REPORTS ON MATHEMATICAL LOGIC
38 (2004), 87–91

Andrzej WROŃSKI

ON PRUCNAL'S MODEL–DETERMINED LOGIC AND DEFINABLE PREDICATES

A b s t r a c t. Prucnal's concept of a logic determined by a model is discussed. It is proved that logics different from the pure first-order one can be determined by models with undecidable theories.

Let $\mathcal{L}$ be a first-order language and let $\mathfrak{M} = \langle M, \mathcal{I}\rangle$ be a model for $\mathcal{L}$. Then every formula $\varphi(x_1, \ldots, x_n)$ of $\mathcal{L}$ determines a unique n-place predicate:

$$\varphi^{\mathfrak{M},n} = \{\langle a_1, \ldots, a_n\rangle \in M^n : \mathfrak{M} \models \varphi[a_1, \ldots, a_n]\}$$

Here, of course, the notation $\varphi(x_1, \ldots, x_n)$ is used to indicate that the free variables of φ constitute a subset of $\{x_1, \ldots, x_n\}$ (see [1] p. 24).

A n-placed relation $R \subseteq M^n$ is said to be *definable* in $\mathfrak{M}$ if there exists a formula $\varphi(x_1, \ldots, x_n)$ in $\mathcal{L}$ such that $R = \varphi^{\mathfrak{M},n}$.

Received February 15, 2000
1991 AMS *Subject Classification* 03C99, 03C07
Research supported by Polish KBN grant 1H01A 001 12

Let $\mathcal{L}_0$ be the pure first-order language in the sense of Church [2] i.e. the language without identity, individual constants and function symbols equipped with $\aleph_0$ predicate symbols with n-places, for every $n > 0$. Let $\mathcal{L}_0(\mathfrak{M})$ be the class of all models for $\mathcal{L}_0$ of the form $\langle M, \mathcal{I}_0 \rangle$ where the interpretation function $\mathcal{I}_0$ maps predicate symbols of $\mathcal{L}_0$ to predicates definable in $\mathfrak{M}$. It is easy to see, that $\mathsf{Th}(\mathcal{L}_0(\mathfrak{M}))$ i.e. the theory in the language $\mathcal{L}_0$ determined by the class $\mathcal{L}_0(\mathfrak{M})$ coincides with the logic of the model $\mathfrak{M}$ in the sense of Prucnal [5]. We shall use the symbol $\mathsf{LP}(\mathfrak{M})$ to denote Prucnal's logic determined by $\mathfrak{M}$ in the above sense.

The fact that every theory of the form $\mathsf{Th}(\mathcal{L}_0(\mathfrak{M}))$ must be closed under the rule of substitution for predicate symbols justifies in a way the term *logic* in this case. Indeed, the property of being closed under the rule of substitution for predicate symbols is one of distinctive features of logic among various theories and consequently any law of the pure first-order predicate logic can be viewed as a law-schema describing a set of similar laws.

The reason why the pure first-order predicate logic is closed under the rule of substitution for predicate symbols is quite simple: a logical law must be true under *every* interpretation in *every* non-empty universe. Thus, any law of the pure first-order logic can be made into a law of the second-order logic just by binding all predicate symbols with universal quantifiers i.e. for every sentence φ of the language $\mathcal{L}_0$, if all predicate symbols of φ are among $P_1, \dots, P_n$ then we have:

Observation 1. *$\forall P_1 \dots \forall P_n \varphi$ is a law of the second-order logic iff $\mathfrak{M} \models \varphi$, for every model $\mathfrak{M}$.*

Despite its simplicity, the above observation yields an important consequence: there is no hope to find a decidable set of axioms and rules of inference for second-order logic.

Often it happens that $\mathsf{LP}(\mathfrak{M})$ coincides with the pure first-order logic. It was first noticed by Dr. Marcin Mostowski during the discussion following

Prucnal's lecture at 39-th Conference on the History of Logic in Cracow, 1993. Answering a question of Tadeusz Prucnal Dr. Mostowski recalled a classical result of Hilbert and Bernays [3], pp. 252–253, which easily yields:

Fact 2. *If φ is a satisfiable formula of $\mathcal{L}_0$ whose predicate symbols are among $P_1, \ldots, P_n$ then it is satisfied by a model of the form: $\langle \omega, R_1, \ldots, R_n \rangle$ where all predicates R_i are definable in the standard model of arithmetic.*

Thus, by virtue of the *metamathematical completeness theorem* of Hilbert and Bernays (see Kleene [4] p. 394), $\mathsf{LP}(\mathfrak{M})$ equals to the pure first-order logic whenever $\mathfrak{M}$ is an expansion of the standard model of arithmetic.

Soon it became clear that such equality cannot occur if $\mathfrak{M}$ has a decidable first-order theory. Indeed, we have:

Fact 3. *Let be the class of all models $\mathfrak{M}$ such that $\mathsf{Th}(\mathfrak{M})$ is decidable. Then the corresponding Prucnal's logic, $\mathsf{LP}() = \bigcap\{\mathsf{LP}(\mathfrak{M}) : \mathfrak{M} \in\}$, exceeds the pure first-order logic* (see [6, 7]).

To see that undecidability of $\mathsf{Th}(\mathfrak{M})$ not necessarily forces $\mathsf{LP}(\mathfrak{M})$ to stay small we prove:

Fact 4. *For every infinite set $X \subseteq \omega$ there exists a mono-unary algebra $\mathfrak{N}_X$ such that the degree of unsolvability of $\mathsf{Th}(\mathfrak{N}_X)$ is at least as big as that of X and $\mathsf{LP}(\mathfrak{N}_X)$ exceeds the pure first-order logic.*

Proof. Let $f : \omega \mapsto \omega$ be a bijection. By the *order* of an element $m \in \omega$ we shall mean cardinality of the subuniverse generated by m in the algebra $\langle \omega, f \rangle$. Since f is a bijection, the order of any element of the subuniverse generated by m must be the same as the order of m itself and, moreover, for every natural number $n > 0$, the discourse language of the algebra $\langle \omega, f \rangle$ contains a formula $\rho_n(x)$ meaning that: *x is an object of order n.* If $m, n \in \omega$ and $n > 0$ then the discourse language of $\langle \omega, f \rangle$ contains also a sentence $\psi_{m,n}$ meaning that: *there exist exactly m objects of order*

n. By discourse language of an algebra we mean of course the first-order language with identity, equipped with symbols of basic operations.

Now let $X \subseteq \omega$ be an infinite set of natural numbers. Without loss of generality we can assume that $0 \notin X$. Let $f_X : \omega \mapsto \omega$ be a bijection such that the following condition is satisfied:

- For every $n \in \omega, n > 0$, the algebra $\mathfrak{N}_X = \langle \omega, f_X \rangle$ has exactly n objects of order n if $n \in X$ and zero objects of order n if $n \notin X$.

It is easy to see that any decision procedure for $\mathsf{Th}(\mathfrak{N}_X)$ can be also used as a decision procedure for X because $n \in X$ iff $\mathfrak{N}_X \models \psi_{n,n}$, for every $n > 0$.

To prove that $\mathsf{LP}(\mathfrak{N}_X)$ exceeds the pure first order logic one can use a sentence φ involving two binary predicate symbols E and P, where E plays the role of identity predicate and P plays the role of a linear ordering without the greatest element, whose domain is the whole universe. We will show that $\neg\varphi \in \mathsf{LP}(\mathfrak{N}_X)$. Indeed, if $\neg\varphi \notin \mathsf{LP}(\mathfrak{N}_X)$ then there exists a model $\mathfrak{M} = \langle \omega, \mathcal{I} \rangle$ where the relations $\mathcal{I}(E), \mathcal{I}(P)$ are definable in $\mathfrak{N}_X$ and $\mathfrak{M} \models \varphi$. Since $\mathcal{I}(E)$ is an equivalence relation on ω then picking a unique element in every equivalence class of $\mathcal{I}(E)$ we can build up a selector $S \subseteq \omega$. Putting now $R = S^2 \cap \mathcal{I}(P)$ we obtain a structure $\langle S, R \rangle$ which is an infinite chain. Since the relation $\mathcal{I}(P)$ is definable in $\mathfrak{N}_X$ then, by virtue of a well-known theorem of Shelah (see [1] p. 505), the theory $\mathsf{Th}(\mathfrak{N}_X)$ should be unstable. This, however, cannot be true because $\mathsf{Th}(\mathfrak{N}_X)$ is categorical in every uncountable power. Thus supposing that $\neg\varphi \notin \mathsf{LP}(\mathfrak{N}_X)$ one gets a contradiction.

The argument used in the above proof yields a result similar to that of Fact 3, namely:

Fact 5. *Let be the class of all models $\mathfrak{M}$ such that $\mathsf{Th}(\mathfrak{M})$ is stable. Then the corresponding Prucnal's logic,* $\mathsf{LP}() = \bigcap\{\mathsf{LP}(\mathfrak{M}) : \mathfrak{M} \in\}$, *exceeds the pure first-order logic.*

Thus we have to end with the following:

Problem. *Characterize models* $\mathfrak{M}$ *such that* $\mathsf{LP}(\mathfrak{M})$ *equals to the pure first-order logic.*

References

[1] Chang C.C., Keisler H.J., *Model Theory*, vol. 73 of Studies in Logic and the Foundations of Mathematics, North-Holland Publishing Co., Amsterdam, third ed., 1990.

[2] Church A., *Introduction to Mathematical Logic*, vol. 1, Princeton Univ. Press, Princeton, 1956.

[3] Hilbert D., Bernays P., *Grundlagen der Mathematik II*, Springer-Verlag, Berlin, 1939.

[4] Kleene S.C., *Introduction to Metamathematics*, North-Holland Publishing Co., Amsterdam, 1967. Fifth reprint.

[5] Prucnal T., *Logics of relational systems*, Bull. of the Section of Logic, Polish Acad. Sci., 19 (1990).

[6] Wroński A., *The logic determined – in the sense of T. Prucnal – by decidable first-order structures.* Streszczenia referatów XLI Konferencji Historii Logiki, Kraków, 24 – 25 październik, 1995.

[7] Wroński A., *Uwagi o logice wyznaczonej – w sensie Tadeusza Prucnala – przez struktury o rozstrzygalnych zawartościach*, in Byt, Logos, Matematyka, J. Perzanowski and A. Pietruszczak, eds., Wydawnictwo Uniwersytetu Mikołaja Kopernika, Toruń, 1997, pp. 233–236. Ksiega pamiatkowa bloku ontologiczno-logicznego VI Polskiego Zjazdu Filozoficznego, Toruń, 5 – 9 września 1995 r.

Department of Logic
Jagiellonian University
Grodzka 52
Kraków 31-044, Poland

e-mail: `uzwronsk@cyfr-kr.edu.pl`