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Hugo Luiz MARIANO and Francisco MIRAGLIA

PROFINITE STRUCTURES ARE RETRACTS OF ULTRAPRODUCTS OF FINITE STRUCTURES

A b s t r a c t. We show that if L is a first-order language with equality, then profinite L-structures, the projective limits of finite L-structures, are retracts of certain ultraproducts of finite L-structures. As a consequence, any elementary class of L-structures axiomatized by L-sentences of the form $\forall\vec{x}(\psi_0(\vec{x}) \rightarrow \psi_1(\vec{x}))$, where $\psi_0(\vec{x}), \psi_1(\vec{x})$ are positive existential L-formulas, is closed under the formation of profinite objects in **L-mod**, the category of L-structures and L-homomorphisms. We also mention some interesting applications of our main result to the Theory of Special Groups that have already appeared in the literature.

The results presented here first appeared in Chapter 2 of [Mrn1] and were announced with proofs in [MM1]. Our motivation came from [KMS], that introduces the class of direct limits of finite abstract order spaces, a theory due to M. Marshall ([Mar1]). The theory of special groups, introduced in [DM2], is a *first-order* axiomatization of the algebraic theory of quadratic forms, and there is a natural categorical duality between the

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category of *reduced* special groups and that of abstract ordered spaces, as shown in Chapter 3 of [DM2] (first established, by a different method, in [Lim]).

Let L be a first-order language with equality and let **L-mod** be the category of L-structures and L-morphisms. As a preliminary to the proof of our main result, Theorem 2.3, the first section recalls the notions of retract and pure morphisms in **L-mod**, as well as some basic material on limits and colimits in this category, together with the relation between colimits and reduced products of L-structures (Proposition 1.8). Although many of these facts are folklore, full proofs of the needed results can be found in [MM2] and in Chapter 17 of [Mir]. Our general references for Category Theory and Model Theory are [Mac] and [CK] or [BS], respectively.

At the end of the paper we mention some interesting applications of our main result to the theory of special groups, some of which have already appeared in the literature.

1. Preliminaries

We recall the following

Definition 1.1 Let $\langle I, \leq \rangle$ be a non-empty partially ordered set (poset). For $i \in I$, set

$$i^{\leftarrow} = \{j \in I : j \leq i\} \quad \text{and} \quad i^{\rightarrow} = \{j \in I : i \leq j\}.$$

a) (1) $\langle I, \leq \rangle$ is **upward directed** (or *filtered*) if I for each $i,\ j \in I$, $i^{\rightarrow} \cap j^{\rightarrow} \neq \emptyset$.

(2) $\langle I, \leq \rangle$ is **downward directed** (or *cofiltered*) if for each $i,\ j \in I$, $i^{\leftarrow} \cap j^{\leftarrow} \neq \emptyset$.

Clearly a poset $\langle I, \leq \rangle$ is upward directed iff its opposite poset $\langle I, \leq \rangle^{op}$ is downward directed and vice-versa. The expression *directed poset* will always refer to upward directed posets.

b) A filter $\mathcal{F}$ on I is **directed** if for all $i \in I$, $i^{\rightarrow} \in \mathcal{F}$.

Lemma 1.2 *If $\langle I, \leq \rangle$ is a directed poset then there is a directed* ultrafilter *in $\langle I, \leq \rangle$.*

Proof. Because $\langle I, \leq \rangle$ is directed we see that the set $S = \{i^{\rightarrow} \colon\ i \in I\}$ has the finite intersection property and so is contained in a proper ultrafilter on I. $\square$

Henceforth, we fix a first-order language with equality, L and write:
$*$ **L-mod** for the category of L-structures and L-morphisms;
$*$ $\exists^+(\boldsymbol{L})$ for the set of L-formulas that are logically equivalent, to a positive existential L-formula;
$*$ $\mathbf{pp}(\boldsymbol{L})$ for the set of L-formulas that are logically equivalent to a positive primitive L-formula (pp-formula), that is, one of the form $\exists\overline{x}\ \varphi$, where φ is a conjunction of atomic formulas.
It is well-known that any formula in $\exists^+(L)$ is logically equivalent to a disjunction of conjunctions of pp-formulas.

Definition 1.3 *Let $f : A \longrightarrow B$ be a L-homomorphism and $n \geq 1$ be an integer. If $\overline{a} = \langle a_1, \ldots, a_n \rangle \in A^n$, write $f(\overline{a})$ for $\langle f(a_1), \ldots, f(a_n) \rangle$.*
a) We say that f is **pure** *if it reflects positive existential L-formulas with parameters in A, that is, if $\exists\overline{v}\varphi(\overline{v};\overline{a})$ is an existential L-formula, with $\overline{a} \in A^n$, then $B \models \exists\overline{v}\varphi(\overline{v}; f(\overline{a})) \quad \Rightarrow \quad A \models \exists\overline{v}\varphi(\overline{v};\overline{a})$.*
b) We say that f has a **retract** *and that* $\boldsymbol{A}$ **is a retract of** $\boldsymbol{B}$*, if there is a L-homomorphism $g : B \longrightarrow A$ such that $g \circ f = Id_A$. It is customary to refer to g as a* **retraction** *and to f as a* **section***.*

Remark 1.4 With notation as in Definition 1.3, the following facts are easily established:
(1) Since L has equality, all pure L-morphisms are embeddings.
(2) Since positive existential formulas are logically equivalent to a disjunction of conjunctions of pp-formulas, $f : A \longrightarrow B$ is a pure embedding iff it reflects pp-formulas with parameters in A.
(3) Any morphism with a retract is a pure embedding.
(4) Let Σ be a set of L-sentences of the form $\forall\vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$, where $\psi_0(\vec{x}),\ \psi_1(\vec{x}) \in \exists^+(L)$. If $f : M \longrightarrow N$ is a pure morphism and N is a model of Σ, then the same is true of M. $\square$

1.5 Filtered Limits and Colimits in L-mod. If $\{M_k : k \in K\}$ is a family of L-structures, let $M = \prod_{k\in K} M_k$ be the product L-structure and, for each $k \in K$, write $\pi_k : M \longrightarrow M_k$ for the natural L-projection onto the k^{th}-coordinate. We now register the following

Definition 1.6 Let $\langle I, \leq \rangle$ be a directed poset.

a) (1) An **inductive system of L-structures over I**,

$$\mathcal{M} \;=\; \langle\, M_i;\ \{f_{ij} : i \leq j \text{ in } I\}\,\rangle,$$

consists of a family of L-structures, $\{M_i : i \in I\}$, together with L-morphisms, $f_{ij} : M_i \longrightarrow M_j$, whenever $i \leq j$ in I, such that $f_{ii} = Id_{M_i}$ and, if $i \leq j \leq k$, the diagram (I) below is commutative:

$$\begin{tikzcd} M_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_{ik}"'] \arrow[drr, phantom, "\mathrm{(I)}"] & & M_j \arrow[dl, "f_{jk}"] \\ & M_k & \end{tikzcd} \qquad \begin{tikzcd} M_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"'] \arrow[drr, phantom, "\mathrm{(II)}"] & & M_j \arrow[dl, "f_j"] \\ & D & \end{tikzcd}$$

$$\begin{tikzcd} M_i \arrow[rr, "f_i"] \arrow[dr, "\eta_i"'] \arrow[drr, phantom, "\mathrm{(III)}"] & & D \arrow[dl, "f"] \\ & E & \end{tikzcd}$$

(2) If $\mathcal{M} = \langle\, M_i;\ \{f_{ij} : i \leq j \text{ in } I\}\,\rangle$ is an inductive system over $\langle\, I, \leq \,\rangle$, a **dual cone over** $\mathcal{M}$, $\langle\, D;\ \{f_i : i \in I\}\,\rangle$, consists of a L-structure, D, together with L-morphisms, $f_i : M_i \longrightarrow D$, such that for all $i \leq j$ in I, diagram (II) above is commutative. Such a dual cone is the **inductive limit or colimit** of $\mathcal{M}$, written

$$D \;=\; \varinjlim \mathcal{M} \quad \text{or} \quad D \;=\; \varinjlim_{i \in I} M_i,$$

if for any dual cone over $\mathcal{M}$, $\langle\, E;\ \{\eta_i : i \in I\}\,\rangle$, there is a **unique** L-morphism, $f : D \longrightarrow E$, such that for all $i \in I$, diagram (III) above is commutative.

b) (1) A **projective system of L-structures over I**,

$$\mathcal{N} \;=\; \langle\, N_i;\ \{g_{ji} : i \leq j \text{ in } I\}\,\rangle,$$

consists of a family of L-structures, $\{N_i : i \in I\}$, together with L -morphisms, $g_{ji} : M_j \longrightarrow M_i$, whenever $i \leq j$, such that $g_{ii} = Id_{N_i}$ and, for $i \leq j \leq k$, diagram (IV) below is commutative:

$$
\begin{tikzcd}
M_k \arrow[rr, "g_{kj}"] \arrow[dr, "g_{ki}"'] & {} \arrow[d, phantom, "\text{(IV)}"] & M_j \arrow[dl, "g_{ji}"] \\
& M_i &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C \arrow[rr, "g_j"] \arrow[dr, "g_i"'] & {} \arrow[d, phantom, "\text{(V)}"] & M_j \arrow[dl, "g_{ji}"] \\
& M_i &
\end{tikzcd}
$$

$$
\begin{tikzcd}
F \arrow[rr, "g"] \arrow[dr, "\varphi_i"'] & {} \arrow[d, phantom, "\text{(VI)}"] & C \arrow[dl, "g_i"] \\
& N_i &
\end{tikzcd}
$$

(2) If $\mathcal{N} = \langle N_i;\ \{g_{ji} : i \leq j \text{ in } I\}\rangle$ is a projective system over I, a **cone over** $\mathcal{N}$, $\langle C;\ \{g_i : i \in I\}\rangle$, consists of a L-structure C and L-morphisms $g_i : C \longrightarrow M_i$, such that for all $i \leq j$ in I, the diagram above right is commutative. Such a cone is the **projective limit or limit of** $\mathcal{N}$, written

$$C = \varprojlim \mathcal{N} \quad \text{or} \quad C = \varprojlim_{i \in I} N_i,$$

if for any cone over $\mathcal{N}$, $\langle F;\ \{\varphi_i : i \in I\}\rangle$, there is a **unique** L-morphism, $g : F \longrightarrow C$, such that for all $i \in I$, diagram (VI) above is commutative.

Now we have (for proofs see [MM2] or Chapter 17 in [Mir])

Theorem 1.7 *The category* **L-mod** *has all filtered limits and colimits. Moreover, if $\langle I, \leq\rangle$ is a directed poset:*

a) If $\mathcal{M} = \langle M_i;\ \{f_{ij} : i \leq j \text{ in } I\}\rangle$ is an inductive system over I, a dual cone $\langle D;\ \{f_i : i \in I\}\rangle$ is (isomorphic to) $\varinjlim \mathcal{M}$ iff the following conditions are satisfied:

[colim 1] : $D = \bigcup\ \{f_i(M_i) : i \in I\}$.

[colim 2] : *If $\varphi(v_1, \ldots, v_n)$ is an atomic formula in L and $\overline{s} \in D^n$,*

$$D \models \varphi[\overline{s}] \quad \Leftrightarrow \quad \begin{cases} \exists\ k \in I \ \text{ and } \ \overline{x} \in M_k^n \text{ such that} \\ s_p = f_k(x_p),\ 1 \leq p \leq n, \ \text{ and } \ M_k \models \varphi[\overline{x}]. \end{cases}$$

b) Let $\mathcal{N} = \langle N_i;\ \{\{g_{ji} : i \leq j \text{ in } I\}\rangle$ be a projective system over I and let $\langle C;\ \{g_i : i \in I\}\rangle$ be a cone over $\mathcal{N}$. Let $g : C \longrightarrow \prod_{i \in I} N_i$ be the unique L-morphism such that for all $i \in I$, $\pi_i \circ g = g_i$. Then, C is (isomorphic to) $\varprojlim \mathcal{N}$ iff the following conditions are verified:

[lim 1] : *The image of* g *in* $\prod_{i\in I} N_i$ *is the set* $\{x \in \prod_{i\in I} N_i$ *: for all* $i \leq j$ *in* I $g_{ji}(\pi_i(x)) = \pi_j(x)\}$.
[lim 2] : *If* $\varphi(v_1, \ldots, v_n)$ *is an atomic* L*-formula and* $\overline{s} \in C^n$, $C \models \varphi[\overline{s}]$ $\Leftrightarrow$ $\forall\, i \in I$, $N_i \models \varphi[g_i(\overline{s})]$. □

We assume that the reader is familiar with reduced products and their basic properties. Recall that if $\{M_i : i \in I\}$ is a family of L-structures and $\mathcal{F}$ is a proper filter on I, $\prod_{i\in I} M_i/\mathcal{F}$ is the *reduced product* of the M_i modulo $\mathcal{F}$, referred to as an *ultraproduct* whenever $\mathcal{F}$ is an ultrafilter. We shall describe, omitting proofs, how reduced products can be seen as colimits, a fact that will be useful in the proof of Theorem 2.3.

If $I \neq \emptyset$ is a set and $\{M_i : i \in I\}$ is a family of L-structures, we set, for $J \subseteq K \subseteq I$:
$*$ $\mathcal{M}(J) = \prod_{j\in J} M_j$; $*$ $\pi_{KJ} : \mathcal{M}(K) \longrightarrow \mathcal{M}(J)$ is the projection that forgets coordinates outside J.

If $\mathcal{F}$ is a proper filter on a set I, note that with opposite of the partial order of inclusion, $\mathcal{F}$ is a directed poset since it is closed under intersections. With these preliminaries, we have

Proposition 1.8 *With notation as above, let* $\mathcal{F}$ *be a proper filter on* I*, let* $M = \prod_{i\in I} M_i$ *and* $P = M/\mathcal{F}$*. Then,*

$$\mathcal{M} = \langle\, \mathcal{M}(J);\ \{\pi_{KJ} : J \subseteq K,\ J \in \mathcal{F}\}\,\rangle$$

is an inductive system over $\langle\, \mathcal{F}, \subseteq^{op} \rangle$ *and* $\varinjlim \mathcal{M} \approx \langle\, P;\ \{\nu_J : J \in \mathcal{F}\}\,\rangle$*, where for* $J \in \mathcal{F}$*,* $\nu_J : \mathcal{M}(J) \longrightarrow P$ *is given by* $x \longmapsto x/\mathcal{F}$ *= the class of any element of* M *that coincides with* x *on* J*.* □

2. Profinite Structures and Ultraproducts

Definition 2.1 A L-structure is **profinite** if it is L-isomorphic to the limit of a projective system of *finite* L-structures over a directed poset.

Remark 2.2 If P is a profinite L-structure then there is a directed poset, $\langle\, I, \leq \rangle$, and a diagram of finite L-structures over I,

$$\mathcal{M} = (M_i, \{f_{ji} : i \leq j\})$$

such that $(P;\ \{\lambda_i : i \in I\}) = \varprojlim \mathcal{M}$. By Theorem 1.7.(b) P is a substructure of the product $M = \prod_{i\in I} M_i$, i.e., there is a natural L-embedding, $\iota : P \longrightarrow M$, such that for all $i \in I$,

(♯) $\lambda_i \ = \ \pi_i \circ \iota.$

$$\begin{tikzcd} P \arrow[rr, "\iota"] \arrow[dr, "\lambda_i"'] & & M \arrow[dl, "\pi_i"] \\ & M_i & \end{tikzcd}$$

where $\pi_i : M \longrightarrow M_i$ is the canonical projection. Furthermore, it follows from [lim 1] in Theorem 1.7.(b) that

(♭) $\forall\, \overline{x} \in P \ \ \forall\, j,k \in I \ \ (\ j \in k^{\rightarrow} \quad \Rightarrow \quad f_{jk}(x_j) \ = \ x_k\ ).$

With notation as in Proposition 1.8, if $\mathcal{F}$ is a filter in I then for each $J \in \mathcal{F}$ there is a natural L-morphism

$$\nu_J : M_{|J} \longrightarrow M/\mathcal{F}, \text{ given by } x \longmapsto x/\mathcal{F},$$

where $M/\mathcal{F}$ is the reduced product $\prod_{i\in I} M_i/\mathcal{F}$. □

We now state

Theorem 2.3 *Profinite L-structures are retracts of ultraproducts of finite L-structures. More precisely, and with the notation in* 2.2, *let* $\langle I, \leq \rangle$ *be a directed poset and*

$$\mathcal{M} \ = \ \langle M_i;\ \{f_{ij} : i \leq j \ in\ I\}\rangle$$

is a projective system of finite L-structures over I. If

$$\varprojlim \mathcal{M} \ = \langle P;\ \{\lambda_i : i \in I\}\rangle,$$

then the composition

$$P \ \xrightarrow{\iota} \ \prod_{i\in I} M_i \ \xrightarrow{\nu_I} \ \prod_{i\in I} M_i/\mathcal{U},$$

is an L-section (Definition 1.3.(b)), *where $\mathcal{U}$ is any directed ultrafilter in I* (Definition 1.1.(b)).

$$\begin{tikzcd} P \arrow[r, "\iota"] \arrow[drr, "Id_P"'] & \prod_{i\in I} M_i \arrow[r, "\nu_I"] \arrow[dr, phantom, "\circlearrowright"] & \prod_{i\in I} M_i/\mathcal{U} \arrow[d, "\gamma^U"] \\ & & P \end{tikzcd}$$

Proof. By Lemma 1.2, there is a directed ultrafilter $\mathcal{U}$ in $\langle I, \leq \rangle$, which will remain fixed throughout the proof. Let $M = \prod_{i \in I} M_i$ be the product L-structure. By Proposition 1.8 (and with the same notation),

$$\prod_{i \in I} M_i/\mathcal{U} \;=\; \langle M/\mathcal{U};\ \{\nu_J : J \in \mathcal{U}\} \rangle$$

is L-isomorphic to

$$\underrightarrow{lim}\ \langle M_{|J};\ \{\pi_{KJ} : J \subseteq K\ ,\ J \in \mathcal{U}\} \rangle.$$

We shall use this fact to construct a L-morphism, $\gamma^U : M/\mathcal{U} \longrightarrow P$, such that

$$\gamma^U \circ (\nu_I \circ \iota) \;=\; Id_P,$$

completing the proof. As $\mathcal{U}$ will remain fixed, we shall write γ in place of γ^U. Moreover, to ease presentation, the proof will be divided into several Facts; all notational conventions remain in force, in particular those in 2.2.

For $J \in \mathcal{U}$, $i \in I$, $\overline{x} \in \mathcal{M}(J) = \prod_{j \in J} M_j$ and $y \in M_i$ we define

$$V_{J,i}(\overline{x}, y) \;=\; \{j \in J \cap i^{\rightarrow} : f_{ji}(x_j) = y\}.$$

Fact 2.4 *For $J \in \mathcal{U}$, $i \in I$, $\overline{x} \in \mathcal{M}(J)$ and $y, z \in M_i$,*

a) $z \neq y \quad \Rightarrow \quad V_{J,i}(\overline{x}, y) \cap V_{J,i}(\overline{x}, z) \;=\; \emptyset.$

b) $J \cap i^{\rightarrow} \;=\; \coprod_{y \in M_i} V_{J,i}(\overline{x}, y)$ ($\coprod$ is disjoint union).

Proof. Item (a) follows immediately from the fact that f_{ji} is a function. For (b), by the definition of $V_{J,i}(\overline{x}, y)$ it is clearly enough to show that the left side of the equality is contained in its right side; but note that if $j \in J \cap i^{\rightarrow}$, then $f_{ji}(x_j) \in M_i$, as required. $\square$

Fact 2.5 *For each $J \in \mathcal{U}$ and $i \in I$ there is a L-morphism $\gamma_{J,i}$: $\mathcal{M}(J) = \prod_{k \in J} M_k \longrightarrow M_i$, such that*

a) If $\overline{x} \in M_{|J}$ and $y \in M_i$ then $\gamma_{J,i}(\overline{x}) \;=\; y$ iff $V_{J,i}(\overline{x}, y) \in \mathcal{U}$.

b) If $J \subseteq K$ are members of $\mathcal{U}$ and $i \in I$, then the diagram (I) *below is commutative:*

$$\begin{tikzcd}
M_{|K} \arrow[rr, "\pi_{KJ}"] \arrow[dr, "\gamma_{J,i}"'] & \arrow[d, phantom, "\mathrm{(I)}"] & M_{|J} \arrow[dl, "\gamma_{K,i}"] \\
& M_i &
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{|J} \arrow[rr, "\gamma_{J,k}"] \arrow[dr, "\gamma_{J,i}"'] & \arrow[d, phantom, "\mathrm{(II)}"] & M_k \arrow[dl, "f_{ki}"] \\
& M_i &
\end{tikzcd}$$

c) For each $J \in \mathcal{U}$ and $i \leq k$ in I, diagram (II) *above is commutative.*
d) For each $k \in I$, $\gamma_{I,k} \circ \iota = \pi_k \circ \iota$, where $\pi_k : M \longrightarrow M_k$ is the canonical projection.

Proof. Because $\mathcal{U}$ is a directed filter in I, for each $J \in \mathcal{U}$ and $i \in I$, we have $J \cap i^{\rightarrow} \in \mathcal{U}$; since $\mathcal{U}$ is an ultrafilter and M_i is finite, Fact 2.4.(b) implies that *there is a unique* $y \in M_i$ such that $V_{J,i}(\overline{x}, y) \in \mathcal{U}$. We define

$$\gamma_{J,i}(\overline{x}) \;=\; \text{the unique } y \in M_i \text{ such that } V_{J,i}(\overline{x}, y) \in \mathcal{U}.$$

Clearly, item (a) is verified. Now, we must show that $\gamma_{J,i}$ is a L-morphism. To ease reading, if $J \in \mathcal{U}$, we shall use an exponent J to indicate the interpretation of the symbols of L in $\mathcal{M}(J)$.

$*$ Let c be a constant in L. Since c^J is the sequence $\langle c^{M_j} \rangle \in \mathcal{M}(J)$ and the f_{ji} are L-morphisms, we get

$$V_{J,i}(c^J, c^{M_i}) \;=\; \{j \in J \cap i^{\rightarrow} : f_{ji}(c^J_j) = c^{M_i}\} \;=\; \{j \in J \cap i^{\rightarrow} : f_{ji}(c^{M_j}) = c^{M_i}\} \;=\; J \cap i^{\rightarrow}$$

that belongs to $\mathcal{U}$. By item (a), $\gamma_{J,i}(c^J) = c^{M_i}$, as needed.

$*$ Let ω be a n-ary function symbol in L. If $\overline{x}_1, \ldots, \overline{x}_n \in \mathcal{M}(J)^n$ and $j \in J$ then, taking into account the interpretation of ω in the product L-structure M, we have

$$\omega^J(\overline{x}_1, \ldots, \overline{x}_n)(j) \;=\; \omega^{M_j}(x_{1j}, \ldots, x_{nj}). \tag{A}$$

Consider

$$\begin{cases} y^p &= \gamma_{J,i}(\overline{x}_p), \qquad 1 \leq p \leq n; \\ z &= \omega^{M_i}(y^1, \ldots, y^n); \\ h &= \omega^J(\overline{x}_1, \ldots, \overline{x}_n) \qquad (\in \mathcal{M}(J)). \end{cases}$$

We will show that

$$\textstyle\bigcap_{p=1}^n V_{J,i}(\overline{x}_p, y^p) \;\subseteq\; V_{J,i}(h, z). \tag{B}$$

If $j \in \bigcap_{p=1}^n V_{J,i}(\overline{x}_p, y^p)$ then the definition of $V_{J,i}$ implies

$$\forall\, 1 \leq p \leq n, \;\; f_{ji}(x_{pj}) \;=\; y^p. \tag{C}$$

Because the f_{ji} are L-morphisms, (A) and (C) yield

$$f_{ji}(h_j) \;=\; f_{ji}(\omega^{M_j}(x_{1j}, \ldots, x_{nj})) \;=\; \omega^{M_i}(f_{ji}(x_{1j}), \ldots, f_{ji}(x_{nj})) \;=\; \omega^{M_i}(y^1, \ldots, y^n) \;=\; z,$$

establishing (B). As the intersection of the left-hand side in (B) belongs to $\mathcal{U}$ we have $V_{J,i}(h, z) \in \mathcal{U}$. By the item (a) of this Fact, this means that

$$\gamma_{J,i}(\omega^J(\overline{x}_1, \ldots, \overline{x}_n)) \;=\; \omega^{M_i}(\gamma_{J,i}(\overline{x}_1, \ldots, \overline{x}_n))$$

showing that $\gamma_{J,i}$ preserves the operation ω;

$*$ Let R be a n-ary relation symbol in L. Consider $\overline{x}_1, \ldots, \overline{x}_n \in \mathcal{M}(J)^n$. The interpretation of R in the product L-structure yields

$$\mathcal{M}(J) \models R[\overline{x}_1, \ldots, \overline{x}_n] \text{ iff } \forall\, j \in J, \ \ M_j \models R[x_{1j}, \ldots, x_{nj}]. \tag{D}$$

As above, let $y^p = \gamma_{J,i}(\overline{x}_p)$, $1 \leq p \leq n$. We must show that

$$\mathcal{M}(J) \models R[\overline{x}_1, \ldots, \overline{x}_n] \quad \Rightarrow \quad M_i \models R[y^1, \ldots, y^n]. \tag{E}$$

Because $\bigcap_{p=1}^n V_{J,i}(\overline{x}_p, y^p) \in \mathcal{U}$, this intersection is non-empty; if j is a member of this intersection, it is clear that (C) holds true. Hence, it follows from (D) and the fact that f_{ji} is a L-morphism that

$$M_{|J} \models R[\overline{x}_1, \ldots, \overline{x}_n] \quad \Rightarrow \quad M_j \models R[x_{1j}, \ldots, x_{nj}] \quad \Rightarrow$$
$$M_i \models R[f_{ji}(x_{1j}), \ldots, f_{ji}(x_{nj})],$$

which together with (C) implies (E), completing the verification that $\gamma_{J,i}$ is a L-morphism.

b) Let $\overline{t} \in M_{|K}$ and $\overline{x} = \pi_{KJ}(\overline{t})$ [1]. If $y = \gamma_{J,i}(\overline{x})$, then

$$V_{J,i}(\overline{x}, y) \subseteq V_{K,i}(\overline{t}, y).$$

Indeed, if $j \in V_{J,i}(\overline{x}, y)$ (obviously contained $K \cap i^{\rightarrow}$), then $f_{ji}(t_j) = f_{ji}(x_j) = y$, as required. Since $V_{J,i}(\overline{x}, y) \in \mathcal{U}$, we have $V_{K,i}(\overline{t}, y) \in \mathcal{U}$ and item (a) ensures that $\gamma_{K,i}(\overline{t}) = y = \gamma_{J,i}(\pi_{KJ}(\overline{t}))$, as desired.

c) Let $\overline{x} \in M_{|J}$ and $z = \gamma_{J,k}(\overline{x})$. Then

$$V_{J,k}(\overline{x}, z) \subseteq V_{J,i}(\overline{x}, f_{ki}(z)). \tag{F}$$

Indeed, if $j \in V_{J,k}(\overline{x}, z)$ (contained in $J \cap i^{\rightarrow}$ because $i \leq k$) then $f_{jk}(x_j) = z$. As $\mathcal{M}$ is a projective system, we get $f_{ji}(x_j) = f_{ki}(f_{jk}(x_j)) = f_{ki}(z)$, showing that $j \in V_{J,i}(\overline{x}, f_{ki}(z))$; but then (F) above guarantees that this set is in U, and so item (a) implies $\gamma_{J,i} = f_{ki} \circ \gamma_{J,k}$, as needed.

d) For each $\overline{x} \in P$ and $k \in I$, note that $\pi_k(\iota(\overline{x})) = x_k$. From relation ($\flat$) in Remark 2.2, it follows that

$$V_{I,k}(\iota(\overline{x}), x_k) = \{j \in k^{\rightarrow} : f_{jk}(x_j) = x_k\} = k^{\rightarrow}.$$

Because $\mathcal{U}$ is a directed ultrafilter, we have $V_{I,k}(\iota(\overline{x}), x_k) \in \mathcal{U}$ and another application of item (a) yields the desired conclusion, ending the proof of Fact 2.5. $\square$

By Proposition 1.8 we have

$$M/\mathcal{U} = \underrightarrow{lim} \langle \mathcal{M}(J); \{ \pi_{KJ} : J \subseteq K , J \in \mathcal{U}\} \rangle.$$

The universal property of colimits and 2.5.(b) yield, for each $i \in I$, *a unique* L-morphism, $\gamma_i : M/\mathcal{U} \longrightarrow M_i$, such that for all $J \in \mathcal{U}$ the diagram below left commutes:

[1] Recall that π_{KJ} is the projection that forgets the coordinates outside K.

$$\begin{tikzcd} \mathcal{M}(J) \arrow[r, "\nu_J"] \arrow[dr, "\gamma_{J,i}"'] & M/\mathcal{U} \arrow[d, "\gamma_i"] \\ & M_i \end{tikzcd} \qquad \begin{tikzcd} \mathrm{M}/\mathcal{U} \arrow[r, "\gamma_k"] \arrow[dr, "\gamma_i"'] & M_k \arrow[d, "f_{ki}"] \\ & M_i \end{tikzcd} \tag{*}$$

Fact 2.6 *For each $i \leq k$ in I, the diagram above right in $(*)$ is commutative.*

Proof. For each $i \leq k$ in I and $J \in \mathcal{U}$, Fact 2.5.(c) gives $\gamma_{J,i} = f_{ki} \circ \gamma_{J,k}$. Then, the commutativity of the diagram above left in (*) – for k and i –, implies that, for all $J \in \mathcal{U}$ we have

$$f_{ki} \circ \gamma_k \circ \nu_J = f_{ki} \circ \gamma_{J,k} = \gamma_{J,i} = \gamma_i \circ \nu_J.$$

Now, the uniqueness of the γ_i making the left diagram commutative implies $f_{ki} \circ \gamma_k = \gamma_i$, as required. $\square$

Fact 2.6 shows that $\langle M/\mathcal{U};\ \{\gamma_i : i \in I\}\rangle$ is a cone over $\mathcal{M}$. Thus, the universal property of limits yields *a unique* L-morphism,

$$\gamma : M/\mathcal{U} \longrightarrow P = \varprojlim \mathcal{M},$$

such that the following diagram is commutative, for all $i \in I$:

$$\begin{tikzcd} \mathrm{M}/\mathcal{U} \arrow[r, "\gamma"] \arrow[dr, "\gamma_i"'] & P \arrow[d, "\lambda_i"] \\ & M_i \end{tikzcd} \tag{**}$$

We shall now check that

$$\gamma \circ \nu_I \circ \iota = Id_P. \tag{G}$$

Since $\langle P;\ \{\lambda_i : i \in I\}\rangle = \varprojlim \mathcal{M}$, the universal property of limits ensures that to prove (G) it is enough to show that for all $k \in I$

$$\lambda_k \circ (\gamma \circ \nu_I \circ \iota) = \lambda_k. \tag{H}$$

Since $\begin{cases} \lambda_k \circ \gamma = \gamma_k & \text{by diagram (**);} \\ \gamma_k \circ \nu_I = \gamma_{I,k} & \text{by the left diagram in (*);} \\ \pi_k \circ \iota = \lambda_k & \text{by } (\sharp) \text{ in Remark 2.2,} \end{cases}$

(H) is equivalent to $\gamma_{I,k} \circ \iota = \pi_k \circ \iota$, precisely the content of the Fact 2.5.(d), completing the proof. □

Theorem 2.3 yield the following result, whose part (b) appears, with different proofs, as Corollary 3.25 in [DM3] and as Proposition 1.9.8 in [Lim].

Corollary 2.7 *a) If T is a theory axiomatized by L-sentences of the form $\forall\vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$, where $\psi_0(\vec{x}), \psi_1(\vec{x})$ are formulas in $\exists^+(L)$, any projective limit of finite models of T is also a model of T.*
b) If $\mathcal{G} = \langle G_i; \{f_{ji} : i \leq j \text{ in } I\} \rangle$ is a projective system of finite special groups over the directed poset $\langle I, \leq \rangle$, then its projective limit in the category of groups, G, has a natural special group structure, with which it is the projective limit of $\mathcal{G}$ in the category of special groups. Moreover, G is reduced iff $R = \{i \in I : G_i \text{ is reduced}\}$ is cofinal in I [2].

Proof. Item (a) follows from item (4) in 1.4, Theorem 2.3 and Łós' Theorem for ultraproducts. Item (b) is a consequence of (a) and the fact that the theory of special groups is a first-order theory, whose axioms have the form $\forall\vec{x}(\psi_0(\vec{x}) \to \psi_1(\vec{x}))$, where $\psi_0(\vec{x}), \psi_1(\vec{x})$ are positive existential formulas (see Definition 1.2, [DM2]). □

Noteworthy applications of Theorem 2.3 are the following:
∗ In Proposition 2.15, [DMM], it is used to show that all reduced profinite special groups are lattice ordered;
∗ In Theorem 5.9, [DM4], it is instrumental in proving that all reduced profinite special groups satisfy a powerful K-theoretic property − the [SMC]-property −, that implies both Marshall's signature conjecture and a generalization of Milnor's Witt-ring conjecture for mod 2 K-theory. The interested reader may consult [DM4] for details.

Since profinite structures, particularly profinite groups, appear frequently in many branches of Mathematics, it is envisaged that Theorem 2.3 might have further applications in obtaining interesting first-order properties of these structures.

[2] I.e., for all $i \in I$, $i^{\to} \cap K \neq \emptyset$.

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Institute of Mathematics and Statistics,
University of São Paulo, São Paulo, Brazil;
{hugomar; miraglia}@ime.usp.br