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INTUITIONISTIC PREDICATE LOGIC WITH DECIDABLE PROPOSITIONAL ATOMS

A b s t r a c t. First-order intuitionistic logic extended with the assumption about decidability of all propositional atoms combines classical and intuitionistic properties. Two classes of formulas on which this extension coincides with classical and intuitionistic logic, respectively, are identified. Constrained Kripke structures are introduced for modeling intuitionistic logic with decidable propositional atoms. The extent of applicability of classical-only laws, the extent of the disjunction and existence properties, decidability issues, and translations are investigated.

1. Introduction

The dispute between supporters of classical and intuitionistic logic revolves around application of the law of excluded middle (LEM) to formulas containing object variables. Note that Brouwer [4] criticized classical logic only because LEM was abstracted from finite situations and extended without

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justification to statements about infinite collections. As Weyl [16] states in reference to Brouwer: "classical logic was abstracted from the mathematics of finite sets and their subsets". Propositional logic can be considered a part of the mathematics of finite sets because of availability of finite models using truth tables. Thus, LEM for propositional formulas is not really a target of intuitionistic criticism of classical logic.

The classical assumption that every propositional atom (symbol) is either true or false but not both is so natural. This assumption translates into decidability of propositional atoms in terms of proof theory. Let us look what happens to first-order intuitionistic logic when assuming that all propositional atoms are decidable. In a formulation including terms, it is reasonable to extend this assumption onto predicate atomic expressions without variables, i.e. whose arguments are all constants. This extended assumption corresponds to computability of functions modeling predicates.

First-order intuitionistic logic with decidable propositional atoms is supposed to be a system somewhat acceptable by both camps: classical and intuitionistic. On one hand, it is fully classical in its propositional part. On the other hand, LEM, double negation elimination and their variants are not extended onto statements about infinite collections. The remainder of this paper is devoted to investigation of properties of this extension of intuitionistic predicate logic. Note that results presented here can be retained in a formulation including terms and having decidable predicate atoms without variables.

In comparison with fibring logics [2], absence of paired connectives makes this approach to combining classical and intuitionistic features more natural and intuitive. This approach does not suffer the collapsing problem either [12]. In contrast to intermediate logics, there is no introduction of non-intuitionistic axioms applicable to prime predicate formulas that are relevant to infinite collections.

Note that research with slightly similar motivations and goals was recently done by H. Kurokawa [8]. He investigated propositional logic with two types of atoms: classical and intuitionistic. And LEM is applicable to classical atoms only. Like in this paper, Kurokawa presents a disjunction property theorem and investigates how Kripke structures can be utilized to get sound and complete models for intuitionistic propositional logic with some classical atoms. Although in contrast to this research, Kurokawa's paper does not cover first-order issues. From the perspective of Kurokawa's research, first-order logic with classical propositional atoms and intuitionistic predicate atoms is investigated here. The disjunction property theorem presented here is stronger than one proved by Kurokawa.

2. Background and Definitions

Investigation of properties of first-order intuitionistic logic with decidable propositional atoms will be based on using sequent calculus as a framework. We use the phrase 'predicate symbol' only in application to symbols whose arity is more than zero because symbols of arity zero are decidable and thus have very different properties than the rest of formulas.

A sequent is an expression $\Gamma \vdash \Lambda$ where Γ is a (possibly empty) sequence of formulas and Λ is either a formula or an empty sequence. Γ is called antecedent, Λ is called succedent. We use Gentzen's formulation LJ for intuitionistic logic [3]. The rules of inference in LJ are: thinning, contraction, exchange, cut, and logical rules for all connectives and quantifiers in the antecedent and succedent. The only axiom schema of LJ is: $A \vdash A$. Typically, two constants - $\top$ and $\bot$ - are introduced by the following axioms: $\vdash \top$; $\bot \vdash$.

Formula F is called decidable if $F \vee \neg F$ holds in a respective theory. From the proof-theoretic point of view, assumption that all propositional atoms are decidable means addition of the axiom schema $\vdash A \vee \neg A$ that holds for all propositional atoms. Now, consider an alternative definition. Let us extend LJ with the following inference rule:

$$\frac{N, \Gamma \vdash \Lambda \qquad \neg N, \Gamma \vdash \Lambda}{\Gamma \vdash \Lambda}$$

Following [10], this rule will be called Rule of Excluded Middle (REM). Only propositional formulas N are allowed in REM. In a formulation including decidability of predicate atoms without variables, the additional axiom schema and REM should hold for these predicate atoms too. This extension will be called LJ+.

LJ+ is equivalent to intuitionistic logic extended with the aforementioned axiom schema for propositional atoms, that is, the sets of derivable sequents are the same. As well-known, decidability of propositional atoms implies LEM for all propositional formulas. This equivalence is proved by replacing the rules of excluded middle with $\vdash N \vee \neg N$ and vice versa.

Every formula derivable in LJ+ is also derivable in LK. Therefore, LJ+ is consistent. The propositional fragment of LJ+ is equivalent to classical propositional logic, that is, to the propositional fragment of LK [3]. Clearly, there exist first-order formulas provable in LJ+ but not provable in LJ, and there exist formulas provable in LK but not in LJ+. Relationship between intuitionistic logic with decidable propositional atoms and pure intuitionistic logic can be characterized as the following. Formula F is derivable in LJ+ if and only if, for some classically derivable propositional formula A, $A \supset F$ is derivable in LJ.

3. Classical Features

If R is a propositional symbol occurring in formula A, then let $A\{R|F\}$ denote the formula obtained from A by replacing all occurrences of R with formula F.

Theorem. If R is a propositional symbol occurring in formula A, then A is derivable in LJ+ if and only if both $A\{R|\top\}$ and $A\{R|\bot\}$ are derivable in LJ+.

Proof. The 'only if' part is proved by replacing all occurrences of R by $\top$ or by $\bot$ in A's derivation, respectively. Now consider derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ and the occurrences of $\top$ and $\bot$ that replaced R. The constants $\top$ and $\bot$ introduced in derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ by other means than the axioms $\vdash \top$ and $\bot \vdash$ can be replaced by R everywhere in the derivations because the only other means are thinning, axioms $\top \vdash \top$, $\bot \vdash \bot$. If $\top$ is introduced by axiom $\vdash \top$ in derivation of $A\{R|\top\}$, let us replace this axiom by axiom $R \vdash R$. If $\bot$ is introduced by axiom $\bot \vdash$ in derivation of $A\{R|\bot\}$, let us replace this axiom by $R \vdash R$.

After that, let us transform these modified derivations of $A\{R|\top\}$ and $A\{R|\bot\}$ so that R ($\neg R$) is added as the rightmost formula to the antecedents of all sequents below the replaced axioms for $\top$ and $\bot$. Also, if a rule has two upper sequents and R ($\neg R$) has been already added as the rightmost formula to the antecedent of one of its upper sequents, then let us add R ($\neg R$) to the counterpart upper sequent by injection of a thinning and exchanges in order to make R ($\neg R$) the rightmost formula of the antecedent of this sequent. As a result of this transformation, both derivations under consideration remain correct. The endsequent in the derivation corresponding to $A\{R|\top\}$ will have one formula R in the antecedent whereas the endsequent in the derivation corresponding to $A\{R|\bot\}$ will have one formula $\neg R$ in the antecedent. These two derivations are merged by application of REM. The resulting endsequent is $\vdash A$. □

Kleene compiled an extensive collection of logical laws [5, 6]. Most laws from this collection hold in intuitionistic logic but still some are classical-only. Many of these laws have two components (A and B). Below are those classical-only laws from the Kleene collection which also hold in LJ+ provided that one of the two components is propositional. The remaining classical-only laws from Kleene's collection do not hold in LJ+ even when one of the two components involved is propositional. Symbol $\sim$ is used for logical equivalence.

a. $A \supset B \sim \neg B \supset \neg A$ - holds if B is propositional
b. $\neg A \& B \sim \neg A \vee \neg B$ - holds if A or B is propositional
c. $\neg A \supset B \sim A \& \neg B$ - holds if A is propositional
d. $A \supset B \sim \neg A \vee B$ - holds if A or B is propositional
e. $A \vee B \sim \neg A \supset B$ - holds if A is propositional
f. $A \vee \forall x B(x) \sim \forall x(A \vee B(x))$ - holds if A is propositional
g. $A \supset \exists x B(x) \sim \exists x(A \supset B(x))$ - holds if A is propositional

Beyond propositional formulas, LJ+ coincides with classical logic on an extension of the class of Harrop formulas [14]. This extension (H+) is defined recursively by the following:

- $\top$, $\bot$, $\neg A$, B are H+ formulas whenever B is propositional
- $A \& B$ is a H+ formula whenever A and B are H+ formulas
- $A \vee B$ is a H+ formula whenever one of A, B is propositional and the other is a H+ formula
- $A \supset B$ is a H+ formula whenever B is a H+ formula
- $\forall x A(x)$ is a H+ formula whenever A(x) is a H+ formula

As known, any formula is classically derivable if and only if its double negation translation is intuitionistically derivable [14]. Once we prove equivalence of $\neg\neg A$ and A in LJ+ for all formulas A from this class, the fact that every classically derivable H+ formula is also derivable in LJ+ will follow from the above. In order to prove that H+ formulas are double negation closed in LJ+, we leverage on the proof of this fact for Harrop formulas in intuitionistic logic. The two additional cases to cover are: propositional formulas and disjunctions. Propositional formulas are, of course, double negation closed in LJ+. The disjunctions from the above definition are double negation closed due to equivalence of $\neg\neg(A \vee B)$ and $\neg\neg A \vee \neg\neg B$ in LJ+ when one of A, B is propositional.

A subformula occurrence in a logical formula is called positive or negative according to an even or odd number of negations containing this occurrence plus implications containing this occurrence as the antecedent. If there are no such negations or implications in the formula, then the subformula occurrence is also called strictly positive. A propositional symbol is called unipolar in a given formula/sequent if its occurrences in this formula/sequent are either all positive or all negative. Otherwise, it is called bipolar. A formula/sequent is called unipolar if all propositional symbols are unipolar in it.

A prenex formula has all its quantifiers at the front, and the rest of the formula is the scope of all the quantifiers. Formula F has a prenex form in LJ+ if the following conditions are satisfied:

- every subformula occurrence $\forall x A(x)$ in F is strictly positive
- every subformula occurrence $\exists x A(x)$ in F is either strictly positive or negative
- every subformula occurrence B&C with quantifiers is either positive or one of B, C is propositional
- every subformula occurrence $B \vee C$ with quantifiers is strictly positive, not containing universal and negative existential quantifiers, or negative, or one of B, C is propositional
- every subformula occurrence $B \supset C$ with quantifiers is strictly positive, not containing existential quantifiers, or negative, or one of B, C is propositional

Overall, existence of a prenex form in LJ+ for this class of formulas is proved as in classical logic. The two last laws dealing with quantifiers (f, g) fill the gap for moving strictly positive quantifiers upward. The difference is that implications with one propositional argument are turned into disjunctions, negations are pushed down, and negative disjunctions/implications are turned into conjunctions.

4. Weak Cut Elimination and Subformula Property

Theorem (Weak cut elimination). Every LJ+ derivation can be transformed into another derivation with the same endsequent and in which the cut rule does not occur.

This theorem is proved by using Gentzen's cut elimination proof for LJ [3] and augmenting it with two transformations that swap REM and the cut rule. One of the transformation lowers the left rank, the other lowers the right rank. Gentzen's proof for the case when both the left and the right rank equal one does not require any augmentation. Note that REM is another discharge rule along with the cut rule, and it cannot be eliminated from LJ+ derivations.

Negri and von Plato prove admissibility of cut in G3ip extended with REM [10]. Every predicate formula in a cut-free derivation in LJ+ is a subformula of some formula in the endsequent but propositional formulas can be discarded by application of REM. Every derivation in LJ+ can be transformed into another derivation in which only negations of propositional atoms can be discarded. Negri and von Plato still call this subformula property [10], and we call this weak subformula property. Actually, the following theorem gives a somewhat stronger property for LJ+ but it is not the strict subformula property of LJ, LK and such.

Theorem (Weak subformula property). Every derivation in LJ+ can be transformed into such derivation with the same endsequent that all subformulas from this derivation occur in the endsequent except for negations of the propositional atoms that are bipolar in the endsequent.

Proof. Consider a derivation with using axioms $\vdash A \vee \neg A$ applied to propositional atoms only. This derivation can be turned into a derivation with the rules of excluded middle that also apply to propositional atoms only. Let us eliminate cut from this derivation. After that, rules of excluded middle still apply exclusively to propositional atoms. All formulas discarded in this derivation have the form N or $\neg N$ where N is a propositional atom.

Suppose propositional symbol N occurs in the derivation but not in the endsequent. Clearly, N is unipolar in both upper sequents of the bottommost rules of excluded middle applied to N. Now suppose N is unipolar in the endsequent. The so-called sign property [6] implies that N can be bipolar in an upper sequent of any rule other than REM only if it is bipolar in its lower sequent. Hence, N is unipolar in the lower sequent and in one of the upper sequents of the bottommost rules of excluded middle applied to N.

Consider any chain of rules of excluded middle $\mathcal{R}_1, \ldots, \mathcal{R}_m$ applied to N and such that $\mathcal{R}_1$ is a bottommost rule, every $\mathcal{R}_{k+1}$ is a bottommost rule in the derivation part rooted at the upper sequent of $\mathcal{R}_k$ in which N is unipolar, and there are no other rules of excluded middle applied to N above the upper sequent of $\mathcal{R}_m$ in which N is unipolar. The sign property guarantees that such chain exists. The derivation above the selected upper sequent ($N, \Gamma \vdash \Lambda$ or $\neg N, \Gamma \vdash \Lambda$) of $\mathcal{R}_m$ cannot contain axiom $N \vdash N$ because N is bipolar in this axiom. Therefore, N can be introduced in the derivation part above $N, \Gamma \vdash \Lambda$ by thinnings only. $\neg N$ can appear in the derivation part above $\neg N, \Gamma \vdash \Lambda$ via thinnings introducing N or $\neg N$.

Let us remove these thinnings and adjust all other rules below these thinnings in the derivation part above the selected upper sequent of $\mathcal{R}_m$. Contractions and exchanges applied to N or $\neg N$, and negation rules applied to N are eliminated. This derivation part remains correct, and its endsequent becomes $\Gamma \vdash \Lambda$. Subsequently, $\mathcal{R}_m$ can be eliminated along with the part above its other upper sequent.

Therefrom, all rules of excluded middle applied to N can be eliminated. It is proved by induction on m. Now, the number of propositional atoms that occur in the derivation but are not bipolar in the endsequent is reduced by one. By induction on the number of these atoms, we can get a transformed derivation in which REM is applied to endsequent's bipolar propositional atoms only. □

5. Constrained Kripke Structures

A Kripke structure (frame) K consists of a partially ordered set K of nodes, a function D assigning to each node k in K a domain D(k), and a valuation T(k) mapping every $k \in K$ to a set of atomic expressions with parameters from D(k). It is said that k forces atomic expressions from T(k). A forcing relation is defined recursively for first-order logical formulas [9].

For modeling LJ+, we consider a subset of Kripke structures defined by the following constraint: if $k \leq k'$ and Q is a propositional symbol, then either both T(k) and T(k') contain Q or they both do not contain Q. Let us call this subset constrained Kripke structures. Sequent $\Gamma \vdash \Lambda$ is called valid if for every node k in every constrained Kripke structure K, either there is such formula A from Γ that k does not force A or k forces Λ provided that Λ is not empty.

Theorem (Soundness of LJ+ with respect to constrained Kripke structures). Any sequent $\Gamma \vdash \Lambda$ derivable in LJ+ is valid.

Corollary. If some k from some constrained Kripke structure K does not force formula A, then A is not derivable in LJ+.

This theorem is proved by considering a derivation in LJ+ without cut. Since it is well known that all other rules that may appear in the derivation preserve validity [9], we only have to prove the same about REM. This property of REM follows from the observation that, for any node k of any constrained Kripke structure K and any propositional symbol N, either k forces N or k forces $\neg N$.

Theorem (Completeness of LJ+ with respect to constrained Kripke structures). If formula F is forced at any node k in any constrained Kripke structure K, then F is derivable in LJ+. If F is not derivable in LJ+, then there exists a countable constrained Kripke structure K and such node k in it that k does not force F.

This theorem is proved by induction on the number of propositional symbols in F. The base case follows from the completeness of LJ with respect to Kripke structures. The induction step reduces F to two formulas each of which contains fewer propositional symbols. We consider $F\{R|\top\}$ and $F\{R|\bot\}$ instead of F. Again, proof of existence of an invalidating countable constrained Kripke structure is leveraged on the strong completeness theorem for LJ [14].

6. Intuitionistic Features

By the weak subformula property, LJ+ coincides with LJ on the class of unipolar formulas. Given that the simplest formulas having both negative and positive occurrences of propositional symbols are LEM and double negation elimination, this class perhaps gives a reasonable syntactically-defined approximation of the boundaries within which LJ+ and LJ coincide. And the next theorem gives a slightly wider class on which LJ+ and LJ coincide.

Definition. Subformula occurrence G is called a block component of formula F if every subformula occurrence above G (i.e. having G as a proper part) is one of the following:

- $\neg A$, $\forall x A(x)$, $\exists x A(x)$, where A is not G
- positive $A\&B$, or negative $A \vee B$, or negative $A \supset B$; not in the scope of non-strictly positive quantifiers except for negative existential ones; if $A\&B/A \vee B/A \supset B$ is in the scope of strictly positive $\exists x$, then one of A, B does not contain x free
- $A\&B$, one of A, B is propositional, if $A\&B$ is negative, then every propositional symbol from both A and B is unipolar in $A\&B$
- $A \vee B$ or $A \supset B$, one of A, B is propositional, if $A \vee B/A \supset B$ is positive, then every propositional symbol from both A and B is unipolar in $A \vee B/A \supset B$
- strictly positive $A \vee B$; not in the scope of universal quantifiers; every propositional symbol from both A and B is unipolar in $A \vee B$

Theorem. If F is derivable in LJ+ and its every minimal block component, i.e. not containing other block components as its parts, is either a unipolar formula or a negative/Harrop formula not containing x free if it is in the scope of strictly positive $\exists x$, then F is also derivable in LJ.

Proof. Let $F_1, \ldots, F_n$ be all the minimal block components of F. Let us first prove this theorem for the following class of formulas: only connectives occurring above any of $F_1, \ldots, F_n$ are & and $\vee$; no quantifier occurs above any of $F_1, \ldots, F_n$; every propositional symbol occurring in both disjuncts of every disjunction above any of $F_1, \ldots, F_n$ is unipolar in the disjunction.

The proof is by induction on the minimal depth of the tips of $F_1, \ldots, F_n$ in F, i.e. minimal length of the paths from the root of the tree representing F to the roots of $F_1, \ldots, F_n$. Base case: the minimal depth is zero. In this case, F is its only block component. Whether F is unipolar or it is Harrop, it is derivable in LJ. Induction step: suppose the theorem holds for formulas in which the minimal depth of the tips of their minimal block components is not more than n, and this minimal depth equals n+1 for F.

Case 1: F is $A\&B$. If $A\&B$ is derivable in LJ+, then both A and B ought to be derivable in LJ+. By the induction hypothesis, A and B are derivable in LJ. Therefore, $A\&B$ is derivable in LJ as well.

Case 2: F is $A \vee B$. It will be proved later in section Disjunction and Existence Properties that if $A \vee B$ is derivable in LJ+ and every propositional symbol occurring in both A and B is unipolar in $A \vee B$, then either

A or B is derivable in LJ+ as well. If A is derivable, then A is derivable in LJ by the induction hypothesis. Consequently, $A \vee B$ is also derivable in LJ. The same argument applies to the case of B.

Now let us convert F to the aforementioned form. First, we replace every $A \supset B$ above any of $F_1, \ldots, F_n$ and such that one of A, B is propositional by $\neg A \vee B$. Second, we move $\neg$, $\exists$, $\forall$ located above any of $F_1, \ldots, F_n$ over disjunctions and conjunctions with one propositional argument. After that, all non-strictly positive quantifiers except for negative existential ones are moved down to $F_1, \ldots, F_n$.

Third, all negations above any of $F_1, \ldots, F_n$ are pushed down to them. Negative implications, disjunctions, and existenstial quantifiers above any of $F_1, \ldots, F_n$ are completely eliminated at the end of this negation propagation down to $F_1, \ldots, F_n$. No implications are left above any of $F_1, \ldots, F_n$ and no negations are left above conjunctions/disjunctions above any of $F_1, \ldots, F_n$ after this. Finally, all quantifiers above any of $F_1, \ldots, F_n$, which are now all strictly positive, can be moved down to them. As a result of all these transformations, $F_1, \ldots, F_n$ may become preceded by some of $\neg$, $\exists$, $\forall$. Denote them $F'_1, \ldots, F'_n$ and denote F' the result of the transformation of F.

Above these prefixed formulas $F'_1, \ldots, F'_n$, only & and $\vee$ may occur. If F_k is unipolar, then F'_k is unipolar as well. For every negative F_k, F'_k is a Harrop formula because it is preceded by a negation and cannot be prefixed by existential quantifiers. Every Harrop F_k remains such when prefixed for the same cause. F' belongs to the class of formulas defined in the beginning of this proof. The above induction proof guarantees that F' is derivable in LJ, and so is F. $\square$

If B is a subformula occurrence in formula A, then let $A_{B|C}$ denote the formula obtained from A by replacing the given occurrence of B by C. If A is a classically provable formula that is not derivable in LJ+, B is a subformula occurrence in A, C is a propositional formula not derivable in LJ+, E is a propositional formula such that $\neg E$ is not derivable in LJ+, all propositional symbols from C and E are different from those of A, then none of $A_{B|B \vee C}$, $A_{B|E \supset B}$, $A_{B|B \& E}$ holds in LJ+. This fact is proved via construction of invalidating constrained Kripke structures.

7. Disjunction and Existence Properties

Obviously, the disjunction property does not hold for the entire LJ+ because it does not hold for classical propositional logic. The existence property does not hold in LJ+ either. The following two theorems give an insight on the extent of the disjunction and existence properties. Note that in our simplistic formulation without functional symbols and even without object constants, the existence property is reduced to a statement about a variable as opposed to a term.

Theorem. If formula $F \vee G$ is derivable in LJ+, and every propositional symbol occurring in both F and G is unipolar in $F \vee G$, then either F or G is derivable in LJ+.

Proof. First, the disjunction property holds for such formulas $F \vee G$ that F and G do not have common propositional symbols. The proof is by induction on the number of propositional symbols occurring in $F \vee G$. Base case: $F \vee G$ has no propositional symbols. In this case, $F \vee G$ is derivable intuitionistically, and hence, either F or G is derivable intuitionistically.

Induction step: suppose the disjunction property holds in LJ+ for formulas having not more than n propositional symbols, and $F \vee G$ has n+1 propositional symbols. Suppose R is one of such propositional symbols. Assume for certainty that R occurs in F. If $F \vee G$ is derivable in LJ+, so are $F \vee G\{R|\top\}$ and $F \vee G\{|\bot\}$. Note that $F \vee G\{R|\top\}$ is the same as $F\{R|\top\} \vee G$. By the induction hypothesis, either $F\{R|\top\}$ or G is derivable in LJ+. If $F\{R|\top\}$ is derivable in LJ+, then by the same argument, $F\{R|\bot\}$ is also derivable in LJ+. By the theorem from section Classical Features, F is derivable in LJ+.

The proof for the case of F and G having common propositional symbols is also by induction. This time, it is on the number of propositional symbols occurring simultaneously in both F and G. Base case: F and G have no common propositional symbols. The base case was proved earlier. Induction step: suppose this theorem holds for formulas having not more than n propositional symbols occurring in both disjuncts, and $F \vee G$ has n+1 propositional symbols occurring in both F and G. Suppose R is one of such propositional symbols, all occurrences of R are positive. The case of negative occurrences is similar to this one. If $F \vee G$ is derivable in LJ+, then $F \vee G\{R|\bot\}$ is derivable as well. By the induction hypothesis, either $F\{R|\bot\}$ or $G\{R|\bot\}$ is derivable in LJ+.

For certainty, suppose $F\{R|\bot\}$ derivable in LJ+. LJ has the following well-known properties: if $A \supset B$ is derivable and A is a positive subformula occurrence in F, then $F \supset F_{A|B}$ is derivable; if $C \supset A$ is derivable and A is a negative subformula occurrence in F, then $F \supset F_{A|C}$ is derivable. Consider such chain of formulas that the first formula is $F\{R|\bot\}$, every next formula in this chain is received by replacing one occurrence of $\bot$ back by R. The occurrences of $\bot$ that have not been generated by the original replacement of R are not involved. The last formula in this chain is F. Since $\bot \supset R$, every formula in this chain except the last one implies the next formula. Hence, $F\{R|\bot\}$ is derivable in LJ+ and so is F. □

Here is a counterexample showing that the existence property does not hold in LJ+. Let A be a unary predicate symbol and N be a propositional symbol. Formula $\exists x((A(a) \vee N)\&(A(b) \vee \neg N) \supset A(x))$ is derivable in LJ+. In order to show that the existence property does not hold for this formula, we have to consider all possibilities for the variable replacing x: a; b; neither a nor b. None of the respective formulas is even classically derivable. It can be shown by building classical models in which these formulas are false.

Theorem. The existence property holds for $\exists x F(x)$ derivable in LJ+ if its every minimal block component is either a unipolar formula or a formula not containing x.

Note that this class of formulas includes the formulas from the theorem in section Intuitionistic Features. Proof of this theorem is similar to the proof of the theorem in section Intuitionistic Features. Interestingly, LJ+ has a property that can be classified as a weaker form of the existence property. If $\exists x A(x)$ is derivable in LJ+, then for some variables $t_1, \ldots, t_n$, $A(t_1) \vee \ldots \vee A(t_n)$ is derivable in LJ+ as well. Note that in a formulation including functions, word term should be used instead of word variable. This property is a direct corollary of the theorem stating that if $\exists x A(x)$ is derivable from B in LJ and if B does not contain an existential strictly positive subformula, then $A(t_1) \vee \ldots \vee A(t_n)$ is derivable from B as well (see Chapter 4 of [15]).

8. Decidability Issues

Like intuitionistic logic and classical logic, LJ+ is undecidable. Since LJ+ lies between intuitionistic and classical logics, any formula is classically derivable if and only if its double negation translation is derivable in LJ+. Therefore, were LJ+ decidable, classical logic would be decidable too. The same argument can be used to prove undecidability of other fragments of LJ+ for which double negation translations belong to the same fragment. For instance, the three-variable fragment of LJ+ is undecidable [13, 7].

According to the following theorem, decidability in LJ+ normally agrees with that in intuitionistic logic. The prenex fragment of intuitionistic logic [11, 1] is decidable, and the prenex fragment of LJ+ is decidable too. The two-variable fragment of intuitionistic logic is known to be undecidable [7], and it is undecidable in LJ+ as well.

Theorem. If a fragment of intuitionistic logic is decidable and closed under substitution of $\top$ and $\bot$, then the same fragment of LJ+ is decidable. If a fragment of intuitionistic logic is undecidable and closed under substitution of $\exists x P(x)$ or $\forall x P(x)$ with any predicate symbol P and some x, then the same fragment of LJ+ is undecidable.

Proof. Consider a decidable fragment of intuitionistic logic. Let F be a formula from this fragment and $R_1, \ldots, R_n$ be all distinct propositional symbols in it. F is derivable in LJ+ if and only if all 2^n formulas of the form $F\{R_1|P_1\}\ldots\{R_n|P_n\}$ are derivable in LJ+. Here, every P_k is either $\top$ or $\bot$, and $\{P_1 \ldots P_n\}$ range over all distinct sequences of $\top$ and $\bot$. Since formulas $F\{R_1|P_1\}\ldots\{R_n|P_n\}$ do not contain propositional symbols, they are derivable in LJ+ if and only if they are derivable in LJ. Since all of $F\{R_1|P_1\}\ldots\{R_n|P_n\}$ belong to the same fragment, this fragment is decidable in LJ+ as well.

For an undecidable fragment of intuitionistic logic, we will prove that its subfragment comprised of formulas without propositional symbols is undecidable in intuitionistic logic. Let F be a formula containing propositional symbol R, P be a one-place predicate symbol not occurring in F. We claim that F is intuitionistically derivable if and only if $F\{R|\forall x P(x)\}$ is derivable for a given x. The 'only if' part follows from closeness under substitution. Consider a derivation of $F\{R|\forall x P(x)\}$ without cut. In this derivation, $P(...)$ can only be a subformula of itself or $\forall x P(x)$. Let us replace P(...) and $\forall x P(x)$ everywhere in this derivation by R and remove the $\forall$ rules applied to $P(...)$ altogether. The derivation remains correct because $P(...)$ cannot be a side formula of any other logical rule than the two $\forall$ rules. Therefore, F is intuitionistically derivable as well. The case of $\exists x P(x)$ is similar.

By induction on the number of propositional symbols in F, F is intuitionistically derivable if and only if the formula obtained from F by replacing all its propositional symbols by $\forall x P(x)$ with distinct P for distinct propositional symbols is derivable. Therefore, decidability of the subfragment under consideration would imply decidability of the original fragment. Since LJ+ coincides with intuitionistic logic on the class of formulas without propositional symbols, this subfragment is undecidable in LJ+, and so is the original fragment. □

9. Translations

From the proof of the weak subformula property, formula F is derivable in LJ+ if and only if $(A_1 \vee \neg A_1)\&\ldots\&(A_k \vee \neg A_k) \supset F$ is derivable in LJ, where $\{A_1, \ldots, A_k\}$ is the set of bipolar propositional atoms from F. This relationship can be viewed as a translation of LJ+ into intuitionistic logic. Yet another translation of LJ+ into intuitionistic logic was actually outlined in the theorem in section Decidability Issues. Formula F is derivable in LJ+ if and only if $\underset{\{P_1,\ldots,P_n\}}{\&} F\{A_1|P_1\}\ldots F\{A_n|P_n\}$ is derivable in LJ. This conjunction is on 2^n vectors $\{P_1, \ldots, P_n\}$ of $\top$ or $\bot$.

Double negation translation works for classical logic and LJ+. It follows from the fact that LJ+ is a superset of intuitionistic logic and a subset of classical logic. Similarly, formula F is classically provable if and only if $\neg\neg F$ is provable in LJ+ extended with the double negation shift. In application to classical logic and LJ+, double negation translation can be made less 'intrusive'. First, propositional formulas do not have to be negated. Second, disjunctions having at least one propositional component do not have to be translated either. The usual proof connecting classical derivability and intuitionistic derivability for double negation translation is by induction on derivations in classical logic. This proof remains unchanged for the less intrusive translation but the fact that the two aforementioned classes of formulas are double negation closed in LJ+ is used (see section Classical Features).

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