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SOME CARDINAL CHARACTERISTICS RELATED TO THE COVERING NUMBER AND THE UNIFORMITY OF THE MEAGRE IDEAL

A b s t r a c t. We extend the concepts of splitting, reaping, and independent families to families of functions and permutations on ω and define associated cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, $\mathfrak{r}_p$, $\mathfrak{i}_f$, and $\mathfrak{i}_p$. We study relationships among $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, and these cardinals. In this paper, we show that $\mathfrak{s}_f = \mathrm{non}(\mathcal{M}) = \mathfrak{s}_p$, $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$, and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_f, \mathfrak{i}_p$.

1. Introduction

The *covering number* of the meagre ideal $\mathcal{M}$, $\mathrm{cov}(\mathcal{M})$, is the smallest size of a family of meagre subsets of ${}^{\omega}\omega$ whose union is ${}^{\omega}\omega$ and the *uniformity*

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of $\mathcal{M}$, $\mathrm{non}(\mathcal{M})$, is the smallest size of a non-meagre subset of ${}^{\omega}\omega$ (see [3] or [7, Chapter III] for more details). It is well-known that $\aleph_1 \leq \mathfrak{p} \leq \mathrm{cov}(\mathcal{M}) \leq \mathfrak{r} \leq \mathfrak{i} \leq \mathfrak{c}$ and $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{s} \leq \mathrm{non}(\mathcal{M}) \leq \mathfrak{c}$, where $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{r}$, and $\mathfrak{i}$ are the *pseudo-intersection*, the *splitting*, the *reaping*, and the *independence numbers* respectively (for more details about these numbers see [3] or [6, Chapter 9]).

The *almost disjoint number* $\mathfrak{a}$ is the smallest size of a maximal almost disjoint family of infinite subsets of ω. It has been shown that both $\mathfrak{a}$ and $\mathrm{non}(\mathcal{M})$ lie between the *bounding number* $\mathfrak{b}$ and $\mathfrak{c}$ (see [3] and [6]). Almost disjoint families of functions and permutations on ω and associated cardinal characteristics, denoted by $\mathfrak{a}_e$ and $\mathfrak{a}_p$ respectively, were studied by Zhang in [9]. Brendle, Spinas, and Zhang showed in [4] that $\mathrm{non}(\mathcal{M})$ is a lower bound of both $\mathfrak{a}_e$ and $\mathfrak{a}_p$ (cf. [4, Theorem 2.2 and Proposition 4.6]).

Independent families of functions and permutations on ω and associated cardinal characteristics $\mathfrak{i}_f$ and $\mathfrak{i}_p$ were studied by us in [8]. We have shown that $\mathfrak{p} \leq \mathfrak{i}_f, \mathfrak{i}_p \leq \mathfrak{i}$ and also mentioned that $\mathrm{cov}(\mathcal{M})$ is a lower bound of both $\mathfrak{i}_f$ and $\mathfrak{i}_p$. In this paper, we give a full direct proof of this fact.

We also extend the concepts of splitting and reaping families to families of functions and permutations on ω and define associated cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, and $\mathfrak{r}_p$. We study relationships among $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, and these cardinals. As mentioned above, $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}$. In this paper, we show that $\mathfrak{s}_f = \mathrm{non}(\mathcal{M}) = \mathfrak{s}_p$ and $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

2. Splitting and reaping families

A set $A \subseteq \omega$ *splits* an infinite set $B \subseteq \omega$ if both $B \cap A$ and $B \setminus A$ are infinite. A *splitting family* $\mathcal{S}$ is a family of infinite subsets of ω such that each infinite set $B \subseteq \omega$ is split by at least one $A \in \mathcal{S}$. A *reaping family* $\mathcal{R}$ is a family of infinite subsets of ω such that there is no infinite subset of ω which splits every member of $\mathcal{R}$. The *splitting number* $\mathfrak{s}$ is the smallest cardinality of any splitting family and the *reaping number* $\mathfrak{r}$ is the smallest cardinality of any reaping family.

We write ${}^{\omega}\omega$ and $\mathrm{Sym}(\omega)$ for the set of functions and the set of permutations, respectively, on ω. We extend the concepts of splitting and reaping families to families of functions and permutations on ω. To be precise, we say $f \in {}^{\omega}\omega$ splits $g \in {}^{\omega}\omega$ if both $g \cap f$ and $g \setminus f$ are infinite. A *splitting family*

$\mathcal{S}$ of functions (permutations) is a family of functions (permutations) on ω such that each $g \in {}^{\omega}\omega$ ($g \in \mathrm{Sym}(\omega)$) is split by an $f \in \mathcal{S}$. A *reaping family* $\mathcal{R}$ of functions (permutations) is a family of functions (permutations) on ω such that there is no function (permutation) on ω which splits every member of $\mathcal{R}$. We define corresponding cardinal characteristics $\mathfrak{s}_f$, $\mathfrak{s}_p$, $\mathfrak{r}_f$, and $\mathfrak{r}_p$ as follows.

$$\begin{aligned}
\mathfrak{s}_f &= \min\{|\mathcal{S}| : \mathcal{S} \subseteq {}^{\omega}\omega \text{ is a splitting family}\},\\
\mathfrak{s}_p &= \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathrm{Sym}(\omega) \text{ is a splitting family}\},\\
\mathfrak{r}_f &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq {}^{\omega}\omega \text{ is a reaping family}\}, \text{ and}\\
\mathfrak{r}_p &= \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathrm{Sym}(\omega) \text{ is a reaping family}\}.
\end{aligned}$$

It is easy to see that the above definitions are well-defined since ${}^{\omega}\omega$ and $\mathrm{Sym}(\omega)$ are splitting and reaping families of functions and permutations respectively.

First, we shall show that $\mathfrak{s}_f = \mathrm{non}(\mathcal{M})$ and $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M})$. The following is Theorem 5.9 in [3]. The first statement is also from [1, Corollary 1.8].

Theorem 2.1.

$$\begin{aligned}
\mathrm{cov}(\mathcal{M}) &= \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^{\omega}\omega \wedge \neg\exists f \in {}^{\omega}\omega\ \forall g \in \mathcal{C}\ (f \cap g \textit{ is infinite})\}, \textit{ and}\\
\mathrm{non}(\mathcal{M}) &= \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^{\omega}\omega \wedge \forall f \in {}^{\omega}\omega\ \exists g \in \mathcal{C}\ (f \cap g \textit{ is infinite})\}.
\end{aligned}$$

Theorem 2.2. $\mathfrak{s}_f = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M})$.

Proof. It follows immediately from the above theorem that $\mathfrak{r}_f \leq \mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M}) \leq \mathfrak{s}_f$. To show that $\mathfrak{s}_f \leq \mathrm{non}(\mathcal{M})$, let $\mathcal{C} \subseteq {}^{\omega}\omega$ be an infinite family such that for all $f \in {}^{\omega}\omega$, there exists a $g \in \mathcal{C}$ such that $f \cap g$ is infinite.

For each $g \in \mathcal{C}$, define $\tilde{g} \in {}^{\omega}\omega$ by

$$\tilde{g}(n) = \begin{cases} g(n) & \text{if } n \text{ is even},\\ g(n)+1 & \text{if } n \text{ is odd}. \end{cases}$$

Let $\mathcal{D} = \mathcal{C} \cup \{\tilde{g} : g \in \mathcal{C}\}$. To show that $\mathcal{D}$ is a splitting family, let $f \in {}^{\omega}\omega$. By the property of $\mathcal{C}$, there is a $g \in \mathcal{C}$ such that $f \cap g$ is infinite. If $f \setminus g$ is finite, then there is an $n_0 < \omega$ such that $f(n) = g(n)$ for all $n \geq n_0$, and hence $\tilde{g}$ splits f. Otherwise, g splits f. Thus $\mathfrak{s}_f \leq |\mathcal{D}| = |\mathcal{C}|$. Since $\mathcal{C}$ is arbitrary, $\mathfrak{s}_f \leq \mathrm{non}(\mathcal{M})$.

To show that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_f$, let $\mathcal{C} \subseteq {}^\omega\omega$ be an infinite family such that $|\mathcal{C}| < \mathrm{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not a reaping family.

For each $g \in \mathcal{C}$, let $g \oplus 1 \in {}^\omega\omega$ be defined by $(g \oplus 1)(n) = g(n) + 1$. Let $\mathcal{D} = \mathcal{C} \cup \{g \oplus 1 : g \in \mathcal{C}\}$. Then $|\mathcal{D}| = |\mathcal{C}| < \mathrm{cov}(\mathcal{M})$. By the above theorem, there is an $f \in {}^\omega\omega$ such that $f \cap h$ is infinite for any $h \in \mathcal{D}$. Consider a $g \in \mathcal{C}$. Since $f \cap (g \oplus 1)$ is infinite, there are infinitely many $k \in \omega$ such that $f(k) \neq g(k)$. Hence $g \setminus f$ is infinite. Since $f \cap g$ is infinite, f splits g. Therefore, $\mathcal{C}$ is not a reaping family. □

Next, we shall show that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$. The proofs make use of Martin's Axiom. We start with some relevant definitions and known facts.

Definition 2.3. $MA_{\mathbb{P}}(\kappa)$ is the statement that whenever $\mathcal{D}$ is a family of dense subsets of a poset $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there exists a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

By the Generic Filter Existence Lemma [7, Lemma III.3.14], we obtain the following theorem.

Theorem 2.4. *$MA_{\mathbb{P}}(\kappa)$ holds for any poset $\mathbb{P}$ and $\kappa \leq \aleph_0$.*

Definition 2.5. A subset C of a poset $\mathbb{P}$ is *centered* if, for any $n \in \omega$ and any $p_1, p_2, ..., p_n \in C$ there is a $q \in \mathbb{P}$ such that $q \leq p_i$ for all i. $\mathbb{P}$ is *σ-centered* if $\mathbb{P}$ is a countable union of centered subsets of $\mathbb{P}$.

Definition 2.6. $\mathfrak{m}_\sigma$ is the least κ such that there is a σ-centered poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails, and $\mathfrak{m}_{\mathrm{ctbl}}$ is the least κ such that there is a countable poset $\mathbb{P}$ for which $MA_{\mathbb{P}}(\kappa)$ fails.

We have shown, in Theorem 2.2, that $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M})$. Now, we show that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$ by using the following theorem which is Proposition (d) in [5].

Theorem 2.7. *$\mathfrak{m}_{ctbl} = \mathrm{cov}(\mathcal{M})$.*

Theorem 2.8. $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

Proof. It suffices to show that $\mathfrak{m}_{\mathrm{ctbl}} \leq \mathfrak{r}_p$. To show this, let $\mathcal{C} \subseteq \mathrm{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_{ctbl}$. Consider the poset $\mathbb{P} = \mathrm{Fn}_{1-1}(\omega, \omega)$, i.e. $\{s \subseteq \omega \times \omega : s \text{ is a finite injection}\}$. For each $n \in \omega$ and $f \in \mathcal{C}$, let

$$A_n = \{p \in \mathbb{P} : n \in \mathrm{dom}(p) \cap \mathrm{ran}(p)\},$$
$$B_{n,f} = \{p \in \mathbb{P} : \exists k \geq n\ \exists \ell \geq n\ (p(k) = f(k) \wedge p(\ell) \neq f(\ell))\}.$$

Then A_n and $B_{n,f}$ are dense in $\mathbb{P}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let
$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_{n,f} : n \in \omega, f \in \mathcal{C}\}.$$
Since $\mathcal{D}$ is of size $< \mathfrak{m}_{\mathrm{ctbl}}$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_{n,f}$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup G$. Then $g \in \mathrm{Sym}(\omega)$ and for any $n \in \omega$ and any $f \in \mathcal{C}$, we have that $g(k) = f(k)$ and $g(\ell) \neq f(\ell)$ for some $k, \ell \geq n$. Hence for any $f \in \mathcal{C}$, $f \cap g$ and $f \setminus g$ are infinite, so g splits f. Thus $\mathcal{C}$ is not a reaping family. $\square$

It is well-known that $\mathfrak{p} \leq \mathfrak{s}$ (cf. [6, Chapter 9]). Now, we shall use the fact below to show that $\mathfrak{p}$ is also a lower bound of $\mathfrak{s}_p$. The following theorem is from Bell ([2]), and is also Theorem III.3.61 in [7].

Theorem 2.9. $\mathfrak{m}_\sigma = \mathfrak{p}$.

Theorem 2.10. $\mathfrak{p} \leq \mathfrak{s}_p$.

Proof. It suffices to show that $\mathfrak{m}_\sigma \leq \mathfrak{s}_p$. To show this, let $\mathcal{C} \subseteq \mathrm{Sym}(\omega)$ be such that $\aleph_0 \leq |\mathcal{C}| < \mathfrak{m}_\sigma$. Define the poset $\mathbb{P} = \mathrm{Fn}_{1-1}(\omega, \omega) \times [\mathcal{C}]^{<\omega}$, where $(s, E) \leq (t, F)$ if and only if
$$s \supseteq t,\ E \supseteq F \text{ and } \forall n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)\ \forall f \in F\ (s(n) \neq f(n)).$$
Clearly this poset is σ-centered, as the set $\{(s, E) \in \mathbb{P} : E \in [\mathcal{C}]^{<\omega}\}$ is centered for any fixed s and $\mathrm{Fn}_{1-1}(\omega, \omega)$ is countable. For each $n \in \omega$ and $f \in \mathcal{C}$, let
$$\begin{aligned} A_n &= \{(s, E) \in \mathbb{P} : n \in \mathrm{dom}(s) \cap \mathrm{ran}(s)\}, \\ B_f &= \{(s, E) \in \mathbb{P} : f \in E\}. \end{aligned}$$
It is easy to see that B_f is dense in $\mathbb{P}$ for all $f \in \mathcal{C}$. To show that A_n is dense in $\mathbb{P}$ for any $n \in \omega$, let $n \in \omega$ and $(s, E) \in \mathbb{P}$. Since s is a finite function and E is a finite set of injections, we can pick $k \in \omega \setminus \mathrm{dom}(s)$ and $\ell \in \omega \setminus \mathrm{ran}(s)$ so that $(k, n), (n, \ell) \notin \bigcup E$. We choose
$$t = \begin{cases} s & \text{if } n \in \mathrm{dom}(s) \cap \mathrm{ran}(s), \\ s \cup \{(k, n)\} & \text{if } n \in \mathrm{dom}(s) \setminus \mathrm{ran}(s), \\ s \cup \{(n, \ell)\} & \text{if } n \in \mathrm{ran}(s) \setminus \mathrm{dom}(s), \\ s \cup \{(k, n), (n, \ell)\} & \text{if } n \notin \mathrm{dom}(s) \cup \mathrm{ran}(s). \end{cases}$$
Then $(t, E) \leq (s, E)$ where $(t, E) \in A_n$. So A_n is dense in $\mathbb{P}$. Let
$$\mathcal{D} = \{A_n : n \in \omega\} \cup \{B_f : f \in \mathcal{C}\}.$$

Since $\mathcal{D}$ is of size $|\mathcal{C}| < \mathfrak{m}_\sigma$, there is a filter G on $\mathbb{P}$ such that $G \cap A_n \neq \emptyset \neq G \cap B_f$ for any $n \in \omega$ and $f \in \mathcal{C}$. Let $g = \bigcup \mathrm{dom}(G)$. Then $g \in \mathrm{Sym}(\omega)$.

To show that $g \cap f$ is finite for any $f \in \mathcal{C}$, let $f \in \mathcal{C}$. Since $G \cap B_f \neq \emptyset$, there is a $(s, E) \in G$ such that $f \in E$. Let $m \in \mathrm{dom}(g) \setminus \mathrm{dom}(s)$. We shall show that $g(m) \neq f(m)$. Since $(m, g(m)) \in g = \bigcup \mathrm{dom}(G)$, there is a $(t, F) \in G$ such that $(m, g(m)) \in t$. Since G is a filter, there is a $(s', E') \in G$ such that $(s', E') \leq (s, E)$ and $(s', E') \leq (t, F)$. Then $m \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)$ and hence $g(m) = t(m) = s'(m) \neq f(m)$. Therefore, $g(m) \neq f(m)$ for any $m \in \mathrm{dom}(g) \setminus \mathrm{dom}(s)$. So $\{m : g(m) = f(m)\} \subseteq \mathrm{dom}(s)$, which implies that $g \cap f$ is finite. Therefore, $\mathcal{C}$ is not a splitting family. □

The above proof shows the relationship between $\mathfrak{p}$ and $\mathfrak{s}_p$ by using the fact that $\mathfrak{m}_\sigma = \mathfrak{p}$. However, since $\mathfrak{p} \leq \mathfrak{s} \leq \mathrm{non}(\mathcal{M})$, a stronger result can be obtained as shown in the following theorem. The notation $\exists^\infty n$ means "there are infinitely many" and $\forall^\infty n$ means "for all but finitely many".

Theorem 2.11. $\mathrm{non}(\mathcal{M}) = \mathfrak{s}_p$.

Proof. We first show that $\mathfrak{s}_p \leq \mathrm{non}(\mathcal{M})$. Note that $\mathrm{Sym}(\omega)$ is homeomorphic to ${}^\omega\omega$, so the notion of "the smallest size of a meagre set" in both (topological) spaces are the same. Let $\mathcal{S} \subseteq \mathrm{Sym}(\omega)$ be such that $|\mathcal{S}| < \mathfrak{s}_p$. We shall show that $\mathcal{S}$ is meagre in $\mathrm{Sym}(\omega)$. By the definition of $\mathfrak{s}_p$, there is a $g \in \mathrm{Sym}(\omega)$ such that, for each $f \in \mathcal{S}$, $\forall^\infty n[g(n) \neq f(n)]$ or $\forall^\infty n[g(n) = f(n)]$. Let $\mathcal{S}_0 = \{f \in \mathcal{S} : \forall^\infty n[g(n) \neq f(n)]\}$. We claim that $\mathcal{S}_0$ is meagre in $\mathrm{Sym}(\omega)$. For each $n < \omega$, let

$$C_n = \{f \in \mathrm{Sym}(\omega) : \forall m > n[g(m) \neq f(m)]\}.$$

It is straightforward to show that C_n is closed nowhere dense and $\mathcal{S}_0 \subseteq \bigcup_{n<\omega} C_n$, and hence $\mathcal{S}_0$ is meagre. Since $\mathcal{S} \setminus \mathcal{S}_0 = \{f \in \mathcal{S} : \forall^\infty n[g(n) = f(n)]\}$ is countable (and hence is meagre), $\mathcal{S} = \mathcal{S}_0 \cup (\mathcal{S} \setminus \mathcal{S}_0)$ is meagre.

We next show that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{s}_p$. Let $\mathcal{S} \subseteq \mathrm{Sym}(\omega)$ be such that $|\mathcal{S}| < \mathrm{non}(\mathcal{M})$, and we shall show that $\mathcal{S}$ is not a splitting family.

Claim. There exists an injection $f \in {}^\omega\omega$ such that $f(n) > n$ for all $n < \omega$ and for all $q \in \mathcal{S}$, $\forall^\infty n[f(n) \neq q(n)]$ and $\forall^\infty n[f(n) \neq q^{-1}(n)]$.

Proof. Let $\pi : \omega^2 \to \omega$ be a one-to-one map such that $\pi(n, m) > n$ for all $n, m < \omega$. For any $q \in \mathrm{Sym}(\omega)$, we define $q^+ \in {}^\omega\omega$ by

$$q^+(n) = \begin{cases} m & \text{if } q(n) = \pi(n, m), \\ 0 & \text{otherwise.} \end{cases}$$

Put $S^{-1} = \{p^{-1} : p \in \mathcal{S}\}$ and $\mathcal{S}^+ = \mathcal{S} \cup \mathcal{S}^{-1} \cup \{q^+ : q \in \mathcal{S} \cup \mathcal{S}^{-1}\}$. Since $|\mathcal{S}| < \mathrm{non}(\mathcal{M})$, by Theorem 2.1, there exists an $\hat{f} \in {}^\omega\omega$ such that for all $g \in \mathcal{S}^+$, $\forall^\infty n[\hat{f}(n) \neq g(n)]$. In particular, for each $q \in \mathcal{S} \cup \mathcal{S}^{-1}$, $\forall^\infty n[\hat{f}(n) \neq q^+(n)]$. Define $f \in {}^\omega\omega$ by $f(n) = \pi(n, \hat{f}(n))$. Clearly f is one-to-one and $f(n) > n$ for all $n < \omega$. Notice that

$$\forall q \in \mathcal{S} \cup \mathcal{S}^{-1}\ \forall n < \omega\ [f(n) = q(n) \to \hat{f}(n) = q^+(n)].$$

Hence, for each $q \in \mathcal{S} \cup \mathcal{S}^{-1}$, $\forall^\infty n[f(n) \neq q(n)]$, and the proof of the claim is complete.

Let $f(k) = n_k$, and note that $n_k > k$ for all k and n_k's are distinct. Define $p \in \mathrm{Sym}(\omega)$ recursively as follows. Suppose we have already defined $p{\restriction}k$. If there exists an $i < k$ such that $p(i) = k$ then put $p(k) = i$; otherwise, put $p(k) = n_k$. Note that, after the construction is done, if $p(x) = y$ then $(x, y) = (k, n_k)$ or $(x, y) = (n_k, k)$ for some k. So $p(p(x)) = x$ for all $x < \omega$, and hence p is bijective.

We finally show that $\forall^\infty k[p(k) \neq q(k)]$ for all $q \in \mathcal{S}$. Suppose to the contrary that there is a $q \in \mathcal{S}$ such that $\exists^\infty k[p(k) = q(k)]$. Let $X = \{k : p(k) = n_k\}$. Note that $\omega \setminus X = \{n_k : p(n_k) = k\}$. Then either

$$\exists^\infty k \in X[p(k) = q(k)] \text{ or } \exists^\infty i \in \omega \setminus X[p(i) = q(i)].$$

In the former case, we have $\exists^\infty k[f(k) = n_k = p(k) = q(k)]$. In the latter case, we have $\exists^\infty k[k = p(n_k) = q(n_k)]$, so $\exists^\infty k[q^{-1}(k) = n_k = f(k)]$. Both cases contradict the above claim. Therefore $\mathcal{S}$ is not a splitting family. $\square$

3. Independent families

An infinite set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is said to be an *independent family* (or shortly i.f.) if, for any disjoint finite sets $A, B \subseteq \mathcal{I}$, $\bigcap A \setminus \bigcup B$ is infinite. We interpret $\bigcap \emptyset = \omega$. The cardinal $\mathfrak{i}$ is defined as the least cardinality of a maximal independent family. We extend the concept of independent families to families of functions and permutations on ω and define corresponding cardinal characteristics $\mathfrak{i}_f$ and $\mathfrak{i}_p$ as follows.

$$\mathfrak{i}_f = \min\{|\mathcal{I}| : \mathcal{I} \subseteq {}^\omega\omega \text{ is a maximal independent family}\} \text{ and}$$
$$\mathfrak{i}_p = \min\{|\mathcal{I}| : \mathcal{I} \subseteq \mathrm{Sym}(\omega) \text{ is a maximal independent family}\}.$$

Since there is an i.f. of permutations of cardinality $\mathfrak{c}$ (see Proposition 2.1 in [8]), $\mathfrak{i}_f$ and $\mathfrak{i}_p$ are well-defined.

We shall show that $\operatorname{cov}(\mathcal{M})$ is a lower bound of $\mathfrak{i}_p$ (and also $\mathfrak{i}_f$). First, we need the following fact.

Fact. $\operatorname{cov}(\mathcal{M})$ is the least cardinality of a family of open dense subsets of ${}^{\omega}\omega$ whose intersection is empty.

This fact follows from Proposition (a) in [5] by viewing a Polish space X as the Baire space ${}^{\omega}\omega$ (with the basic open sets of the form $[p] = \{f \in {}^{\omega}\omega : f \supseteq p\}$ for $p \in {}^{<\omega}\omega$) together with the fact from topology that "a subset O of a topological space X is open dense if and only if $X \setminus O$ is closed nowhere-dense". Note the fact that $D \subseteq \mathbb{P}$ is dense in the Cohen poset $\mathbb{P} = {}^{<\omega}\omega$ if and only if $[D] = \{f \in {}^{\omega}\omega : f \supseteq p \text{ for some } p \in D\}$ is open dense in the Baire space ${}^{\omega}\omega$.

For an infinite family $\mathcal{C} \subseteq {}^{\omega}\omega$, let

$$\operatorname{bc}(\mathcal{C}) = \left\{ \bigcap A \setminus \bigcup B : A, B \in \operatorname{fin}(\mathcal{C}), A \cap B = \emptyset \text{ and } A \neq \emptyset \right\}.$$

Then each member of $\operatorname{bc}(\mathcal{C})$ is a function and is an injection if $\mathcal{C}$ is a family of permutations. Notice that $\mathcal{C}$ is an independent family if and only if every member of $\operatorname{bc}(\mathcal{C})$ is infinite.

Theorem 3.1. $\operatorname{cov}(\mathcal{M}) \leq \mathfrak{i}_p$.

Proof. Let $\mathcal{C} \subseteq \operatorname{Sym}(\omega)$ be an i.f. of permutations such that $\aleph_0 \leq |\mathcal{C}| < \operatorname{cov}(\mathcal{M})$. We shall show that $\mathcal{C}$ is not maximal.

For any $p \in {}^{<\omega}\omega \cup {}^{\omega}\omega$, let $\hat{p}$ be the one-to-one sequence obtained from p by removing all repetitions of each occurrence of $p(i)$ except its first one. Let $\mathbb{P} = {}^{<\omega}\omega$ and for each $x \in \operatorname{bc}(\mathcal{C})$ and $n < \omega$, let

$$\begin{aligned} D_{x,n} =& \big\{p \in \mathbb{P} : \exists k, \ell \geq n \big(k, \ell \in \operatorname{dom}(\hat{p}) \cap \operatorname{dom}(x) \wedge \hat{p}(k) = x(k) \\ & \wedge \hat{p}(\ell) \neq x(\ell)\big)\big\}, \\ A_n =& \{p \in \mathbb{P} : n \in \operatorname{ran}(p)\}. \end{aligned}$$

It is easy to see that each A_n is dense in $\mathbb{P}$. To show that each $D_{x,n}$ is dense in $\mathbb{P}$, let $x \in \operatorname{bc}(\mathcal{C}), n < \omega$, and $p \in \mathbb{P}$. Pick distinct $k, \ell \geq \max\{n, \operatorname{dom}(p)\}$ such that $k, \ell \in \operatorname{dom}(x)$ and $k < \ell$ where $x(k)$ and $x(\ell)$ are not in $\operatorname{ran}(p)$. Choose a $q \in \mathbb{P}$ such that $q \supseteq p$ and the k-th and the ℓ-th unrepeated elements are equal to $x(k)$ and not equal to $x(\ell)$, respectively. Rigorously, let $s = \operatorname{dom}(p)$, $t = |\operatorname{ran}(p)|$, pick distinct $a_0, a_1, ..., a_{k-t-1}, b_0, b_1, ..., b_{\ell-k-1} \in \omega \setminus (\operatorname{ran}(p) \cup \{x(k), x(\ell)\})$, and define $q = p \cup \{(s+i, a_i) : i < k-t\} \cup \{(s-t+k, x(k))\} \cup \{(s-t+k+1+j, b_j) : j < \ell - k\}$. Thus $\hat{q}(k) = x(k)$ and $\hat{q}(\ell) \neq x(\ell)$, so $q \in D_{x,n}$. Let

$$\mathcal{D} = \{[D_{x,n}] : x \in \mathrm{bc}(\mathcal{C}) \text{ and } n < \omega\} \cup \{[A_n] : n < \omega\}$$

Then $\mathcal{D}$ is a family of open dense subsets of the Baire space ${}^\omega\omega$ where $|\mathcal{D}| \leq |\mathcal{C}| < \mathrm{cov}(\mathcal{M})$. By the above fact, $\bigcap \mathcal{D} \neq \emptyset$, and we can pick an element $f \in \bigcap \mathcal{D}$. Thus $x \cap \hat{f}$ and $x \setminus \hat{f}$ are infinite where $\hat{f} \in \mathrm{Sym}(\omega)$. So $\mathcal{C} \cup \{\hat{f}\}$ is an i.f. of permutations. □

Another way to prove the above theorem is, by using the fact that $\mathfrak{m}_{\text{ctbl}} = \mathrm{cov}(\mathcal{M})$ and showing that $\mathfrak{m}_{ctbl} \leq \mathfrak{i}_p$ instead. This can be done by consider the countable poset $\mathrm{Fn}_{1-1}(\omega, \omega)$. We leave the details for the reader.

By simplifying the proof of Theorem 3.1, we can show that $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_f$. However, the following theorem gives a better lower bound of $\mathfrak{i}_f$. Recall that the cardinal $\mathfrak{d}$ is the *dominating number*, the smallest size of a dominating family of functions on ω.

Theorem 3.2. $\mathfrak{d} \leq \mathfrak{i}_f$.

Proof. Suppose $\mathcal{I} \subseteq {}^\omega\omega$ is an independent family with $\aleph_1 \leq |\mathcal{I}| < \mathfrak{d}$. We shall show that $\mathcal{I}$ is not maximal.

Take a model M of sufficiently large finite fragment of ZFC with $\mathcal{I} \in M$ and $|M| = |\mathcal{I}|$.

Claim. There is a strictly increasing sequence $\{n_k : k < \omega\} \subseteq \omega$ with $n_0 = 0$ so that for any $g \in M \cap {}^\omega\omega$, there are infinitely many k such that $g(n_k) < n_{k+1}$.

Proof. Since $|M| < \mathfrak{d}$, ${}^\omega\omega \cap M$ is not a dominating family. Hence there is a strictly increasing function $f \in {}^\omega\omega$ such that $\exists^\infty n[g(n) < f(n)]$ for all $g \in {}^\omega\omega \cap M$. Define $n_0 = 0$ and $n_{k+1} = f(n_k)$ for each $k < \omega$.

Let $g \in {}^\omega\omega \cap M$. We shall show that $\exists^\infty k[g(n_k) < n_{k+1}]$. In M, define $G \in {}^\omega\omega \cap M$ by $G(0) = 1$ and

$$G(n+1) = \max\left(\{g(i) : i \leq G(n)\} \cup \{G(n)\}\right) + 1.$$

If there is an $\ell < \omega$ such that $|\mathrm{ran}(G) \cap [n_k, n_{k+1})| \leq 1$ for all $k \geq \ell$, then $G(k) \geq n_{k+1} = f(n_k) \geq f(k)$ for all $k \geq n_\ell + 1$, which is impossible by the property of f. So there are infinitely many k such that $|\mathrm{ran}(G) \cap [n_k, n_{k+1})| \geq 2$. For such a k, there is an a_k such that $n_k \leq G(a_k) < G(a_k + 1) < n_{k+1}$ and hence, by the definition of G, $g(n_k) \leq G(a_k + 1) < n_{k+1}$, and the proof of the claim is done.

Let $\{f_k : k < \omega\} \subseteq \mathcal{I}$ be a sequence in M without repetitions. Define

$$h = \bigcup_{k<\omega} f_k{\restriction}[n_k, n_{k+1}).$$

We shall show that $\mathcal{I} \cup \{h\}$ is an independent family and $h \notin \mathcal{I}$. To show this, let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$ be disjoint finite sets. Note that h and f_k agree on $[n_k, n_{k+1})$ for each k. It suffices to show that

$$\exists^\infty k < \omega\ \exists a \in [n_k, n_{k+1})\ \ [\forall f \in \mathcal{A}[f(a) = f_k(a)] \wedge \forall g \in \mathcal{B}[g(a) \neq f_k(a)]]\,.$$

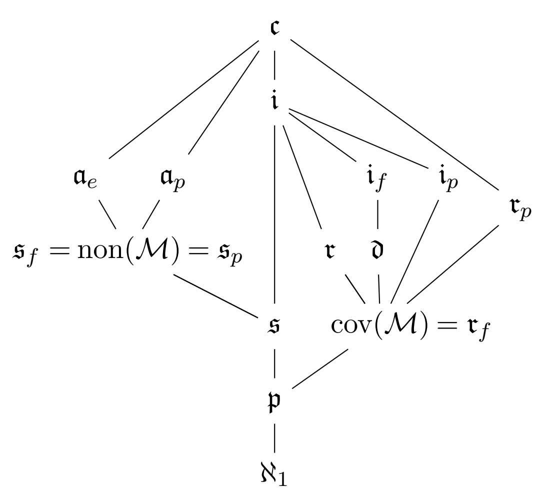
Choose an $\ell < \omega$ so that $f_k \notin \mathcal{A} \cup \mathcal{B}$ for all $k > \ell$. Since, for any n with $\ell < n < \omega$, three sets $\mathcal{A}$, $\mathcal{B}$ and $\{f_k : \ell < k \leq n\}$ are disjoint subset of an independent family $\mathcal{I}$, working in M, we can construct a $d \in {}^\omega\omega \cap M$ such that for any n, k with $\ell < k \leq n$,

$$\exists a \in [n, d(n))\ \ [\forall f \in \mathcal{A}[f(a) = f_k(a)] \wedge \forall g \in \mathcal{B}[g(a) \neq f_k(a)]].$$

Since there are infinitely many k such that $d(n_k) < n_{k+1}$ (by the above claim), we are done. □

4. Summary and open problems

We summarise relationships among the cardinals studied in this paper and other well-known ones in the following diagram. A line connecting two cardinals indicates that the lower cardinal is less than or equal to the upper one.



By Cohen forcing, we have that $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.3, pages 472–473]). Therefore, the following statement is consistent with ZFC:

$$\aleph_1 = \mathfrak{p} = \mathfrak{s} = \mathfrak{s}_f = \mathfrak{s}_p = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{r} = \mathfrak{r}_f = \mathfrak{r}_p = \mathfrak{i}_f = \mathfrak{i}_p = \mathfrak{i} = \mathfrak{c}.$$

By Random forcing, we have that $\aleph_1 = \mathfrak{s} = \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{c}$ is relatively consistent with ZFC (cf. [3, Section 11.4, pages 473–474]). Thus it is relatively consistent with ZFC that

$$\aleph_1 = \mathfrak{p} = \mathfrak{r}_f = \mathfrak{s} = \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M}) = \mathfrak{r} = \mathfrak{s}_f = \mathfrak{s}_p = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{i} = \mathfrak{c}.$$

Since there are many models of ZFC in which $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{d} = \aleph_2$, e.g., Laver, Mathias, or Miller forcing (cf. [3, Sections 11.7-11.9, pages 478–479]), by Theorem 3.2, $\mathrm{cov}(\mathcal{M}) < \mathfrak{i}_f$ in these models.

From the above results, there are some interesting open problems below.

1. Is $\mathfrak{r}_p = \mathrm{cov}(\mathcal{M})$?
2. Is $\mathfrak{d}$ a lower bound of $\mathfrak{i}_p$?
3. Is there any model of ZFC in which $\mathrm{cov}(\mathcal{M}) < \mathfrak{i}_p$?

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References

[1] T. Bartoszyński, Combinatorial aspects of measure and category, *Fundamenta Mathematicae*, **127** (1987), 225–239.

[2] M. G. Bell, On the combinatorial principle $P(c)$, *Fundamenta Mathematicae*, **114** (1981), 149–157.

[3] A. Blass, Combinatorial cardinal characteristics of the continuum, In: *Handbook of Set Theory*, vol. 1, M. Foreman, A. Kanamori (eds.), (Springer, Berlin, 2010), 395–490.

[4] J. Brendle, O. Spinas, Y. Zhang, Uniformity of the meager ideal and maximal cofinitary groups, *Journal of Algebra*, **232** (2000), 209–225.

[5] D. H. Fremlin, S. Shelah, On partitions of the real line, *Israel Journal of Mathematics*, **32** (1979), 299–304.

[6] L. J. Halbeisen, Combinatorial set theory: With a gentle introduction to forcing, 2 ed., Springer International Publishing (2017).

[7] K. Kunen, Set theory, vol. 34 of Studies in logic, College Publications (2011).

[8] N. Sonpanow, P. Vejjajiva, Independent families of functions and permutations, *Mathematical Logic Quarterly*, **66** (2020), 311–315.

[9] Y. Zhang, On a class of m.a.d. families, *Journal of Symbolic Logic*, **64** (1999), 737–746.

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