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$\mathbf{n \times m}$–VALUED ŁUKASIEWICZ ALGEBRAS WITH NEGATION

A b s t r a c t. Matrix Łukasiewicz algebras were introduced by W. Suchoń in 1975 (*Matrix Łukasiewicz Algebras*, Reports on Mathematical Logic 4 (1975), 91-104). In this paper $n \times m$–valued Łukasiewicz algebras with negation (or $NS_{n\times m}$–algebras) are defined and investigated. These algebras constitute an extension of those given by W. Suchoń and in $m = 2$ case they coincide with n–valued Łukasiewicz algebras. Firstly, some of the main results established for matrix Łukasiewicz algebras are extended to $NS_{n\times m}$–algebras. In particular, a functional representation theorem is given. Next, $NS_{n\times m}$–congruences are determined by taking into account an implication operation which is defined on these algebras. In addition, it is proved that the class of $NS_{n\times m}$–algebras is a variety. Besides, subdirectly irreducible algebras are characterized. As a consequence, it is shown that this variety is semisimple and locally finite. Finally, the algebra which generates the variety of $NS_{n\times m}$–algebras is obtained and an equational base for the latter is determined.

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1. Introduction

In 1940, G. Moisil introduced 3–valued and 4–valued Łukasiewicz algebras with the purpose to obtain the algebraic counterpart of the corresponding Łukasiewicz logics. A year later, he generalized these algebras by defining n–valued Łukasiewicz algebras ([18]) and he studied them from the algebraic point of view. It is well-known that these algebras are not the algebraic counterpart of n–valued Łukasiewicz propositional calculi for $n \geq 5$ (see [4, 7]). This problem was solved by R. Cignoli ([8, 9]) by adding to the basic operations of n–valued Łukasiewicz algebras certain binary operations, and the systems obtained in this way were called proper n–valued Łukasiewicz algebras. For a general account of the origins of Łukasiewicz many valued logics and Łukasiewicz algebras the reader is referred to [4, 10, 11].

On the other hand, in 1975 W. Suchoń ([24]) defined matrix Łukasiewicz algebras in order to generalize n–valued Łukasiewicz algebras without negation. The only paper we know about these algebras is the one mentioned above and a brief reference to them can be found in [4, page 121]. In the present paper, we introduce $n \times m$–valued Łukasiewicz algebras with negation (or $NS_{n\times m}$–algebras). These algebras constitute an extension of those given by Suchoń and in $m = 2$ case they coincide with n–valued Łukasiewicz algebras.

In the example that we shall develop next, we will find the required motivation in order to legitimate the study of this new class of algebras. To this end, according to that quoted in [14], let us recall that Belnap's 4-valued logic ([2], [3]) is a logical system well-known for its many applications, in particular in the study of deductive data-bases and distributed logic programs, handling information that may contain conflicts or gaps. Belnap's idea is simple: Faced with a situation (for examples, see the quoted papers) where one has several conflicting pieces of information on the truth of a sentence, or where one has no information about it, the classical truth-values (true and false) must be treated as being mutually independent, thus giving birth to four non-classical epistemic values: $1 :=$ true and not false; $0 :=$ false and not true (these values are to some extent identifiable with the classical ones); $n :=$ neither true nor false, the well-known "undetermined" value of some $3-$valued logics and $b :=$ both true and false, also called "overdetermined", the value corresponding to the situation where several (probably independent) sources assign a different classical value to a sentence. These values can be ordered by means of the lattice illustrated in Figure 1.

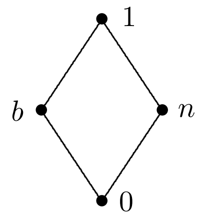

Figure 1

Besides, on this lattice which we shall denote by T_4, Belnap considered a negation operation $\neg$ defined as: $\neg 0 = 1$, $\neg n = n$, $\neg b = b$ and $\neg 1 = 0$.

By taking into account the system described above, we have considered the following one which extends it: Faced with a situation like the one analysed by Belnap, we shall distinguish the classical truth values of a sentence, from the information about it, which can be *positive* or *negative*. Then, we shall consider the classical values $1 :=$ true; $0 :=$ false and the epistemic ones, similar to those considered by Belnap, $a :=$ all the information is negative and none is positive; $b :=$ some information is positive and some is negative; $c :=$ there is neither positive nor negative information and $d :=$ all information is positive and none is negative. All these values can be ordered from false to true by means of the lattice $S_{3\times 3}$ illustrated in Figure 2.

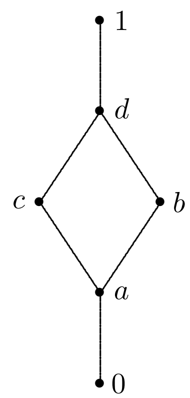

Figure 2

We shall also define on $S_{3\times 3}$ the De Morgan negation $\sim$ indicated in Table 1.

On the other hand, in 1978 A. Monteiro extended Belnap's base algebra $(T_4, \neg, 1)$, by adding the modal operator $\Box$ defined by $\Box 1 = 1$ and $\Box x = 0$, for all $x \neq 1$. The algebra thus obtained is the one that generates the tetravalent modal algebras, which were studied by I. Loureiro in [16, 17] (see also [12, 13]). Later, in [14], for a given sentence ϕ the operator $\Box$ was interpreted as

$\Box\phi :=$ *the available information confirms that* ϕ *is true.*

In a similar way to the above described on Belnap's base algebra, we extend the algebra $(S_{3\times 3}, \sim, 1)$ by defining certain possibility operators σ_{ij}, $1 \leq i, j \leq 2$. Then for every value in $S_{3\times 3}$ we have the possibility of adopting different decision criteria, depending on the available information of a sentence. So, for every pair i, j, $1 \leq i, j \leq 2$, the operator σ_{ij} is the one defined in Table 1.

x	$\sim x$	$\sigma_{11}x$	$\sigma_{12}x$	$\sigma_{21}x$	$\sigma_{22}x$
0	1	0	0	0	0
a	d	0	0	0	1
b	b	0	1	0	1
c	c	0	0	1	1
d	a	0	1	1	1
1	0	1	1	1	1

Table 1

In order to give an interpretation to each of these operators, let us consider, for example, a manager who has to make a decision based on the information given by his advisors. Hence, this manager could be considered according to the decision σ_{ij} he makes as: *conservative and distrustful* (σ_{11}); *conservative but risky* (σ_{12}); *risky* (σ_{21}) or *excessively risky* (σ_{22}).

Then, for each sentence ϕ, the operators σ_{ij}, $1 \leq i, j \leq 2$ can be interpreted as

$\sigma_{11}\phi :=$ *the available information confirms that* ϕ *is true,*

$\sigma_{12}\phi :=$ *the available information allows to consider* ϕ *as true,*

$\sigma_{21}\phi$:= *the available information does not allow to consider* ϕ *as false,*

$\sigma_{22}\phi$:= *the available information does not confirm* ϕ *as false.*

Thus, in this context the sentence

$\sigma_{11}\phi$ is true only when ϕ is true, while it is false in all other cases,

$\sigma_{12}\phi$ is considered true when ϕ is true or when there is some positive information about ϕ (disregarding if at the same time there is some negative information about ϕ), while it is considered false in all other cases,

$\sigma_{21}\phi$ is considered true when ϕ is true or when there is no negative information about ϕ (disregarding if at the same time there is no positive information about ϕ), while it is considered false in all other cases,

$\sigma_{22}\phi$ is considered true in all cases except the one in which ϕ is false.

Therefore, we obtain the characteristic matrix $(S_{3\times3}, \sim, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, 1)$ of a logic which we shall call 3×3-valued Suchoń logic. Next, we shall generalize the above situation by using as starting point, the main results obtained by W. Suchoń in [24].

The paper is organized as follows. In section 1 we briefly summarize the main definitions and results needed throughout this article. In section 2 we introduce $n \times m$–valued Łukasiewicz algebras with negation and we show their most important properties which are necessary for further development. We also extend some of the results established in [24] to these algebras. In particular, we give a functional representation theorem and we determine a necessary and sufficient condition under which such embedded is onto. In section 3 we define an implication operation on these algebras, which allows us to determine the congruence lattice. By taking into account this result, we prove that the class of $NS_{n\times m}$–algebras is a variety. In section 4 we characterize the subdirectly irreducible algebras. As a consequence, we show that this variety is semisimple and locally finite. Besides, we establish the relationship between the possibility operations and a special family of prime filters of a subdirectly irreducible $NS_{n\times m}$–algebra.

Furthermore, we obtain the system which determines each subdirectly irreducible $NS_{n\times m}$–algebra. Finally, we obtain the algebra which generates the variety of $NS_{n\times m}$–algebras and we determine an equational base for the latter.

2. Preliminaries

We refer the reader to the bibliography listed here as [1, 5, 15, 22, 21] for specific details of the many basic notions and results of universal algebra including distributive lattices, De Morgan algebras, Kleene algebras and n–valued Łukasiewicz algebras without negation considered in this paper.

In 1969, R. Cignoli ([7]) defined n–valued Łukasiewicz algebras in an equivalent way to that given by G. Moisil ([18, 19, 20]) as indicated below.

An n–valued Łukasiewicz algebra, in which n is an integer, $n \geq 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{s_i\}_{i\in\{1,\ldots,n-1\}}, 0, 1\rangle$ where the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{s_i\}_{i\in\{1,\ldots,n-1\}}$ is a family of unary operations on L which fulfills the following conditions:

(L1) $s_i(x \vee y) = s_i x \vee s_i y$,

(L2) $s_i x \vee \sim s_i x = 1$,

(L3) $s_i(s_j x) = s_j x$,

(L4) $s_i(\sim x) = \sim s_{n-i} x$,

(L5) $s_1 x \leq s_2 x \leq \ldots \leq s_{n-1} x$,

(L6) $s_i x = s_i y$ for all $i \in \{1, \cdots, n-1\}$ imply $x = y$.

In what follows, we shall denote this algebra by its underlying set or by $(L, \sim, \{s_i\}_{i\in\{1,\ldots,n-1\}})$. For further information on these algebras the reader is referred to [4, 6, 7, 18, 19, 20, 21].

On the other hand, in 1975, W. Suchoń ([24]) introduced $n \times m$–valued Łukasiewicz algebras, in which n and m are integers, $n \geq 2$, $m \geq 2$, as algebras $\langle L, \wedge, \vee, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of lattice endomorphisms on L pair-wise different which fulfills these conditions:

(S1) $\sigma_{ij} : L \to B(L)$ where $B(L)$ is the set of Boolean elements of L,

(S2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,

(S3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,

(S4) $\sigma_{ij}(\sigma_{rs}x) = \sigma_{rs}x$,

(S5) $\sigma_{ij}0 = 0$, $\sigma_{ij}1 = 1$,

(S6) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i,j) \in (n \times m)$ imply $x = y$.

It is worth mentioning that there are examples of these algebras which contain sets closed under all the operations of the algebra but the operations σ_{ij} restricted to those sets are not pair-wise different.

The notions and results announced here for $n \times m$–valued Łukasiewicz algebras will be used throughout the paper.

(S7) Representation Theorem: Let $B(L) \uparrow^{(n\times m)} = \{f : (n \times m) \longrightarrow B(L)$ such that for arbitraries i, j if $r \leq s$, then $f(r,j) \leq f(s,j)$ and $f(i,r) \leq f(i,s)\}$. Then $B(L) \uparrow^{(n\times m)}$ is an $n \times m$–valued Łukasiewicz algebra where for each $f \in B(L) \uparrow^{(n\times m)}$ and $(i,j) \in (n \times m)$ the operation σ_{ij} is defined by the prescription $(\sigma_{ij}f)(r,s) = f(i,j)$ for all $(r,s) \in (n \times m)$ and the remaining operations are defined componentwise. Besides, L can be embedded into $B(L) \uparrow^{(n\times m)}$ ([24, Theorem 1]).

(S8) L is centred if for each $(i,j) \in (n \times m)$ there exists $c_{ij} \in L$ such that

$$\sigma_{rs}c_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s \\ 1 & \text{if } i \leq r \text{ and } j \leq s \end{cases} \quad \text{([24, Definition 4]).}$$

The element c_{ij} is called the (i,j)–centre of L. It follows from (S6) that the (i,j)–centre is unique.

(S9) An element x is

(i) vertically increasing if for every $i \in \{1, \ldots, n-2\}$, $\sigma_{i(m-1)}x \leq \sigma_{(i+1)1}x$,

(ii) horizontally increasing if for every $j \in \{1, \ldots, m-2\}$, $\sigma_{(n-1)j}x \leq \sigma_{1(j+1)}x$,

(iii) increasing if it is vertically and horizontally increasing ([24, Definition 2]).

Each of the sets of all vertically increasing, horizontally increasing and increasing elements of L are denoted by $P_i(L)$, $P_o(L)$ and $C(L)$, respectively.

Although in [24] the congruence lattice of these algebras was not described, in order to determine a necessary and sufficient condition under which an $n \times m$–valued Łukasiewicz algebra is isomorphic to the cartesian product of an n–valued and an m–valued Łukasiewicz algebra both without negation, a useful congruence was introduced as indicated in (S10).

(S10) For each $z \in B(L)$, let S_z be the relation defined on L by: $x\ S_z\ y$ if and only if $x \wedge z = y \wedge z$. Then S_z is a congruence on L ([24, p. 94]).

3. n × m–valued Łukasiewicz algebras with negation

The class of algebras which is of our concern now, rises from $n \times m$–valued Łukasiewicz algebras without the restriction that the endomorphisms be pair-wise different and endowed with a De Morgan negation.

Definition 3.1. An $n \times m$–valued Łukasiewicz algebra with negation (or $NS_{n\times m}$–algebra), in which n and m are integers, $n \geq 2$, $m \geq 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfills these conditions:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(C2) $\sigma_{ij}x \leq \sigma_{(i+1)j}x$,

(C3) $\sigma_{ij}x \leq \sigma_{i(j+1)}x$,

(C4) $\sigma_{ij}(\sigma_{rs}x) = \sigma_{rs}x$,

(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i,j) \in (n \times m)$ imply $x = y$,

(C6) $\sigma_{ij}x \vee \sim \sigma_{ij}x = 1$, (C7) $\sigma_{ij}(\sim x) = \sim \sigma_{(n-i)(m-j)}x$.

In what follows we shall indicate with $\boldsymbol{NS_{n\times m}}$ the class of $NS_{n\times m}$–algebras and we shall denote them by L or $(L, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)})$.

In Lemma 3.1 we summarize the most important properties of these algebras necessary in what follows.

Lemma 3.1. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then the following properties are satisfied:*

(C8) $\sigma_{ij}(x \wedge y) = \sigma_{ij}x \wedge \sigma_{ij}y$,

(C9) $\sigma_{ij}x \wedge \sim \sigma_{ij}x = 0$,

(C10) $x \leq y$ *if and only if* $\sigma_{ij}x \leq \sigma_{ij}y$ *for all* $(i,j) \in (n \times m)$,

(C11) $x \leq \sigma_{(n-1)(m-1)}x$,

(C12) $\sigma_{ij}0 = 0$, $\sigma_{ij}1 = 1$,

(C13) $\sigma_{11}x \leq x$,

(C14) $\sim x \vee \sigma_{(n-1)(m-1)}x = 1$,

(C15) $x \vee \sim \sigma_{11}x = 1$.

Proof. It is routine. □

Remark 3.1. (i) From (C6) and (C9) we deduce that for all $(i,j) \in (n \times m)$, the Boolean complement of $\sigma_{ij}x$, which we shall denote by $(\sigma_{ij}x)'$, coincides with its De Morgan negation. Therefore, $\sigma_{ij}x \in B(L)$ for all $x \in L$.

(ii) If the operations σ_{ij} of an $NS_{n\times m}$–algebra L are pair-wise different for $(i,j) \in (n \times m)$, then from (i) and (C12) we have that the reduct $\langle L, \wedge, \vee, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ is an $n \times m$–valued Łukasiewicz algebra.

(iii) By identifying the set $\{1\} \times \{1, \ldots, m-1\}$ with $\{1, \ldots, m-1\}$ we infer that every $NS_{2\times m}$–algebra is isomorphic to an m–valued Łukasiewicz algebra.

(iv) Unlike what happens in n–valued Łukasiewicz algebras, generally the De Morgan reducts of $NS_{n\times m}$–algebras are not Kleene algebras. To this end, let us consider the $NS_{3\times 3}$–algebra $S_{3\times 3}$ described in Figure 2 where the operations are defined in Table 1. Then $b \wedge \sim b \not\leq c \vee \sim c$.

On the other hand, bearing in mind the definition of i–invariant element given in [6, Definition 2.2] for n–valued Łukasiewicz algebras we introduce the following notion.

Definition 3.2. An element x of an $NS_{n\times m}$–algebra L is (i,j)-invariant if $\sigma_{ij}x = x$.

We shall denote by $\sigma_{ij}(L)$ the set of all (i,j)-invariant elements of L. These elements play an important role in the study of these algebras since in particular, they coincide with the Boolean elements as we shall prove in Proposition 3.1.

Proposition 3.1. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then $\sigma_{ij}(L) = B(L)$ for all $(i,j) \in (n \times m)$.*

Proof. By (i) from Remark 3.1 we have that $\sigma_{11}(L) \subseteq B(L)$. Conversely, if $x \in B(L)$, from (C13) it results that $\sigma_{11}x' \leq x'$; and since $(\sigma_{11}x)' = \sigma_{11}x'$ we have that $\sigma_{11}x = x$. Thus, (1) $B(L) = \sigma_{11}L$. On the other hand, (2) $\sigma_{ij}(L) = \sigma_{rs}(L)$ for all $(i,j),(r,s) \in (n \times m)$. Indeed, let $x \in \sigma_{ij}(L)$. Then $\sigma_{ij}x = x$ and by (C4) we deduce that $x \in \sigma_{rs}(L)$. Hence, $\sigma_{ij}(L) \subseteq \sigma_{rs}(L)$. The reverse inclusion is similar. From (1) and (2) we conclude the proof. □

From now until the end of this section, our attention is focused on extending some of the main results given in [24] for matrix Łukasiewicz algebras to $NS_{n\times m}$–algebras.

Proposition 3.2. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then*

$$\langle B(L)\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, O, I\rangle$$

is an $NS_{n\times m}$–algebra where for all $f \in B(L)\uparrow^{(n\times m)}$ and $(i,j) \in (n \times m)$ the operation $\sim$ is defined by the prescription $(\sim f)(i,j) = (f(n-i, m-j))'$ and the remaining operations are those defined in (S7)*.*

Proof. The statement follows from (S7) and the definition of $\sim$. □

Theorem 3.1. *Every $NS_{n\times m}$–algebra L can be embedded into the algebra $B(L)\uparrow^{(n\times m)}$.*

Proof. Taking into account [24, Theorem 1], the application $\tau : L \to B(L) \uparrow^{(n \times m)}$ defined by the prescription $\tau(a)(i,j) = \sigma_{ij}a$ for each $a \in L$ and $(i,j) \in (n \times m)$ is a one-to-one homomorphism of bounded lattices which commutes with σ_{ij} for all $(i,j) \in (n \times m)$. Besides, from (C7) we have that $(\sim \tau(a))(i,j) = (\tau(a)(n-i, m-j))' = \sim \sigma_{(n-i)(m-j)}a = \sigma_{ij}(\sim a) = (\tau(\sim a))(i,j)$ and so, $\sim \tau(a) = \tau(\sim a)$. □

The notion of centred $NS_{n\times m}$–algebras is analogous to that defined in (S8) for matrix Łukasiewicz algebras.

Corollary 3.1. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then the following conditions are equivalent:*

(i) *L is centred,*

(ii) *L is isomorphic to $B(L) \uparrow^{(n\times m)}$.*

Proof. It is a direct consequence of [24, Theorem 8] and Theorem 3.1. □

Next, we shall prove that each of the sets $P_i(L)$, $P_o(L)$ and $C(L)$ defined in (S9) can be embedded into a k–valued Łukasiewicz algebra for some particular integer k. More precisely,

Proposition 3.3. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then*

(i) *$P_i(L)$, $P_o(L)$ and $C(L)$ are subalgebras of L,*

(ii) *$P_i(L)$ and $P_o(L)$ can be embedded into a $((n-1)(m-1)+1)$–valued Łukasiewicz algebra.*

Proof. We shall only prove (ii). From Proposition 3.2 we have that $B(L) \uparrow^{(2\times((n-1)(m-1)+1))}$ is an $NS_{2\times((n-1)(m-1)+1)}$–algebra and by (iii) from Remark 3.1, $B(L) \uparrow^{\{1,\ldots,(n-1)(m-1)\}}$ is an $((n-1)(m-1)+1)$–valued Łukasiewicz algebra where in this case (1) $(\sim f)(j) = (f(((n-1)(m-1)+1)-j))'$. On the other hand, for each $k \in \{1,\ldots,(n-1)(m-1)\}$ there is an only pair i, j with $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$ such that $k = (i-1)(m-1)+j$. If $k = q(m-1)$ with $1 \leq q \leq n-1$, then $i = q$ and $j = m-1$.

Let $\alpha : P_i(L) \to B(L) \uparrow^{\{1,\ldots,(n-1)(m-1)\}}$ be the application defined in [24, Theorem 7] by the prescription $\alpha(x) = f$ if and only if $\sigma_{ij}x = f((i-$

$1)(m-1)+j)$ for all $(i,j) \in (n \times m)$. Then (2) $\alpha(\sim x) = \sim \alpha(x)$. Indeed, from (1), (C7) and (i) from Remark 3.1 it results that $(\sim \alpha(x))((i-1)(m-1)+j) = (\alpha(x)(((n-1)(m-1)+1)-((i-1)(m-1)+j)))' = (\alpha(x)((n-i)(m-1)+(-j+1)))' = (\alpha(x)(((n-i)-1)(m-1)+(m-j)))' = (\sigma_{(n-i)(m-j)}x)' = \sigma_{ij}(\sim x) = (\alpha(\sim x))((i-1)(m-1)+j)$. Hence, (2) holds true. From this last assertion and [24, Theorem 7] we infer that α is a one-to-one $NS_{n\times m}$–homomorphism. Similarly, $P_o(L)$ can be embedded into $B(L)\uparrow^{\{1,\ldots,(n-1)(m-1)\}}$. □

4. Congruences on $\mathbf{NS_{n\times m}}$–algebras

Our next task is to describe the congruence lattices of $NS_{n\times m}$–algebras. In order to do so, we define an implication operation on them as follows:

$$x \to y = \sigma_{(n-1)(m-1)}(\sim x) \vee y.$$

Lemma 4.1. *The implication $\to$ satisfies the following properties:*

(W1) $x \to (y \to x) = 1$,

(W2) $(x \to y) \to (x \to z) = x \to (y \to z)$,

(W3) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(W4) $x \to (y \to z) = (x \wedge y) \to z$,

(W5) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(W6) $(x \to y) \to x = x$,

(W7) $1 \to x = x$,

(W8) $x \leq y$ *implies* $x \to y = 1$,

(W9) $x \leq y$ *if and only if* $\sigma_{ij}x \to \sigma_{ij}y = 1$ *for all* $(i,j) \in (n \times m)$,

(W10) $x \to \sigma_{11}x = 1$,

(W11) $(x \to y) \to y = \sigma_{11}x \vee y$,

(W12) $x \to (y \to z) = y \to (x \to z)$.

Proof. It is routine. □

Definition 4.1. A subset D of an $NS_{n\times m}$–algebra L is a deductive system (d.s.) of L, if it satisfies that $1 \in D$ and the hypothesis $x \in D$ and $x \to y \in D$ imply $y \in D$.

We shall denote by $\mathcal{D}(L)$ the set of all d.s. of L.

On the other hand, let us recall that a filter F of a bounded distributive lattice L is a Stone filter (or s–filter), if for all $x \in F$ there is $y \in F \cap B(L)$ such that $y \leq x$. The set of all s–filters of L will be denoted by $\mathcal{F}_s(L)$.

It is well-known that there exists an isomorphism between $\mathcal{F}_s(L)$ and the set $\mathcal{F}(B(L))$ of all filters of $B(L)$, both ordered by set inclusion, defining the applications

(SF) $F \longmapsto F^* = F \cap B(L)$ and $T \longmapsto F_T = \{x \in L : b \leq x,$ for some $b \in T\}$.

In $NS_{n\times m}$–algebras the s–filters are characterized as follows.

Proposition 4.1. *Let F be a filter of an $NS_{n\times m}$–algebra L. Then the following conditions are equivalent:*

(i) *F is an s–filter of L,*

(ii) *F satisfies: $x \in F$ implies $\sigma_{11}x \in F$.*

Proof. It is a direct consequence of (C10), (C13) and Proposition 3.1. □

In Proposition 4.2 the relationship between the notions of s–filter and deductive system in these algebras is determined.

Proposition 4.2. *Let $L \in \boldsymbol{NS_{n\times m}}$ and $D \subseteq L$. Then the following conditions are equivalent:*

(i) *D is a d.s. of L,*

(ii) *D is an s–filter of L.*

Proof. (i) $\Rightarrow$ (ii): Let x, $y \in D$. Then from (W1), (W3) and (W8) we have that $x \to (x \wedge y) \in D$. Thus, from the hypothesis we infer that $x \wedge y \in D$. On the other hand, if $x, y \in L$ are such that $x \in D$ and $x \leq y$ then by (W8), $y \in D$. Besides, if $x \in D$ by (W10) we conclude that $\sigma_{11}x \in D$.

(ii) $\Rightarrow$ (i): Let x, $x \to y \in D$. Then from Proposition 4.1 we have that $\sigma_{11}x$, $\sigma_{11}(x \to y) \in D$. Hence, by (C1), (C4) and (C7) we deduce that $\sigma_{11}x \wedge \sigma_{11}y = \sigma_{11}x \wedge \sigma_{11}(x \to y) \in D$. So, from (ii) and (C13) we conclude that $y \in D$. $\Box$

Corollary 4.1. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then there is an isomorphism between $\mathcal{D}(L)$ and $\mathcal{F}(B(L))$, both ordered by set inclusion.*

Proof. It is a direct consequence of Proposition 4.2, taking into account the applications defined in (SF). $\Box$

From now on, we shall denote by $Con(L)$ the congruence lattice of L and by L/R the quotient algebra of L by R for all $R \in Con(L)$. Besides, for $x \in L$ the equivalence class of x modulo R shall be denoted by $[x]_R$.

Proposition 4.3. *Let $L \in \boldsymbol{NS_{n\times m}}$, $D \in \mathcal{D}(L)$ and $R(D) = \{(x,y) \in L\times L :$ there exists $d \in D$ such that $x\wedge d = y\wedge d\}$. Then $R(D) \in Con(L)$ and $[1]_{R(D)} = D$.*

Proof. We shall only prove that $R(D)$ is compatible with $\sim$ and σ_{ij}. Let $(x,y) \in R(D)$. Then there is $d \in D$ such that (1) $x \wedge d = y \wedge d$. Thus, (2) $\sigma_{11}d \in D$ and $(\sim x\vee \sim d) \wedge \sigma_{11}d = (\sim y\vee \sim d) \wedge \sigma_{11}d$. From this last assertion and (C15), we get that $\sim x \wedge \sigma_{11}d = \sim y \wedge \sigma_{11}d$. Hence, by (2) we obtain that $(\sim x, \sim y) \in R(D)$. On the other hand, from (1) and (C8) we have that (3) $\sigma_{ij}x \wedge \sigma_{ij}d = \sigma_{ij}y \wedge \sigma_{ij}d$ for all $(i,j) \in (n \times m)$. Besides, from (2), (C2) and (C3) we deduce that $\sigma_{ij}d \in D$ for all $(i,j) \in (n \times m)$. Therefore, from (3) we conclude that $(\sigma_{ij}x, \sigma_{ij}y) \in R(D)$ for all $(i,j) \in (n \times m)$. $\Box$

Proposition 4.4. *Let $L \in \boldsymbol{NS_{n\times m}}$ and $\theta \in Con(L)$. Then $[1]_\theta \in \mathcal{D}(L)$ and $R([1]_\theta) = \theta$.*

Proof. Suppose that (1) x, $x \to y \in [1]_\theta$. Then $(x, x \to y) \in \theta$ and from the hypothesis we have that $(y \to x, y \to (x \to y)) \in \theta$. Thus, by (W1) and (1) it follows that $(x \to y, y \to x) \in \theta$. Therefore, $((x \to y) \to y, (y \to x) \to y) \in \theta$. From this last statement, (W6) and (W11) we deduce that (2) $(\sigma_{11}x \vee y, y) \in \theta$. Since by (1), $(\sigma_{11}x \vee y, 1) \in \theta$ we conclude from (2) that $y \in [1]_\theta$.

On the other hand, let $(x,y) \in \theta$. Then $(\sigma_{ij}x, \sigma_{ij}y) \in \theta$ for all $(i,j) \in (n \times m)$ and so, we get that $d_{ij} = (\sim \sigma_{ij}x \vee \sigma_{ij}y) \wedge (\sim \sigma_{ij}y \vee \sigma_{ij}x) \in [1]_\theta$

for all $(i,j) \in (n \times m)$. Therefore, from Proposition 4.2 we have that (3) $d = \bigwedge\limits_{(i,j)\in(n\times m)} d_{ij} \in [1]_\theta$. Since $\sigma_{rs} d = d$ for all $(r,s) \in (n \times m)$, by (C8) we infer that $\sigma_{rs}(x \wedge d) = \sigma_{rs}(y \wedge d)$ for all $(r,s) \in (n \times m)$. From this last assertion, (C5) and (3), we conclude that $(x,y) \in R([1]_\theta)$. Hence, $\theta \subseteq R([1]_\theta)$. The other inclusion is straightforward. □

Theorem 4.1 follows as an immediate consequence of Propositions 4.3 and 4.4.

Theorem 4.1. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then the following statements hold:*

(i) $Con(L) = \{R(D) : \ D \in \mathcal{D}(L)\}$, *where* $R(D) = \{(x,y) \in L \times L :$ *there exists* $d \in D$ *such that* $x \wedge d = y \wedge d\}$,

(ii) *the lattices $Con(L)$ and $\mathcal{D}(L)$ are isomorphic considering the applications $\theta \longmapsto [1]_\theta$ and $D \longmapsto R(D)$, which are inverse to one another.*

Proposition 4.5. *Let $L \in \boldsymbol{NS_{n\times m}}$ and $R \in Con(L)$. Then L/R is an $NS_{n\times m}$–algebra where the operations are defined componentwise.*

Proof. We shall only prove (C5). Let $[x]_R, [y]_R \in L/R$ be such that $\sigma_{ij}[x]_R = \sigma_{ij}[y]_R$ for all $(i,j) \in (n \times m)$. Then by Theorem 4.1 for each $(i,j) \in (n \times m)$ there exists $k_{ij} \in [1]_R$ such that $\sigma_{ij} x \wedge k_{ij} = \sigma_{ij} y \wedge k_{ij}$. Therefore, $k = \bigwedge\limits_{(i,j)\in(n\times m)} \sigma_{11} k_{ij} \in [1]_R$ and from (C13) we have that $\sigma_{ij} x \wedge \sigma_{11} k_{ij} = \sigma_{ij} y \wedge \sigma_{11} k_{ij}$ for all $(i,j) \in (n \times m)$. Then from (C8) and (C4) we deduce that $\sigma_{rs}(x \wedge k) = \sigma_{rs}(y \wedge k)$ for all $(r,s) \in (n \times m)$. From this last assertion and (C5) we infer that $x \wedge k = y \wedge k$ and so, $[x]_R = [y]_R$. □

The above proposition and well-known results of universal algebra allow us to conclude Theorem 4.2.

Theorem 4.2. *$\boldsymbol{NS_{n\times m}}$ is a variety.*

Furtheron, in section 5 we shall give an equational base for $\boldsymbol{NS_{n\times m}}$.

On the other hand, Remark 4.1 and Lemma 4.2 are fundamental in order to prove Theorem 4.3 which extends [24, Theorem 4] to $NS_{n\times m}$–algebras.

Remark 4.1. Let $(L_1, \sim_1, \{s_i^1\}_{i\in\{1,\ldots,n-1\}})$ and $(L_2, \sim_2, \{s_j^2\}_{j\in\{1,\ldots,m-1\}})$ be an n–valued and an m–valued Łukasiewicz algebra respectively. It is straightforward to prove that $(L_1 \times L_2, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)})$ is an $NS_{n\times m}$–algebra where for each $(i,j) \in (n \times m)$ the operation σ_{ij} is defined by the prescription $\sigma_{ij}(x,y) = (s_i^1 x, s_j^2 y)$ and the remaining operations are defined componentwise.

From now on, if X is a non-empty subset of L we shall denote by $[X)$ the filter of L generated by X. If $X = \{a\}$ we shall write $[a)$ instead of $[\{a\})$.

Lemma 4.2. *Let $L \in \boldsymbol{NS_{n\times m}}$ and $z \in B(L)$. Then the following hold:*

(i) *$[z)$ is an s–filter of L,*

(ii) *$S_z = R([z))$ where S_z is the relation defined in* (S10).

Proof. It is routine. □

Theorem 4.3. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then the following conditions are equivalent:*

(i) *L is isomorphic to the cartesian product of an n–valued and an m–valued Łukasiewicz algebra,*

(ii) *there exists $z \in B(L)$ such that for every $x \in L$ and every pair i, j, $1 \leq i \leq n-1$, $1 \leq j \leq m-1$ the following conditions are satisfied:*

   (a) $[\sigma_{i1}x]_{S_z} = \ldots = [\sigma_{i(m-1)}x]_{S_z}$,

   (b) $[\sigma_{1j}x]_{S_{z'}} = \ldots = [\sigma_{(n-1)j}x]_{S_{z'}}$.

Proof. (i)⇒ (ii): It is a direct consequence of [24, Theorem 4].

(ii)⇒ (i): Let $L_1 = L/S_z$ and $L_2 = L/S_{z'}$. By defining on L_1 the operations s_i by the prescriptions $s_i([x]_{S_z}) = [\sigma_{i1}x]_{S_z}$ for all i, $1 \leq i \leq n-1$ and the remaining operations componentwise, we have that L_1 is a De Morgan algebra and that (L2) holds true. Besides, (L4) is satisfied. Indeed, from (C7) and (a) it follows that $s_i(\sim [x]_{S_z}) = [\sigma_{i1}(\sim x)]_{S_z} = [\sim \sigma_{(n-i)(m-1)}x]_{S_z} =\sim [\sigma_{(n-i)(m-1)}x]_{S_z} =\sim [\sigma_{(n-i)1}x]_{S_z} =\sim s_{n-i}([x]_{S_z})$. Then, bearing in mind [24, Theorem 4] we conclude that L_1 is an n–valued Łukasiewicz algebra. Similarly we proved that L_2 is an m–valued Łukasiewicz algebra where the operations s_j on L_2 are defined by $s_j([x]_{S_{z'}}) = [\sigma_{1j}x]_{S_{z'}}$ for all j, $1 \leq j \leq m-1$ and the remaining operations are defined componentwise. So, in view of Remark 4.1 we have that $L_1 \times L_2$ is an $NS_{n\times m}$–algebra.

Let $w : L_1 \times L_2 \to L$ be the application defined in [24, Theorem 4] by the prescription $w([x]_{S_z}, [y]_{S_{z'}}) = v$ where v is the unique element of $[x]_{S_z} \cap [y]_{S_{z'}}$. Then (1) $w(\sim ([x]_{S_z}, [y]_{S_{z'}})) = \sim w([x]_{S_z}, [y]_{S_{z'}})$. Indeed, let (2) $w([x]_{S_z}, [y]_{S_{z'}}) = v$. Hence, $(v, x) \in S_z$ and $(v, y) \in S_{z'}$. From (2) and these last assertions we infer that (3) $\sim w([x]_{S_z}, [y]_{S_{z'}}) = \sim v$, $(\sim v, \sim x) \in S_z$ and $(\sim v, \sim y) \in S_{z'}$. Thus, $\sim v \in [\sim x]_{S_z} \cap [\sim y]_{S_{z'}}$ from which we obtain that $w(\sim ([x]_{S_z}, [y]_{S_{z'}})) = w([\sim x]_{S_z}, [\sim y]_{S_{z'}}) = \sim v$ and so by (3), we get (1). Therefore, taking into account [24, Theorem 4] we conclude that w is an $NS_{n\times m}$–isomorphism. □

5. Subdirectly irreducible $\mathbf{NS_{n\times m}}$–algebras

Theorem 5.1. *Let L be a non-trivial $NS_{n\times m}$–algebra. Then the following conditions are equivalent:*

(i) *L is subdirectly irreducible,*

(ii) *$B(L) = \{0, 1\}$,*

(iii) *L is simple.*

Proof. (i) ⇒ (ii): From Theorem 4.1 and Corollary 4.1 we have that $B(L)$ is a subdirectly irreducible Boolean algebra and so, (ii) holds true.

(ii) ⇒ (iii): From the hypothesis and Proposition 3.1, we have that (1) $\sigma_{11}(L) = \{0, 1\}$. Let F be an s–filter of L, $F \neq \{1\}$. Then there is $x \in F$, $x \neq 1$. Thus, from (1) and (C13) it follows that $0 \in F$, from which we infer that $F = L$. This means that L is simple. □

Corollary 5.1. $\boldsymbol{NS_{n\times m}}$ *is semisimple.*

Proof. It is a direct consequence of Theorems 4.2 and 5.1 and well-known results of universal algebra (see [5, Lemma 12.2]). □

Corollaries 5.2 and 5.3 are immediate consequences of Theorems 3.1 and 5.1.

Corollary 5.2. *Every simple $NS_{n\times m}$–algebra is finite.*

Corollary 5.3. *There is a finite number of simple non-isomorphic $NS_{n\times m}$–algebras.*

Theorem 5.2 allows us to conclude that $\boldsymbol{NS_{n\times m}}$ is locally finite.

Theorem 5.2. *Every finitely generated $NS_{n\times m}$–algebra L is finite.*

Proof. Let $\mathcal{E}(L)$ be the set of all maximal deductive systems of L. Then from Corollary 5.1 L is isomorphic to $\prod\limits_{M\in\mathcal{E}(L)} L/M$. On the other hand, from Corollary 5.2 we have that L/M is finite. Taking into account Corollary 5.3, let $\{S_p\}_{1\leq p\leq K}$ be the set of all simple non-isomorphic $NS_{n\times m}$–algebras such that each S_p is isomorphic to at least one of the algebras L/M with $M\in\mathcal{E}(L)$. Then $\mathcal{E}(L)=\bigcup\limits_{p=1}^{K}\mathcal{M}_p$ where $\mathcal{M}_p=\{M\in \mathcal{E}(L): L/M\simeq S_p\}$ and $\mathcal{M}_p\cap\mathcal{M}_q=\emptyset$ for all $p\neq q$. Let $Epi(L,S_p)$ be the set of all $NS_{n\times m}$–epimorphisms from L to S_p and $\alpha_p: Epi(L,S_p)\to\mathcal{M}_p$ the application defined by the prescription $\alpha_p(h)=Ker(h)$. Hence, for each $M\in\mathcal{M}_p$ we have that $h=\theta_M\circ q_M\in Epi(L,S_p)$ and $Ker(h)=M$ where q_M is the natural application and θ_M is the $NS_{n\times m}$–isomorphism from L/M to S_p. Thus, α_p is onto. Therefore, if G is a finite set of generators of L, we obtain that $|\mathcal{M}_p|\leq|Epi(L,S_p)|\leq|(S_p)^G|$ from which we infer that $\mathcal{M}_p$ is finite and so, $\mathcal{E}(L)$ is finite. This completes the proof. $\square$

The characterization given in Theorem 5.1 for subdirectly irreducible algebras allows us to prove Theorem 5.3 which establishes the relationship between the operations σ_{ij} with $(i,j)\in(n\times m)$ and a special family of prime filters of a subdirectly irreducible $NS_{n\times m}$–algebra.

Theorem 5.3. *Let $(L,\sim)$ be a De Morgan algebra. Then the following conditions are equivalent:*

(i) *there is a family $\{P_{ij}\}_{(i,j)\in(n\times m)}$ of prime filters of L fulfilling these properties:*

   (a) $P_{ij}\subseteq P_{i(j+1)}$ *and* $P_{ij}\subseteq P_{(i+1)j}$,

   (b) $x\in P_{ij}$ *if and only if* $\sim x\notin P_{(n-i)(m-j)}$,

   (c) *if* $x,y\in L$ *are such that* $x,y\in\bigcap\limits_{(r,s)\in C}P_{rs}\setminus\bigcup\limits_{(r,s)\in(n\times m)\setminus C}P_{rs}$ *for some* $C\subseteq(n\times m)$, *then* $x=y$,

(ii) *there is a family* $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ *of unary operations on* L *such that* $(L, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)})$ *is a subdirectly irreducible* $NS_{n\times m}$*–algebra.*

Proof. (i) $\Rightarrow$ (ii): For every $(i,j) \in (n \times m)$ let σ_{ij} be the unary operation on L defined by $\sigma_{ij}x = 1$ if $x \in P_{ij}$ and $\sigma_{ij}x = 0$ otherwise. In order to see that $(L, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)})$ is an $NS_{n\times m}$–algebra, we shall only prove (C5). Let us suppose that $\sigma_{ij}x = \sigma_{ij}y$ for all $(i,j) \in (n \times m)$. Then from the definition of σ_{ij}, there exists $C \subseteq (n\times m)$ such that $\sigma_{ij}x = \sigma_{ij}y = 1$ for all $(i,j) \in C$ and $\sigma_{ij}x = \sigma_{ij}y = 0$ for all $(i,j) \in (n \times m) \setminus C$. From these last assertions we have that

$$x, y \in \bigcap_{(r,s)\in C} P_{rs} \setminus \bigcup_{(r,s)\in(n\times m)\setminus C} P_{rs}$$

and consequently, from (c) we conclude that $x = y$.

On the other hand, from the definition of the operations σ_{ij} and Proposition 3.1 we have that $B(L) = \{0,1\}$ and so, by Theorem 5.1 we conclude that L is subdirectly irreducible.

(ii) $\Rightarrow$ (i): For every $(i,j) \in (n \times m)$ let $P_{ij} = \{x \in L : \sigma_{ij}x = 1\}$. Then it is easy to check that $\{P_{ij}\}_{(i,j)\in(n\times m)}$ is a family of prime filters of L which satisfies properties (a), (b) and (c). $\square$

Remark 5.1. It is worth mentioning that on a given De Morgan algebra, different structures of $NS_{n\times m}$–algebras can be defined. Indeed, let us consider the De Morgan algebra $(L, \sim)$ illustrated in Figure 3 where $\sim 0 = 1$ and $\sim a = u$.

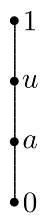

Figure 3

Then by defining on L the families $\{\sigma_{ij}\}_{(i,j)\in(4\times3)}$ and $\{\Phi_{ij}\}_{(i,j)\in(4\times3)}$ as in Tables 2 and 3 respectively, we obtain that $(L, \sim, \{\sigma_{ij}\}_{(i,j)\in(4\times3)})$ and $(L, \sim, \{\Phi_{ij}\}_{(i,j)\in(4\times3)})$ are different $NS_{4\times3}$–algebras.

x	$\sigma_{11}x$	$\sigma_{12}x$	$\sigma_{21}x$	$\sigma_{22}x$	$\sigma_{31}x$	$\sigma_{32}x$
0	0	0	0	0	0	0
a	0	0	0	0	0	1
u	0	1	1	1	1	1
1	1	1	1	1	1	1

Table 2

x	$\Phi_{11}x$	$\Phi_{12}x$	$\Phi_{21}x$	$\Phi_{22}x$	$\Phi_{31}x$	$\Phi_{32}x$
0	0	0	0	0	0	0
a	0	0	0	1	0	1
u	0	1	0	1	1	1
1	1	1	1	1	1	1

Table 3

In what follows we shall denote by $\Pi(L)$ the set of all join irreducible elements of L.

Theorem 5.4. *Let L be a subdirectly irreducible $NS_{n\times m}$–algebra. For each $(i,j) \in (n\times m)$, let $P_{ij} = \{x \in L : \sigma_{ij}x = 1\}$. Then it holds*

(i) *$p \in \Pi(L)$ if and only if $P_{ij} = [p)$ for some $(i,j) \in (n\times m)$,*

(ii) *$\varphi(P_{ij}) = P_{(n-i)(m-j)}$ where φ is the Birula-Rasiowa transformation.*

Proof. Let $p \in \Pi(L)$. Then from Theorem 5.1, we have that $\sigma_{ij}p \in \{0,1\}$. Besides, from (C5) there exists $(r,s) \in (n\times m)$ such that $\sigma_{rs}p \neq 0$ and so, there is a non-empty set $C \subseteq (n\times m)$ such that

$$p \in \bigcap_{(i,j)\in C} P_{ij} \setminus \bigcup_{(i,j)\in(n\times m)\setminus C} P_{ij}. \tag{1}$$

As L is finite, $P_{ij} = [p_{ij})$ where $p_{ij} \in \Pi(L)$. Then from (1) and taking into account that $\bigvee\limits_{(i,j)\in C} p_{ij} \in \bigcap\limits_{(i,j)\in C} P_{ij} \setminus \bigcup\limits_{(i,j)\in(n\times m)\setminus C} P_{ij}$, we obtain from (c) of Theorem 5.3 that $p = p_{ij}$ for some $(i,j) \in C$. Hence, (i) holds true. On the other hand, property (ii) is a direct consequence of (C7). $\square$

Theorem 5.4 allows us to conclude that $(\Pi(L),\psi)$ is the determinant system of the De Morgan reduct of a subdirectly irreducible $NS_{n\times m}$–algebra L where ψ is the well-known decreasing involution associated with φ ([23, 22]). Besides, from Theorem 5.3 it follows that for every $(i,j) \in (n\times m)$ the operation σ_{ij} is defined L by $\sigma_{ij}x = 1$ if $x \in P_{ij}$ and $\sigma_{ij}x = 0$ otherwise. So, these operations are determined from the family $\{P_{ij}\}_{(i,j)\in(n\times m)}$. Therefore, we shall consider the triple $(\Pi(L), \psi, \{P_{ij}\}_{(i,j)\in(n\times m)})$ as the determinant system of L.

Proposition 5.1. *Let L be a simple centred $NS_{n\times m}$–algebra and $p \in L$. Then the following conditions are equivalent:*

(i) *p is a join irreducible element of L,*

(ii) *p is a (i,j)–centre of L for some $(i,j) \in (n \times m)$.*

Proof. For each $(i,j) \in (n \times m)$, let c_{ij} be the (i,j)–centre of L.

(i) $\Rightarrow$ (ii): Since L is centred it follows that (1) $p = \bigvee\limits_{i=1}^{n-1} \bigvee\limits_{j=1}^{m-1} (c_{ij} \wedge \sigma_{ij} p)$. Let $T \subseteq (n \times m)$ be such that $\sigma_{ij} p = 1$ for all $(i,j) \in T$. From the hypothesis and (C5) it is simple to verify that T is a non–emptyset. So, $\sigma_{ij} p = 0$ for all $(i,j) \in (n \times m) \setminus T$. Therefore, from (1) we have that $p = \bigvee\limits_{(i,j)\in T} c_{ij}$. From this last assertion and (i) we conclude that $p = c_{ij}$ for some $(i,j) \in T \subseteq (n \times m)$.

(ii) $\Rightarrow$ (i): Let $p = c_{ij}$ for some $(i,j) \in (n \times m)$ and let $a, b \in L$ be such that $c_{ij} = a \vee b$. Then $\sigma_{rs} c_{ij} = \sigma_{rs} a \vee \sigma_{rs} b$ for all $(r,s) \in (n \times m)$ from which it results that (1) $\sigma_{rs} a = \sigma_{rs} b = 0 = \sigma_{rs} c_{ij}$ for all $i > r$ or $j > s$. Besides, since $\sigma_{ij} c_{ij} = 1$ it follows that $\sigma_{ij} a = 1$ or $\sigma_{ij} b = 1$. Hence, from (C2) and (C3) we have that $\sigma_{rs} a = 1$ for $i \leq r$ and $j \leq s$ or $\sigma_{rs} b = 1$ for $i \leq r$ and $j \leq s$. From these last assertions, (1) and (C5), we conclude the proof. □

Corollary 5.4. *Let L be a simple centred $NS_{n\times m}$–algebra. Then it holds*

(i) *$\Pi(L)$ has $(n-1)\cdot(m-1)$ elements,*

(ii) *$\Pi(L) = \{c_{ij}\}_{(i,j)\in(n\times m)}$ where $c_{ij} \leq c_{rs}$ if and only if $r \leq i$ and $s \leq j$. Furthermore, $P_{ij} = [c_{ij})$ for all $(i,j) \in (n \times m)$.*

Proof. It follows from Proposition 5.1 and Theorem 5.4. □

Since the $NS_{n\times m}$–algebra $\mathbb{S}_{n\times m} = \{0,1\} \uparrow^{(n\times m)}$ is simple and centred, from Corollary 5.4 we have that the ordered set $\Pi(\mathbb{S}_{n\times m})$ is the cartesian product of two chains with $n-1$ and $m-1$ elements respectively. Besides, Theorem 3.1 allows us to conclude that every simple $NS_{n\times m}$–algebra is a subalgebra of $\mathbb{S}_{n\times m}$. Thus, we have proved.

Theorem 5.5. *$\mathbb{S}_{n\times m}$ generates $\boldsymbol{NS_{n\times m}}$.*

Finally, we shall give the announced equational base for $\boldsymbol{NS_{n\times m}}$. For this purpose, Proposition 5.2 will be fundamental.

Proposition 5.2. *Let L be a subdirectly irreducible $NS_{n\times m}$–algebra and x, $y \in L$. Then the following identity is satisfied:*

(C16) $(\sigma_{(n-1)(m-1)}y \to \sigma_{(n-1)(m-1)}x) \to ((\sigma_{(n-1)(m-1)}x \to \sigma_{(n-1)(m-1)}y) \to (\cdots \to ((\sigma_{11}y \to \sigma_{11}x) \to ((\sigma_{11}x \to \sigma_{11}y) \to x)\cdots) = (\sigma_{(n-1)(m-1)}y \to \sigma_{(n-1)(m-1)}x) \to ((\sigma_{(n-1)(m-1)}x \to \sigma_{(n-1)(m-1)}y) \to (\cdots \to ((\sigma_{11}y \to \sigma_{11}x) \to ((\sigma_{11}x \to \sigma_{11}y) \to y)\cdots)$.

Proof. From the hypothesis and Theorem 5.1 we have that (1) $\sigma_{ij}x \to \sigma_{ij}y$, $\sigma_{ij}y \to \sigma_{ij}x \in \{0,1\}$ for all $(i,j) \in (n\times m)$. If $\sigma_{ij}x \to \sigma_{ij}y = \sigma_{ij}y \to \sigma_{ij}x = 1$ for all $(i,j) \in (n\times m)$, then from (W9) and (C5) we infer that $x = y$. Therefore, (C16) is satisfied. On the other hand, if $\sigma_{rs}x \to \sigma_{rs}y \neq 1$ or $\sigma_{kt}y \to \sigma_{kt}x \neq 1$ for some (r,s), $(k,t) \in (n\times m)$, then from (1) we get that $\sigma_{rs}x \to \sigma_{rs}y = 0$ or $\sigma_{kt}y \to \sigma_{kt}x = 0$ from which by (W7) and (W8) we obtain that each member of (C16) is 1. Hence, the proof is complete. □

Corollary 5.5. *Let $L \in \boldsymbol{NS_{n\times m}}$. Then identity* (C16) *of Proposition 5.2 is satisfied.*

Proof. It is a direct consequence of Proposition 5.2 and a well-known result by G. Birkhoff (see [5, Theorem 8.6]). □

Theorem 5.6. *A system $\langle L,\wedge,\vee,\sim,\{\sigma_{ij}\}_{(i,j)\in(n\times m)},0,1\rangle$ is an $NS_{n\times m}$–algebra if and only if $\langle L,\wedge,\vee,\sim,0,1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfill properties* (C1), (C2), (C3), (C4), (C6), (C7), (C12) *and* (C16).

Proof. Let $\sigma_{ij}x = \sigma_{ij}y$ for all $(i,j) \in (n\times m)$. Then from (C7), (C4) and (C6) we have that (1) $\sigma_{ij}x \to \sigma_{ij}y = \sigma_{ij}y \to \sigma_{ij}x = 1$ for all $(i,j) \in (n\times m)$. Besides, from (C12) we get that $1 \to a = a$ for all $a \in L$. Thus, taking into account (1) and (C16) we conclude that $x = y$. Hence, (C5) holds true. The converse follows from Corollary 5.5. □

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