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APPLICATION OF THE CANTOR-BENDIXON CONSTRUCTION TO THE PROBLEM OF FREELY GENERATED FILTERS

A b s t r a c t In this paper we prove that using the Cantor-Bendixon construction, we can reduce the problem of existence of an independent set of generators of a filter in a Boolean algebra to the same problem in an atomless algebra.

In [1] and [2] we have discussed the problem of existence of an independent set of generators for a countably generated filter in an atomless Boolean algebra and for any filter in a free Boolean algebra. In particular, we have proved that if F is a filter in a free Boolean algebra then F is freely generated provided the minimal cardinality of the set of generators of F is not a singular cardinal with a countable cofinality. Moreover, it has been proved in [1] that every countably generated filter in an atomless Boolean algebra is freely generated in this algebra.

Now, we would like to consider the problem of freely generated filters in Boolean algebras containing atoms. It is easy to observe (see [1]) that

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if a Boolean algebra contains infinitely many atoms there are filters in it without any independent set of generators.

We shall prove that the problem of atoms in Boolean algebras can be avoided by the use of the Cantor-Bendixon construction. It turns out that this construction preserves Boolean algebra independence. This observation enables us to formulate some necessary and sufficient conditions for the existence of an independent set of generators for a filter in a Boolean algebra containing atoms.

1. Basic notions

Let $\mathcal{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ be a Boolean algebra. For any finite set $K \subseteq B$ we denote by $\bigwedge K, \bigvee K$, respectively, the meet and the join of all elements of K. In particular, if $K = \emptyset$ then $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$.

An element a is an *atom* of $\mathcal{B}$ if it is a minimal element of the partially ordered set $B\backslash\{0\}$. Let us observe that $a \in B$ is an atom of $\mathcal{B}$ if $b \wedge a = 0$ or $b \wedge a = a$ for any $b \in B$. If a is a maximal element of $B\backslash\{1\}$ then we call a a *coatom* of $\mathcal{B}$. An element a is a coatom of $\mathcal{B}$ if and only if $\neg a$ is an atom of this algebra.

Let $At(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$ and $\neg At(\mathcal{B})$ be the set of all its coatoms. The filter of $\mathcal{B}$ generated by $\neg At(\mathcal{B})$ is denoted by $\nabla(\mathcal{B})$.

If $At(\mathcal{B}) = \emptyset$ then $\mathcal{B}$ is called an *atomless* Boolean algebra. The algebra $\mathcal{B}$ is called *atomic* if for every $b \in B$ there is an atom $a \in At(\mathcal{B})$ such that $b \geq a$, and $\mathcal{B}$ is called *superatomic* if every homomorphic image of $\mathcal{B}$ is atomic.

For every ordinal α we define by induction the filter ∇_α of the Boolean algebra $\mathcal{B}$ and the Boolean algebra $\mathcal{B}_\alpha$ (which will be called the *αth Cantor-Bendixon derivative* of $\mathcal{B}$ - see [4]) in the following way:

1. $\nabla_\alpha = \{1\}$ for $\alpha = 0$.

2. $\mathcal{B}_\alpha = \mathcal{B}/\nabla_\alpha$ for any ordinal α.
3. $\nabla_{\alpha+1} = \{a \in B; [a]_\alpha \in \nabla(\mathcal{B}_\alpha)\}$, where $[a]_\alpha$ denotes the equivalence class containing a and induced by ∇_α.
4. $\nabla_\gamma = \bigcup_{\alpha<\gamma} \nabla_\alpha$ if γ is a limit ordinal.

It is easy to notice that $\mathcal{B}_\alpha = \mathcal{B}_{\alpha+1}$ holds if and only if $\mathcal{B}_\alpha$ is atomless. Let $\gamma(\mathcal{B})$ be the least ordinal γ such that $\mathcal{B}_\gamma = \mathcal{B}_{\gamma+1}$. It follows from the definition that for every $\beta \geq \gamma$ we have $\mathcal{B}_\beta = \mathcal{B}_\gamma$ and the algebra $\mathcal{B}_\gamma$ is either a degenerated Boolean algebra or an atomless Boolean algebra.

A set A is *an independent (or free)* subset of the Boolean algebra $\mathcal{B}$ if and only if $\sigma a_1 \wedge \ldots \wedge \sigma a_n \neq 0$ for any sequence $a_1, \ldots, a_n$ of different elements of A and any function σ such that $\sigma a \in \{a, \neg a\}$ for every $a \in A$. In this paper we shall consider only infinite independent sets.

We call A a set of *generators* of a filter F if for each $b \in F$ there are $a_1, \ldots, a_n \in A$ such that $a_1 \wedge \ldots \wedge a_n \leq b$. We shall call a filter F of a Boolean algebra $\mathcal{B}$ *freely generated* if it has an independent set of generators.

2. Cantor–Bendixon construction preserves independence

Let $\mathcal{B}$ be a Boolean algebra and F be a filter in $\mathcal{B}$. First of all let us observe the following fact:

Lemma 1. *If $\{[a_i]_F\}_{i\in I}$ is an independent set in the quotient Boolean algebra $\mathcal{B}/F$ then $\{a_i\}_{i\in I}$ is an independent set in the algebra $\mathcal{B}$.*

Proof: Let $\{[a_i]_F\}_{i\in I}$ be an independent set and let

$$a_1 \wedge \ldots \wedge a_k \leq a_{k+1} \vee \ldots \vee a_n.$$

Thus

$$[a_1 \wedge \ldots \wedge a_k]_F \leq [a_{k+1} \vee \ldots \vee a_n]_F.$$

Hence

$$[a_1]_F \wedge \ldots \wedge [a_k]_F \leq [a_{k+1}]_F \vee \ldots \vee [a_n]_F$$

and this contradicts the assumption. ∎

The converse of this lemma is not true in the general case. If, for example, we consider any ultrafilter F of $\mathcal{B}$ and any independent subset $\{a_i\}_{i\in I}$ of $\mathcal{B}$ then we get $\{[a_i]_F\}_{i\in I} \subseteq \{[0]_F, [1]_F\}$ and this means that the set $\{[a_i]_F\}_{i\in I}$ is not independent in the quotient algebra $\mathcal{B}/F$. However, if we take for F the filter generated by the set of all coatoms of the given Boolean algebra then the converse of Lemma 1 is true.

Lemma 2. *Let I be an infinite set. If $\{a_i\}_{i\in I}$ is an independent set in a Boolean algebra $\mathcal{B}$ then for every ordinal α the set $\{[a_i]_\alpha\}_{i\in I}$ is independent in the αth Cantor-Bendixon derivative $\mathcal{B}_\alpha$ of the algebra $\mathcal{B}$.*

Proof: (induction on α). For $\alpha = 0$ we have $\mathcal{B}_\alpha = \mathcal{B}$ by the definition. Assume Lemma 2 is true for any ordinal $\gamma < \beta$ and let us suppose that the set $\{[a_i]_\beta\}_{i\in I}$ is not independent. We have to consider the following cases:

- $\beta = \alpha + 1$ for some ordinal α.

  For simplicity, we can assume that

$$[a_1]_\beta \wedge \ldots \wedge [a_n]_\beta \leq [a_{n+1}]_\beta \vee \ldots \vee [a_k]_\beta.$$

Thus

$$[\neg a_1 \vee \ldots \vee \neg a_n \vee a_{n+1} \vee \ldots \vee a_k]_\beta = [1]_\beta$$

hence

$$\neg a_1 \vee \ldots \vee \neg a_n \vee a_{n+1} \vee \ldots \vee a_k \in \nabla_{\alpha+1},$$

and then, by the definition of the filter $\nabla_{\alpha+1}$:

$$[\neg a_1 \vee \ldots \vee \neg a_n \vee a_{n+1} \vee \ldots \vee a_k]_\alpha \in \nabla(\mathcal{B}_\alpha),$$

which means that

$$\neg[a_1]_\alpha \vee \ldots \vee \neg[a_n]_\alpha \vee [a_{n+1}]_\alpha \vee \ldots \vee [a_k]_\alpha \geq [b_1]_\alpha \vee \ldots \vee [b_t]_\alpha,$$

where $[b_i]_\alpha$ for $i = 1, \ldots, t$ are coatoms of the Boolean algebra $\mathcal{B}_\alpha$. For every coatom $[b_i]_\alpha$ we can find a function $\sigma_i : \mathcal{B}_\alpha \to \mathcal{B}_\alpha$, such that $\sigma_i[a]_\alpha \in \{[a]_\alpha, \neg[a]_\alpha\}$ and $\sigma_i[a]_\alpha \vee [b_i]_\alpha = [1]_\alpha$ for every $[a]_\alpha \in \mathcal{B}_\alpha$. Let σ be a function such that $\sigma[a_{k+i}]_\alpha = \sigma_i[a_{k+i}]_\alpha$ for any $i = 1, \ldots, t$. Then we get

$$\neg[a_i]_\alpha \vee \ldots \vee \neg[a_n]_\alpha \vee [a_{n+1}]_\alpha \vee [a_k]_\alpha \vee \sigma[a_{k+1}]_\alpha \vee \ldots \vee \sigma[a_{k+t}]_\alpha \geq$$

$$\geq ([b_1]_\alpha \vee \sigma[a_{k+1}]_\alpha) \vee \ldots \vee ([b_t]_\alpha \vee \sigma[a_{k+t}]_\alpha) = [1]_\alpha,$$

and this means that $\{[a_i]_\alpha\}_{i \in I}$ is not independent and hence, due to the inductive assumption, the set $\{a_i\}_{i \in I}$ is not independent, either.

- β is a limit ordinal.
  If
  $$[a_1]_\beta \wedge \ldots \wedge [a_n]_\beta \leq [a_{n+1}]_\beta \vee \ldots \vee [a_k]_\beta,$$
  then
  $$\neg a_1 \vee \ldots \vee \neg a_n \vee a_{n+1} \vee \ldots \vee a_k \in \nabla_\beta,$$
  hence, by the definition of ∇_β, there is an ordinal $\alpha < \beta$ such that
  $$\neg a_1 \vee \ldots \vee \neg a_n \vee a_{n+1} \vee \ldots \vee a_k \in \nabla_\alpha.$$
  So
  $$[a_1]_\alpha \wedge \ldots \wedge [a_n]_\alpha \leq [a_{n+1}]_\alpha \vee \ldots \vee [a_k]_\alpha,$$
  and this means that the set $\{[a_i]_\alpha\}_{i \in I}$ is not independent and then, by the inductive assumption, the set $\{a_i\}_{i \in I}$ is not independent, either. ∎

Corollary 1. *If a Boolean algebra $\mathcal{B}$ contains an infinite independent set then $\mathcal{B}_{\gamma(\mathcal{B})}$ is a non-degenerated atomless Boolean algebra.*

3. Freely generated filters in Boolean algebras containing atoms

Let F be a filter of a Boolean algebra $\mathcal{B}$. We shall denote by $m(F)$ the minimal cardinality of a set of generators of F. If $m(F) = 1$ then F is a principal filter and it is obvious that it is freely generated if and only if $F \neq B$ and $F \neq \{1\}$. If $m(F) = \omega$ we call F *countably generated.* It has been proved in [1] that

Theorem 3. *Every countably generated filter containing only finitely many coatoms of a Boolean algebra is freely generated in this algebra.*

Let us observe that there is a natural connection between $m(F)$ and the cardinality of an independent set of generators of F:

Lemma 4. *If A is an independent set of generators of F then $m(F) =$ card A.*

Proof: Let G be a set of generators of F and card $G <$ card A. Since each $b \in G$ contains the meet of finitely many elements of A, it follows that there is a subset $C \subseteq A$ of a cardinality less than card A which generates F. Let $a \in A \backslash C$. Since $a \in F$, a contains the meet of finitely many elements of C, and as all of these are different from a, we get a contradiction with the independence of A. ∎

Denote by F_α the image of the filter F in the αth Cantor-Bendixon derivative $\mathcal{B}_\alpha$ of the algebra $\mathcal{B}$, i.e.

$$F_\alpha = F/\nabla_\alpha = \{[a]_\alpha \in \mathcal{B}_\alpha; a \in F\}.$$

It is obvious that $m(F_\alpha) \leq m(F)$. Henceforth, let $\mathcal{B}_\gamma$ be the atomless Cantor-Bendixon derivative of the algebra $\mathcal{B}$.

Theorem 5. *If the filter F is freely generated then $m(F_\alpha) = m(F)$ for any ordinal α. In particular, $m(F_\gamma) = m(F)$.*

Proof: Let A be an independent set of generators of F. As, by Lemma 4, card $A = m(F)$, then

$$m(F) = card\, A = card\, [A]_\alpha = m(F_\alpha),$$

by Lemma 2. ∎

¿From the above theorem we can immediately get the following known result:

Corollary 2. *No filter generated by an infinite set of coatoms is freely generated.*

Proof: If F is a filter of $\mathcal{B}$ generated by any infinite subset of $\neg At(\mathcal{B})$ then $F_\gamma = \{[1]_\gamma\}$ and hence $m(F) > m(F_\gamma) = 1$. Thus, according to Theorem 4 the filter F is not freely generated. ∎

Corollary 3. *If F is a nonprincipal filter in a superatomic Boolean algebra $\mathcal{B}$ then F is not freely generated.*

Proof: A Boolean algebra $\mathcal{B}$ is superatomic if and only if its atomless Cantor-Bendixon derivative $\mathcal{B}_\gamma$ is degenerated (see [4]) and then we have $m(F_\gamma) = 1$ for every F. ∎

The second corollary is equivalent to the known fact (see [4]) that superatomic algebras do not contain infinite independent sets. Now, we shall prove that we can (under some conditions) reverse Theorem 5.

Theorem 6. *Let F be a filter in a Boolean algebra $\mathcal{B}$. If $m(F_\gamma) = m(F)$ and the filter F_γ is freely generated in the atomless Cantor-Bendixon derivative $\mathcal{B}_\gamma$ of $\mathcal{B}$ then F is freely generated in the algebra $\mathcal{B}$.*

Proof: Let $\{b_i\}_{i\in I}$ be a set of pairwise different generators of the filter F in the algebra $\mathcal{B}$ and let card $I = m(F)$.

Suppose $\{[a_i]_\gamma\}_{i\in I}$ is an independent set of generators of the filter F_γ and $a_i \in F$ for every $i \in I$. Then, according to Lemma 1, the set $A = \{a_i\}_{i\in I}$ is also independent and for every $i \in I$ there is a finite set $K_i \subseteq \{a_i\}_{i\in I}$ such that

$$[b_i]_\gamma \geq [\bigwedge K_i]_\gamma .$$

Hence

$$\neg \bigwedge K_i \vee b_i \in \nabla_\gamma$$

and so

$$[\neg \bigwedge K_i \vee b_i]_\gamma = [1]_\gamma \ , \qquad \text{for every } i \in I. \tag{1}$$

Let us consider the set $C = \{(\neg \bigwedge K_i \vee b_i) \wedge a_i\}_{i\in I}$.

Of course $C \subseteq F$. We will show that C is an independent set of generators for the filter F.

Suppose C is not independent. Then

$$\bigwedge_{i\in L} ((\neg \bigwedge K_i \vee b_i) \wedge a_i) \leq \bigvee_{i\in M} ((\neg \bigwedge K_i \vee b_i) \wedge a_i),$$

where L, M are finite and disjoint subsets of I. Hence

$$\bigwedge_{i\in L} ([\neg \bigwedge K_i \vee b_i]_\gamma \wedge [a_i]_\gamma) \leq \bigvee_{i\in M} ([\neg \bigwedge K_i \vee b_i]_\gamma \wedge [a_i]_\gamma)$$

and then, by (1)

$$\bigwedge_{i\in L} [a_i]_\gamma \leq \bigvee_{i\in M} [a_i]_\gamma ,$$

which contradicts the independence of the set $\{[a_i]_\gamma\}_{i\in I}$ and the assumption $L \cap M = \emptyset$.

Thus C is independent. Moreover, if $d \in F$ then $d \geq \bigwedge_{i\in J} b_i$ for some finite subset J of I. However,

$$\bigwedge_{i\in J} b_i \geq \bigwedge_{i\in J} ((\neg \bigwedge K_i \vee b_i) \wedge a_i) \wedge \bigwedge_{i\in J} K_i ,$$

hence

$$d \geq \bigwedge_{i \in J} b_i \geq \bigwedge_{i \in M} ((\neg \bigwedge K_i \vee b_i) \wedge a_i), \text{ where}$$

$$M = \{j \in I : a_j \in \bigcup_{i \in J} K_i\} \cup J.$$

As M is a finite set, C is a set of generators of the filter F. ∎

Using Lemma 2 and Theorem 5 and 6, we reduce the problem of freely generated filters to atomless Boolean algebras:

Theorem 7. *A filter F is freely generated if and only if $m(F) = m(F_\gamma)$ and the filter F_γ is freely generated in the atomless Cantor-Bendixon derivative $\mathcal{B}_\gamma$ of the algebra $\mathcal{B}$.*

In particular, using Theorem 3 we can immediately get a necessary and sufficient condition for the existence of an independent set of generators of a countably generated filter in a Boolean algebra containing atoms:

Theorem 8. *Let F be a countably generated filter in a Boolean algebra $\mathcal{B}$. Then F is freely generated if and only if the filter F_γ is not a principal filter in the atomless Cantor-Bendixon derivative $\mathcal{B}_\gamma$ of the algebra $\mathcal{B}$.*

4. Freely generated ultrafilters

In conclusion we would like to apply the above theorems to the problem of the existence of independent sets of generators for ultrafilters.

First of all, let us observe that any ultrafilter in a free Boolean algebra has an independent set of generators, but it is not so for any ultrafilter in all types of Boolean algebras. However, we have the following result:

Theorem 9. *If the first Cantor-Bendixon derivative of a Boolean algebra $\mathcal{B}$ is infinite and atomless then every countably generated ultrafilter in $\mathcal{B}$ is freely generated.*

Proof: Let $\mathcal{B}_1$ be the nondegenerated and atomless Cantor-Bendixon derivative of a Boolean algebra $\mathcal{B}$. Assume F is a countably generated ultrafilter in $\mathcal{B}$. Then F_1 - the image of the filter F in the algebra $\mathcal{B}_1$ is either an ultrafilter in this algebra or $F_1 = B_1$. In the second case we have $0 \in [x]_1$ for some $x \in F$, hence $\neg x \in \nabla(\mathcal{B})$. Thus

$$\neg x \geq \neg a_1 \wedge \ldots \wedge \neg a_n \text{ for some } a_1, \ldots, a_n \in At(\mathcal{B}).$$

In other words $x \leq a_1 \vee \ldots \vee a_n$ and then

$$x = x \wedge (a_1 \vee \ldots \vee a_n) = (x \wedge a_1) \vee \ldots \vee (x \wedge a_n).$$

However, by the definition of atoms $x \wedge a_1 = a_i$ or $x \wedge a_i = 0$ for all $i = 1, \ldots, n$, so we can assume that $x = a_1 \vee \ldots \vee a_k$ for some $a_1, \ldots, a_k \in At(\mathcal{B})$. Then $a_1 \vee \ldots \vee a_k \in F$, hence $At(\mathcal{B}) \cap F \neq \emptyset$ and this means that F is a principal filter and we get a contradiction with the assumption.

Thus F_1 is an ultrafilter in the algebra $\mathcal{B}_1$. Moreover, as $\mathcal{B}_1$ is atomless, F_1 cannot be principal. Then, according to Theorem 8, the filter F is freely generated. ∎

The above theorem is not true in the case $\gamma(\mathcal{B}) > 1$.

Corollary 4. *In every Boolean algebra $\mathcal{B}$ such that $\gamma(\mathcal{B}) > 1$ there is an ultrafilter without an independent set of generators.*

Proof: Let F_1 be a principal ultrafilter in the Boolean algebra $\mathcal{B}_1$, i.e. the first Cantor-Bendixon derivative of an algebra $\mathcal{B}$. Since $\gamma(\mathcal{B}) > 1$, $\mathcal{B}_1$ contains atoms and F_1 exists. Moreover, F_1 is an image of some ultrafilter F in $\mathcal{B}$. F cannot be principal as $F_1 \neq B_1$. At the same time $F_2 = B_2$, hence F_2 is the principal filter generated by 0 of the algebra $\mathcal{B}_2$. This means, according to Theorem 5, that F is not freely generated. ∎

Thus, in particular, there are countably generated ultrafilters without an independent set of generators.

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