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REMARKS ON THE ARRANGEMENT OF IRREDUCIBLE ELEMENTS OF FINITELY GENERATED DISTRIBUTIVE LATTICES IN A BOOLEAN CUBE

A b s t r a c t. We consider the problem of embedding of freely generated distributive lattices into Boolean cubes.

Suppose that $\mathbf{K}$ is a finite distributive lattice and let $N_{\mathbf{K}}$ be the set of all its irreducible, different from 1 elements. That is

$$N_{\mathbf{K}} = \{a \in K; a \neq 1 \wedge \forall_{x,y \in K}(x \cap y = a \to x = a \vee y = a)\}.$$

It is well known that the closure of the set $N_{\mathbf{K}} \cup \{1\}$ on finite intersections equals to the set of all elements of $\mathbf{K}$.

Suppose that $R_1, R_2, \ldots, R_m$ is a decomposition of the set $N_{\mathbf{K}}$ onto chains and let $|R_i| = r_i$, $1 \leq i \leq m$. Let

$$P_{r_1 \ldots r_m} = \prod(\{0, 1, \ldots, r_i\};\ 1 \leq i \leq m).$$

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Then we define two binary operations $\cup$ and $\cap$ on the set $P_{r_1\ldots r_n}$ as follows:

$$(x_1,\ldots,x_m)\cup(y_1,\ldots,y_m)=(\max(x_1,y_1),\ldots,\max(x_m,y_m)),$$

$$(x_1,\ldots,x_m)\cap(y_1,\ldots,y_m)=(\min(x_1,y_1),\ldots,\min(x_m,y_m)).$$

It is clear that

$$\mathcal{P}_{r_1\ldots r_m}=\langle P_{r_1\ldots r_m},\cup,\cap\rangle$$

is a finite distributive lattice.

Let $f: N_K \to \{0,1\}^n$ be a mapping such that

$$(f(a))(i)=1 \Leftrightarrow a\in R_i, \quad 1\le i\le m$$

and let $I: K\to P_{r_1\ldots r_m}$ be defined as follows:

$$I(x)=(r_1,\ldots,r_m)-\sum(f(a);\ x\le a\wedge a\in N_K).$$

Theorem 1. *The mapping I is an embedding of the lattice K into the lattice $\mathcal{P}_{r_1\ldots r_m}$.*

Let F_n be a free lattices with n free generators $a_1,a_2,\ldots,a_n$. Clearly,

$$N_{F_n}=\{a_{i_1}\cup\ldots\cup a_{i_k};\ 1\le i_1<\ldots<i_k\le n\wedge 1\le k\le n-1\}$$

and $|N_{F_n}|=2^n-2$.

Corollary 2. *a) F_n is embeddable into the Boolean cube*

$$\mathcal{P}_{1\ldots 1}=\langle\{0,1\}^{2^n-2},\cup,\cap\rangle.$$

b) $I(a_i)$ lies in the plane

$$x_1+\ldots+x_{2^n-2}=2^{n-1}-1, \qquad 1\le i\le n.$$

If we identify each element x of the lattice F_n with its image $I(x)$ in the Boolean cube $\mathcal{P}_{1\ldots 1}=\{0,1\}^{2^n-2}$, then the set N_{F_n} can be considered as a matrix whose rows are elements of the set. There is only a finite number of such matrices. They depend on the ordering of N_{F_n} and the generators of F_n which occur in the plane, see Corollary 2b. Let us prove that one can find such matrices $\mathfrak{M}^{(n)}$ that $\mathfrak{M}^{(n+1)}$ can be built up from segments of $\mathfrak{M}^{(n)}$.

Let G_n be a matrix with n rows and $2^n - 2$ columns such that

1) the first $\binom{n}{1}$ columns are $\{0,1\}$ words, ordered lexicographically, and contain 1 only once;

2) the next $\binom{n}{2}$ columns are $\{0,1\}$ words, ordered lexicographically, and contain 1 twice;

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n - 1) the last $\binom{n}{n-1}$ columns are $\{0,1\}$ words, ordered lexicographically, and contain 0 only once.

Let us assume that elements of the form $a_{i_1} \cup \ldots \cup a_{i_j}$ for each $1 \leq j \leq n-1$ are ordered lexicographically with respect to their indices $i_1, \ldots, i_j$, and let us assume that $a_{i_1} \cup \ldots \cup a_{i_j}$ proceeds $a_{k_1} \cup \ldots \cup a_{k_l}$ if $j > l$. Such orderings on N_{F_n} are said to be *quasi-natural.* Let $\mathfrak{M}^{(n)}$ be a matrix such that its k-th row is the k-th element in a given quasi-natural ordering on N_{F_n} with $1 \leq k \leq 2^n - 2$.

Theorem 3. *The matrix G_n and the matrix consisting of the last $\binom{n}{n-1}$ rows of the matrix $\mathfrak{M}^{(n)}$ are the same and hence the rows of G_n are generators of the lattice F_n.*

One can divide the set of all rows of $\mathfrak{M}^{(n)}$ into n-1 classes. The first one contains all generators of the lattice F_n , the second all elements of the form $a_i \cup a_j$ etc. The set of all columns of $\mathfrak{M}^{(n)}$ can also be divided into $n-1$ classes. The first one contains the first $\binom{n}{1}$ columns of $\mathfrak{M}^{(n)}$, the second contains the next $\binom{n}{2}$ columns etc. Let $\mathfrak{M}^{(n)}_{i,k}$ denote the submatrix of $\mathfrak{M}^{(n)}$ containing those elements which occur in the i-th set of the rows of $\mathfrak{M}^{(n)}$ and in the k-th set of its columns.

Theorem 4. *For each natural number $n \geq 2$ and each $1 \leq i,\ k \leq n-1$,*

(a) $\mathfrak{M}^{(n)}_{i,k} = [1]$ *for* $i + k > n$ *where* $[1]$ *is the matrix in which all elements are equal to* 1;

(b) $$\mathfrak{M}^{(n)}_{i,k} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \text{ for } i + k = n;$$

(c) $$\mathfrak{M}^{(n)}_{1,1} = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix};$$

(d) $$\mathfrak{M}^{(n+1)}_{1,k} = \left(\begin{array}{c|c} 0 & 1 \\ \hline \mathfrak{M}^{(n)}_{1,k} & \mathfrak{M}^{(n)}_{1,k-1} \end{array}\right) \text{ for } 1 < k < n-1;$$

(e) $$\mathfrak{M}^{(n+1)}_{i,1} = \left(\begin{array}{c|c} \mathfrak{M}^{(n)}_{i-1,1} & 1 \\ \hline \mathfrak{M}^{(n)}_{i,1} & 0 \end{array}\right) \text{ for } 1 \leq i < n-1;$$

(f) $$\mathfrak{M}^{(n+1)}_{i,k} = \left(\begin{array}{c|c} \mathfrak{M}^{(n)}_{i,k} & 1 \\ \hline \mathfrak{M}^{(n)}_{i-1,k} & \mathfrak{M}^{(n)}_{i-1,k-1} \end{array}\right) \text{ for } i+k < n \text{ and } i \neq 1 \neq k.$$

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