_2)\varphi^{-1} = (e_2e_2)\varphi^{-1} = e_2(e_2)\varphi^{-1} = e_2be_1,\\
&e_1 = (e_1)\varphi\varphi^{-1} = (ae_2)\varphi^{-1} = a(e_2)\varphi^{-1} = abe_1 = ae_2be_1,\\
&e_2 = (e_2)\varphi^{-1}\varphi = (be_1)\varphi = b(e_1)\varphi = bae_2 = be_1ae_2.
\end{aligned}$$

In a similar way, one can obtain that

$$cf_2 = f_1cf_2, df_1 = f_2df_1, f_1 = cf_2df_1 \text{ and } f_2 = df_1cf_2.$$

Take $u = (e_2)\varphi^{-1} + (f_2)\psi^{-1} = be_1 + df_1$. It is a direct check that u is a unit of R with inverse $u^{-1} = (e_1)\varphi + (f_1)\psi = ae_2 + cf_2$. Calculate $u^{-1}e_2u$ as follows:

$$\begin{aligned} u^{-1}e_2u =&(ae_2 + cf_2)e_2(be_1 + df_1) \\ =&ae_2(be_1 + df_1) + cf_2e_2(be_1 + df_1) \\ =&ae_2be_1 + ae_2df_1 = ae_2be_1 + ae_2f_2df_1 = ae_2be_1 = e_1. \end{aligned}$$

(2) ⇒ (1). Assume that $u^{-1}e_2u = e_1$ for some unit u in R. Define an R-module homomorphism $\varphi : Re_1 \to Re_2$ by sending e_1 to $u^{-1}e_2$. Then

$$(re_1)\varphi = r(e_1)\varphi = ru^{-1}e_2.$$

On the one hand, $(rue_1)\varphi = ruu^{-1}e_2 = re_2$, this means that φ is onto. On the other hand, $(re_1)\varphi = ru^{-1}e_2 = 0_R$ implies that $re_1 = ru^{-1}e_2u = 0_R$, so φ is one-to-one. This shows that $\varphi : Re_1 \to Re_2$ is a left R-module isomorphism.

Note that $u^{-1}f_2u = u^{-1}(1_R - e_2)u = 1_R - u^{-1}e_2u = 1_R - e_1 = f_1$. As in the paragraph above, one can prove that $Rf_1 \cong Rf_2$ as left R-modules. □

We are ready to provide an effective proof for the characterization of semiperfect rings by using lemmas and propositions developed above.

Theorem 3.7. *For a ring R with the Jacobson radical $J := Jac(R)$, the following conditions are equivalent over* RCA_0.

(1) R is a semiperfect ring.

(2) $1_R = e_1 + \cdots + e_n$ for some pairwise orthogonal idempotents $e_i (1 \le i \le n)$ of R with each Re_i strongly indecomposable.

Proof. (1) ⇒ (2). Suppose that R is semiperfect. That is, $J = Jac(R)$ exists and $\overline{R} := R/J = \{\overline{r} : r \in R\}$ is a semisimple ring such that idempotents of $\overline{R}$ can be lifted to R. First, as $\overline{R}$ is a semisimple ring, there are mutually orthogonal idempotents $x_1, \cdots, x_n$ of $\overline{R}$ such that $\overline{1_R} = x_1 + \cdots + x_n$ and $\overline{R}x_i$ is a simple left $\overline{R}$-module for all $1 \le i \le n$. Second, there are idempotents $e_i \in R$ such that $x_i = \overline{e_i}$ for all $1 \le i \le n$. It is a direct check that each $Re_i (1 \le i \le n)$ is indecomposable. By Proposition 3.5, Re_i is strongly indecomposable. Let $e = e_1 + \cdots + e_n$. In $\overline{R}$, we have

$$\overline{1_R} = x_1 + \cdots + x_n = \overline{e_1} + \cdots + \overline{e_n} = \overline{e_1 + \cdots + e_n} = \overline{e}.$$

Then $f := 1_R - e \in J$ and $e = 1_R - f$ is left invertible. Let $ve = 1_R$ for some $v \in R$, we see that $f = vef = 0_R$ and $e = 1_R$. So $1_R = e_1 + \cdots + e_n$. (2) holds.

(2) $\Rightarrow$ (1). Let $1_R = e_1 + \cdots + e_n$ with $e_1, \cdots, e_n$ pairwise orthogonal idempotents of R and Re_i strongly indecomposable for all $i = 1, \cdots, n$. For each i, e_iRe_i is a left local ring, that is, $Jac(e_iRe_i) = e_iJac(R)e_i = e_iJe_i$ equals the set of non left invertible elements of the ring e_iRe_i. By Lemma 2.5, $e_iRe_i/Jac(e_iRe_i) = e_iRe_i/e_iJe_i \cong \overline{e_i}\overline{R}\overline{e_i}$ is a division ring.

Claim. For any nonzero $\overline{r} \in \overline{R}$, there is an element $s \in R$ such that $\overline{rsr} \neq \overline{0_R}$.

Proof of the claim. We first show that $Jac(\overline{R}) = \{\overline{0_R}\}$. Let $x \in Jac(\overline{R})$ with $x = \overline{t}$ for some $t \in R$, i.e., $x = \overline{t}$ meets the condition: for any $y \in \overline{R}$, there is a $z \in \overline{R}$ such that $z(\overline{1_R} - yx) = \overline{1_R}$. It remains to check that $t \in Jac(R)$. For any $r_1 \in R$, $\overline{1_R} - \overline{r}_1\, \overline{t} = \overline{1_R - r_1t}$ is left invertible in $\overline{R}$. Let $\overline{r_2}\ \overline{1_R - r_1t} = \overline{1_R}$ with $r_2 \in R$. Then $r_2(1_R - r_1t) - 1_R \in Jac(R)$ and we see that $r_2(1_R - r_1t)$ is left invertible in R; so is $1_R - r_1t$. Hence, $t \in Jac(R)$ by definition, and $x = \overline{t} = \overline{0_R}$.

For the given nonzero $\overline{r}$, we have $\overline{r} \notin Jac(\overline{R})$, i.e., there is an element $s \in R$ such that $\overline{1_R} - \overline{sr}$ is not left invertible in $\overline{R}$. For the particular element $\overline{1_R} + \overline{sr}$,

$$(\overline{1_R} + \overline{sr})\ (\overline{1_R} - \overline{sr}) = \overline{1_R} - \overline{srsr} \neq \overline{1_R},$$

Then $\overline{srsr} \neq \overline{0_R}$, and thus, $\overline{rsr} \neq \overline{0_R}$.

This ends the proof of the claim.

We are ready to show that $\overline{R}\overline{e_i}$ is a simple left $\overline{R}$-module. Let $\overline{r}\ \overline{e_i} = \overline{re_i}$ be a nonzero element in $\overline{R}\overline{e_i}$. By the Claim, there is an element $s \in R$ such that $\overline{re_isre_i}$ is nonzero in $\overline{R}$. Clearly, $\overline{e_isre_i}$ is nonzero in the division ring $\overline{e_i}\overline{R}\overline{e_i}$. Then $\overline{e_isre_i}$ is left invertible in the ring $\overline{e_i}\overline{R}\overline{e_i}$, i.e., there is a nonzero $\overline{u} \in \overline{R}$ such that

$$\overline{e_i} = \overline{e_iue_i}\ \overline{e_isre_i} = \overline{e_iue_is}\ \overline{re_i} \in \overline{R}\overline{re_i}.$$

Then, as left $\overline{R}$-modules, $\overline{R}\overline{e_i} = \overline{R}\overline{re_i}$. This means that $\overline{re_i}$ generates the left $\overline{R}$-module $\overline{R}\overline{e_i}$. By definition, $\overline{R}\overline{e_i}$ is simple as a left $\overline{R}$-module.

By $\overline{1_R} = \overline{e_1} + \cdots + \overline{e_n}$, we see that $\overline{R} = \overline{R}\overline{e_1} \oplus \cdots \oplus \overline{R}\overline{e_n}$ is a direct decomposition of the left regular module of $\overline{R}$ into simple left $\overline{R}$-modules. That is, $\overline{R}$ is a left semisimple ring. To prove that R is a semiperfect ring, it remains to show that idempotents of $\overline{R}$ can be lifted to R.

Let x be an idempotent of $\overline{R}$. Since $\overline{R} = \overline{R}\overline{e_1} \oplus \cdots \oplus \overline{R}\overline{e_n}$ is semisimple, for the left $\overline{R}$-module $\overline{R}x$, by Lemma 2.2, there is a set $A \subseteq \{1, \cdots, n\}$ such that

$$\overline{R}x \cong \bigoplus_{i\in A} \overline{R}\overline{e_i} = \overline{R}(\sum_{i\in A} \overline{e_i}).$$

By Proposition 3.6, there is a unit $z = \overline{u} \in \overline{R}$ for some $u \in R$ such that $x = z^{-1}(\sum_{i\in A} \overline{e_i})z$. Let $z^{-1} = \overline{v}$ with $v \in R$. Now $\overline{uv} = \overline{vu} = \overline{1_R}$, i.e., $1_R - uv, 1_R - vu \in J = Jac(R)$. Then $uv = 1_R - (1_R - uv)$ and $vu = 1_R - (1_R - vu)$ are left invertible in R.

Clearly, u is left invertible in R. We now show that u is right invertible too. Since uv is left invertible, $suv = 1_R$ for some $s \in R$. Now $s = 1_R + s(1_R - uv) \in 1_R + Jac(R)$ is left invertible and let $ts = 1_R$ with $t \in R$. Then

$$uvs = (ts)uvs = t(suv)s = ts = 1_R$$

and u is right invertible. Now u is invertible, let u^{-1} be the inverse of u in R. Set $e = \sum_{i\in A} u^{-1}e_iu$. Since $\sum_{i\in A} e_i$ is an idempotent of R, so is $e = u^{-1}(\sum_{i\in A} e_i)u$; furthermore, we have

$$\overline{e} = \overline{\sum_{i\in A} u^{-1}e_iu} = \sum_{i\in A} \overline{u^{-1}}\overline{e_i}\ \overline{u} = \overline{u^{-1}}(\sum_{i\in A} \overline{e_i})\overline{u} = z^{-1}(\sum_{i\in A} \overline{e_i})z = x.$$

That is, e is an idempotent of R that lifts to the idempotent x of $\overline{R}$. Hence, R is semiperfect. (1) holds. □

4 Semiperfect rings R with $R/Jac(R)$ simple

Semiperfect rings R with $R/Jac(R)$ simple are isomorphic to matrix rings over local rings. In this section, we will develop an effective proof for the result in RCA_0.

We first review the structure of rings that are both semisimple and simple.

Definition 4.1. (RCA_0) A ring R is simple if R is generated by any nonzero element as a two-sided ideal. That is, the Π^0_2 condition

$$\forall x \in R \setminus \{0_R\} \forall y \in R \exists r_1, s_1, \cdots, r_n, s_n \in R[y = r_1xs_1 + \cdots + r_nxs_n]$$

holds.

Typical examples of simple rings are matrix rings over division rings. For semisimple rings, we have obtained in [22] that RCA_0 proves the structure theorem: semisimple rings are isomorphic to finite direct products of matrix rings over division rings. Then semisimple and simple rings are always isomorphic to matrix rings over division rings. For completeness, we now sketch the proof.

Lemma 4.2. (RCA_0) *Let R be a semisimple and simple ring with $R = Re_1 \oplus \cdots \oplus Re_n$ for simple left R-modules $Re_i (1 \le i \le n)$. Then $Re_1 \cong \cdots \cong Re_n$, and R is isomorphic to the matrix ring $M_n(e_1Re_1)$ over the division ring e_1Re_1.*

Proof. For $1 \le i, j \le n$, we have that $Re_i \cong Re_j \Leftrightarrow \exists r, s \in R[e_i = rs \wedge e_j = sr]$. So $Re_i \cong Re_j$ is a Σ^0_1 relation on $\{1, 2, \cdots, n\}$. We can collect isomorphic direct summands of R in the decomposition $R = Re_1 \oplus \cdots \oplus Re_n$ as follows:

(1) Let $i_1 = 1$. By bounded Σ^0_1 comprehension, $X_1 = \{j : 1 \le j \le n, Re_j \cong Re_{i_1}\}$ exists, one can count the size $|X_1|$ of X_1.

(2) Let i_2 be the least number in $\{1, 2, \cdots, n\} \setminus X_1$. Similarly, $X_2 = \{j : 1 \le j \le n, Re_j \cong Re_{i_2}\}$ exists, one can count the size $|X_2|$ of X_2.

(3) Continue the process for no more than n many steps, we obtain numbers $i_1, \dots, i_t$ such that $X_1, \dots, X_t$ form a partition of $\{1, 2, \dots, n\}$, where $X_k = \{j : 1 \le j \le n, Re_j \cong Re_{i_k}\}$ for $1 \le k \le t$.

For each $1 \le k \le t$, the size of X_k is the number of Re_j's isomorphic to Re_{i_k}, let $n_k = |X_k|$. Then $R \cong (Re_{i_1})^{n_1} \oplus \cdots \oplus (Re_{i_t})^{n_t}$ as left R-modules. As shown in Lemma 3 of [21], $R \cong M_{n_1}(e_{i_1} Re_{i_1}) \times \cdots \times M_{n_t}(e_{i_t} Re_{i_t})$ as rings. As R is a simple ring, $t = 1$, so $Re_1 \cong \cdots \cong Re_n$, and we have $R \cong M_n(e_1 Re_1)$. □

To prove the structure theorem of semiperfect rings in RCA_0, we need to verify the effectiveness of several properties related to matrix rings.

Proposition 4.3. (RCA_0) *Let R be a ring with the Jacobson radical $Jac(R)$, then $Jac(M_n(R)) = M_n(Jac(R))$, where $M_n(R)$ is the matrix ring over R.*

Proof. We first show that $Jac(M_n(R)) \subseteq M_n(Jac(R))$. Let $A = (a_{ij})$ be a matrix in $Jac(M_n(R))$, i.e., for any $B \in M_n(R)$, there is a matrix $C \in M_n(R)$ such that $C(I_n - BA) = I_n$, where I_n is the identity matrix in $M_n(R)$ with the (i,i)-entry 1_R for all $1 \le i \le n$ and other entries 0_R. For all $1 \le i, j \le n$, let E_{ij} be the matrix with the (i,j)-entry 1_R and other entries 0_R. Let $b \in R$. For the matrix $B_b = bE_{ij}$, there is a matrix $C = (c_{ij}) \in M_n(R)$ such that $C(I_n - B_b A) = I_n$. By multiplying out the matrices on the left side of the equation, we see that the (i,i)-entry is $c_{ii}(1_R - ba_{ji}) = 1_R$. This implies that $a_{ji} \in Jac(R)$. Thus, $A \in M_n(Jac(R))$.

We next show that $M_n(Jac(R)) \subseteq Jac(M_n(R))$. Let $A = (a_{ij})$ be a matrix in $M_n(Jac(R))$. To argue that $A \in Jac(M_n(R))$, for any $B = (b_{ij}) \in M_n(R)$, we need to find a left inverse for $I_n - BA$ in the matrix ring $M_n(R)$. As before, we only need to show that the row echelon form of the matrix $I_n - BA$ is the identity matrix I_n.

Let $D := (d_{ij}) = I_n - BA$. For all $1 \le i \le n$, the (i,i)-entry of D is $d_{ii} = 1_R - \sum_{k=1}^n b_{ik} a_{ki}$, which is left invertible in R because $\sum_{k=1}^n b_{ik} a_{ki} \in Jac(R)$; for all $1 \le i \ne j \le n$, the (i,j)-entry is $d_{ij} = -\sum_{k=1}^n b_{ik} a_{kj} \in Jac(R)$.

- After performing the elementary operation "adding a left multiple of one row to another row" on D for finitely many times, we obtain a matrix U whose (i,i)-entries are left invertible and other entries are zero. For instance, left multiply the first row of D by $-d_{21} u_{11}$ with u_{11} a left inverse of d_{11} in R, and then add the multiplied first row to the second row of D, we obtain a matrix whose $(2,1)$-entry is zero, (i,i)-entries are still left invertible for all i and (i,j)-entries are still inside $Jac(R)$ for all $i \ne j$.

- After performing the elementary operation "left multiplying one row of a matrix" on U for finitely many times, we obtain the identity matrix I_n. Now the row echelon form of D is I_n.

Hence, $D = I_n - BA$ is left invertible in $M_n(R)$. By definition, we have $A \in Jac(M_n(R))$. □

Proposition 4.4. (RCA_0) *Semisimple rings and local rings are semiperfect.*

Proof. By Proposition 4.3, $Jac(M_n(D)) = M_n(Jac(D)) = \{0_{M_n(D)}\}$ for a division ring D. So for a semisimple ring R, $Jac(R) = \{0_R\}$ and $\overline{R} = R/Jac(R) = R$. This means that semisimple rings are semiperfect.

By Lemma 2.5, for a local ring R, $\overline{R} = R/Jac(R)$ is a division ring, and thus semisimple. Moreover, $\overline{R}$ contains only trivial idempotents, the condition (2) in the definition of semiperfect rings holds. This shows that local rings are semiperfect. □

We also need Corollary 23.9, [11], which says that matrix rings over semiperfect rings are always semiperfect, the classical proof there depends on properties of endomorphism rings of strongly indecomposable modules. We now provide a direct proof for it.

Proposition 4.5. (RCA_0) *For a semiperfect ring R, $M_n(R)$ is semiperfect.*

Proof. First, $Jac(M_n(R)) = M_n(Jac(R))$ exists. As R is semiperfect, by Theorem 3.7, $1_R = e_1 + \cdots + e_n$, where $e_1, \cdots, e_n$ are pairwise orthogonal idempotents of R with each e_iRe_i left local. Consider the identity I_n of the matrix ring $M_n(R)$. $I_n = \sum_{i=1}^n E_{ii}$, where E_{ii} is the square matrix of size n with the (i,i)-entry 1_R and other entries 0_R. Based on the decomposition of 1_R, $E_{ii} = e_1E_{ii} + \cdots + e_nE_{ii}$, and then $I_n = \sum_{i=1}^n \sum_{j=1}^n e_jE_{ii}$. To see why $M_n(R)$ is semiperfect, using Theorem 3.7 for the matrix ring $M_n(R)$, we only need to show that each

$$e_jE_{ii}M_n(R)e_jE_{ii} = \{e_jE_{ii}Ae_jE_{ii} : A \in M_n(R)\}$$

is a left local ring.

For each $A = (a_{ij}) \in M_n(R)$, we have $e_jE_{ii}Ae_jE_{ii} = e_ja_{ii}e_jE_{ii}$, i.e., the matrix whose (i,i)-entry is $e_ja_{ii}e_j$ and all other entries are 0_R. The identity of the ring $e_jE_{ii}M_n(R)e_jE_{ii}$ is just e_jE_{ii}. The matrix $e_ja_{ii}e_jE_{ii}$ is left invertible in the ring $e_jE_{ii}M_n(R)e_jE_{ii}$ if and only if the (i,i)-entry $e_ja_{ii}e_j$ is left invertible in the ring e_jRe_j. By assumption, e_jRe_j is a left local ring, so the non left invertible elements of e_jRe_j are closed under the addition of e_jRe_j. This implies that the non left invertible matrices in the matrix ring $e_jE_{ii}M_n(R)e_jE_{ii}$ are also closed under the addition of the matrix ring. By definition, $e_jE_{ii}M_n(R)e_jE_{ii}$ is a left local ring. □

We are ready to prove the structure theorem for semiperfect rings R with $R/Jac(R)$ simple.

Theorem 4.6. *The following conditions are equivalent over* RCA_0 *for a ring R.*

(1) R is a semiperfect ring with $R/Jac(R)$ simple.

(2) $R \cong M_n(S)$ *for some* $n \geq 1$ *and local ring* S*.*

Proof. (2) $\Rightarrow$ (1). Let S be a local ring, and $\varphi : R \to M_n(S)$ be the isomorphism. Since local rings are semiperfect, by Proposition 4.5, $M_n(S)$ is semiperfect. Then $Jac(R)$ exists because for any $r \in R$, $r \in Jac(R) \Leftrightarrow (r)\varphi \in Jac(M_n(S))$, and R is semiperfect. By Proposition 4.3,

$$R/Jac(R) \cong M_n(S)/Jac(M_n(S)) = M_n(S)/M_n(Jac(S)) \cong M_n(S/Jac(S)).$$

Since $S/Jac(S)$ is a division ring and matrix rings over division rings are simple, $M_n(S/Jac(S))$ is simple. So is $R/Jac(R)$.

(1) $\Rightarrow$ (2). Let R be a semiperfect ring. By Theorem 3.7, $1_R = e_1 + \cdots + e_n$ for some mutually orthogonal idempotents $e_1, \cdots, e_n$ of R with each e_iRe_i a local ring. Then $R = Re_1 \oplus \cdots \oplus Re_n$, and the semisimple ring $\overline{R} = R/Jac(R)$ has a decomposition $\overline{R} = \overline{R}\overline{e}_1 \oplus \cdots \oplus \overline{R}\overline{e}_n$ with each $\overline{R}\overline{e}_i$ simple submodules. Since the semisimple ring $\overline{R}$ is actually simple, by Lemma 4.2, the direct summands $\overline{R}\overline{e}_i (1 \leq i \leq n)$ are all isomorphic to each other. Furthermore, since $Re_i (1 \leq i \leq n)$ are projective R-modules, by Lemma 3.3, $Re_1 \cong Re_2 \cong \cdots \cong Re_n$ as left R-modules, and thus, $R = Re_1 \oplus \cdots \oplus Re_n \cong (Re_1)^n$ as left R-modules. Again, this implies that $R \cong M_n(e_1Re_1)$ as rings with e_1Re_1 a local ring. □

5 Perfect rings R with $R/Jac(R)$ simple

In this section, we study the structure theorem for one-sided perfect rings R with $R/Jac(R)$ simple. We first introduce left (resp., right) T-nilpotent ideals (see page 341, [11]), which are closely related to nil ideals and nilpotent ideals.

Definition 5.1. (RCA_0) Let I be an (two-sided) ideal of a ring R.

(1) I is nil if for any $a \in I$, there is a number n such that $a^n = 0_R$.

(2) I is nilpotent if there is a number n such that for any $a_1, \ldots, a_n \in I$, $a_1 \cdots a_n = 0_R$.

(3) I is left (resp., right) T-nilpotent if for any sequence $\langle a_n : n \in \mathbb{N} \rangle$ of elements of I, there is a number n such that $a_1a_2 \cdots a_n = 0_R$ (resp., $a_n \cdots a_2a_1 = 0_R$).

Nil ideals are weaker than nilpotent ideals. It is not hard to see that one-sided T-nilpotent ideals are strictly between nil ideals and nilpotent ideals.

Definition 5.2. (RCA_0) A ring R with the Jacobson radical $Jac(R)$ is left (resp., right) perfect if it satisfies the following conditions:

(1) $R/Jac(R)$ is semisimple;

(2) $Jac(R)$ is left (resp., right) T-nilpotent.

Unlike semisimple rings, local rings and semiperfect rings, left perfect rings and right perfect rings are distinct notions (see Example 23.22, [11]), and perfect rings refer to rings that are both left perfect and right perfect.

For a proof of Proposition 5.3 below, refer to the classical proof appeared in Theorem 21.28, [11], which works in RCA_0.

Proposition 5.3. (RCA_0) *If I is a nil ideal of a ring R, then idempotents of the quotient ring R/I can be lifted to R.*

For a left (resp., right) perfect ring R, idempotents of $R/Jac(R)$ can be lifted to R because left (resp., right) T-nilpotent ideals are nil. So left perfect rings or right perfect rings are always semiperfect within RCA_0, and we will prove the structure theorem of one-sided perfect rings based on the corresponding theorem of semiperfect rings.

To prove the structure theorem of one-sided perfect rings, we need a key lemma.

Lemma 5.4. (WKL_0) *For an ideal I of a ring R, the following are equivalent.*

(1) I is a left (resp., right) T-nilpotent ideal of R.

(2) $M_n(I)$ is a left (resp., right) T-nilpotent ideal of the matrix ring $M_n(R)$.

Proof. We only prove the case of left T-nilpotent ideals.

(2) $\Rightarrow$ (1). We reason in RCA_0. Assume that $M_n(I)$ is a left T-nilpotent ideal of $M_n(R)$. Let $\langle a_m : m \in \mathbb{N}\rangle$ be a sequence of elements in I. Then $\langle a_m E_{11} : m \in \mathbb{N}\rangle$ is a sequence of matrices in $M_n(R)$, where E_{11} is the n square matrix with $(1,1)$-entry 1_R and other entries 0_R. By (2), there is a number m such that $a_1E_{11}\cdots a_mE_{11} = 0_{M_n(R)}$, the zero matrix in $M_n(R)$; thus, $a_1\cdots a_m = 0_R$. I is left T-nilpotent.

(1) $\Rightarrow$ (2). We reason in WKL_0. A tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ is *bounded* if there is a function $g : \mathbb{N} \to \mathbb{N}$ such that for each node $\tau = \tau(0)^\frown\tau(1)\cdots{}^\frown\tau(|\tau|-1) \in S$ and $i < |\tau|$, $\tau(i) < g(i)$. Here, $|\tau|$ stands for the length of τ. By Lemma IV.1.4, [15], WKL_0 is equivalent to RCA_0 plus *Bounded König's Lemma*: every bounded infinite tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ has an infinite path.

We now use Bounded König's Lemma to prove (1) $\Rightarrow$ (2). Assume that $M_n(I)$ is not a left T-nilpotent ideal of $M_n(R)$. Then there is a sequence of matrices $\langle A_m : m \in \mathbb{N}\rangle$ in $M_n(I)$ such that for any m, $A_0A_1\cdots A_m \neq 0_{M_n(R)}$. Define a bounded tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ based on the sequence $\langle A_m : m \in \mathbb{N}\rangle$ of matrices in $M_n(I)$ as follows:

(1) For any $m \geq 0$ and $1 \leq i,j \leq n$, let a^m_{ij} be the (i,j)-entry of A_m. Also fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$.

(2) A node $\sigma = \sigma(0)^\frown \cdots ^\frown \sigma(m) \in \mathbb{N}^{<\mathbb{N}}$ of length $m+1$ belongs to S if for any $0 \leq k \leq m$, $\sigma(k) = \langle i_k, j_k \rangle$ subject to the condition

$$a^0_{i_0j_0} a^1_{i_1j_1} \cdots a^m_{i_mj_m} \neq 0_R,$$

where $a^k_{i_kj_k}$ is the (i_k, j_k)-entry of the matrix A_k for $0 \leq k \leq m$.

Since nodes of S are chosen from finite sequences of numbers from the set $\{\langle i, j\rangle : 1 \leq i, j \leq n\}$, S is a bounded tree. Moreover, for any m, $A_0 \cdots A_m \neq 0_{M_n(R)}$ implies that there is a nonzero entry of $A_0 \cdots A_m$, which is a sum of multiplications of the form $a^0_{i_0j_0} \cdots a^m_{i_mj_m}$ in R. So there is a node of length $m+1$ in S for any m, and S is an infinite tree. By Bounded König's Lemma, S contains an infinite path f. For any $m \geq 0$, let $f(m) = \langle i_m, j_m\rangle$ with $1 \leq i_m, j_m \leq n$. Then $\langle a^m_{i_mj_m} : m \in \mathbb{N}\rangle$ is a sequence of elements of I such that $a^0_{i_0j_0} \cdots a^m_{i_mj_m} \neq 0_R$ for all m. By definition, I is not a left T-nilpotent ideal of R. □

Using a different characterization of right T-nilpotent ideals (i.e., Theorem 23.16, [11]), Lam's book provides a proof for the structure theorem of right perfect rings R with $R/Jac(R)$ simple (i.e., Theorem 23.23, [11]). Since the characterization of right T-nilpotent ideals in [11] holds in ACA_0, the classical proof of the structure theorem there works in ACA_0. Using Lemma 5.4 above, we can provide a proof for the structure theorem in WKL_0.

Theorem 5.5. (WKL_0) *The following conditions are equivalent for a ring R.*

(1) R is a left (resp., right) perfect ring with $R/Jac(R)$ simple.

(2) $R \cong M_n(S)$ for some $n \geq 1$, and some local ring S whose maximal ideal is left (resp., right) T-nilpotent.

Proof. We only prove the case of left perfect rings. Then the proof for right perfect rings is clear.

(1) ⇒ (2). We reason in RCA_0. If R is a left perfect ring, then R is semiperfect. By Theorem 4.6, there is a local ring S with a maximal ideal $Jac(S)$ such that $R \cong M_n(S)$. Then $M_n(Jac(S)) = Jac(M_n(S)) \cong Jac(R)$. As $Jac(R)$ is a left T-nilpotent ideal of R, $M_n(Jac(S))$ is a left T-nilpotent ideal of $M_n(S)$. Then RCA_0 proves that $Jac(S)$ is a left T-nilpotent ideal of S.

(2) ⇒ (1). We reason in WKL_0. If $R \cong M_n(S)$ for some local ring S with $Jac(S)$ a left T-nilpotent ideal, then by Theorem 4.6, $R/Jac(R)$ is simple, and by Lemma 5.4, WKL_0 proves that $Jac(M_n(S)) = M_n(Jac(S))$ is a left T-nilpotent ideal of $M_n(S)$. Then $Jac(R)$ is a left T-nilpotent ideal of R, and R is a left perfect ring. □

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School of Information Science
Beijing Language and Culture University
15 Xueyuan Road, Haidian District, Beijing 100083
huishanwu@blcu.edu.cn