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A REPRESENTATION THEOREM FOR CO-DIAGONALIZABLE ALGEBRAS

A b s t r a c t. Tadeusz Prucnal [5] published a proof of the following representation theorem: for every atomic co-diagonalizable algebra D there exists an embedding h from D into the field of all subsets of a topological space X such that, for all $a \in D$, $h(\Delta(a))$ is the derivative of the set $h(a)$. He presented this result on a conference in Poland in 1983 and left open the question of it could be generalized to all co-diagonizable algebras. It was my observation that some ideas of measure theory (extending a measure to a complete measure) could be applied to define an embedding of any co-diagonalizable algebra into an atomic co-diagonalizable algebra. Consequently, Prucnal's representation theorem holds true for arbitrary co-diagonalizable algebras, which has been published in our common paper [3]. Two years later, I've published a short note [2] examining a more general situation of embeddability of modal structures into atomic modal structures. This paper surveys these results with modified proofs and additional comments.

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Introduction

Co-diagonalizable algebras are dual to diagonalizable algebras, the latter being algebraic counterparts of provability logic, investigated by Solovay [6] (see Boolos [1], Magari [4], Smoryński [7]). In provability logic, the arithmetical predicate Pr ($Pr(x)$ means that x is the Gödel number of a provable formula) is formalized as a modal operator $\Box$. The axioms and rules of provability logic are:

(CL) all tautologies of classical propositional logic in the modal language,

(A1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

(A2) $\Box\varphi \to \Box\Box\varphi$,

(A3) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb theorem),

(MP) $\varphi \to \psi$, φ/ψ,

(N) $\varphi \,/\, \Box\varphi$.

A diagonalizable algebra is an algebra $\mathcal{D} = (D, \cup, \cap, -, 0, 1, d)$ such that $(D, \cup, \cap, -, 0, 1)$ is a Boolean algebra, and d is a unary operation on D, satisfying the equalities:

(d1) $d(1) = 1$,

(d2) $d(a \cap b) = d(a) \cap d(b)$,

(d3) $d(-d(a) \cup a) = d(a)$,

for all $a, b \in D$.

A co-diagonalizable algebra is an algebra $\mathcal{D} = (D, \cup, \cap, -, 0, 1, \Delta)$ such that $(D, \cup, \cap, -, 0, 1)$ is a Boolean algebra, and Δ is a unary operation on D, satisfying the equalities:

(CD1) $\Delta(0) = 0$,

(CD2) $\Delta(a \cup b) = \Delta(a) \cup \Delta(b)$,

(CD3) $\Delta(a - \Delta(a)) = \Delta(a)$,

for all $a, b \in D$. Clearly, Δ is definable from d, by $\Delta(a) = -d(-a)$, and conversely. Notice that $d(a) \leq d(d(a))$ follows from (d2) and (d3), and similarly, $\Delta(\Delta(a)) \leq \Delta(a)$ follows from (CD2) and (CD3), where $a \leq b$ means $a \cup b = b$. (For the $\Delta-$case, first prove $\Delta(a) = \Delta(a \cup \Delta(a))$; the inclusion $\subseteq$ holds by the monotonicity of Δ, and the converse inclusion holds by (CD3), (CD2), De Morgan laws, and monotonicity again.)

An element $a \in D$ is called an *atom*, if $a \neq 0$, and there is no element $0 < b < a$. An algebra $\mathcal{D}$ is said to be *atomic*, if, for every $a \in D$, $a \neq 0$, there is an atom $b \in D$ such that $b \leq a$.

Prucnal theorem

Let C denote the closure operator in a topological space X. An element $x \in X$ is called an *accumulation point* of a set $A \subseteq X$, if $x \in C(A - \{x\})$. The set of all accumulation points of set A is called the *derivative* of A and denoted A^d. $P(X)$ denotes the powerset of the set X.

The following theorem is due to Prucnal (5).

Theorem 1. *For every atomic co-diagonalizable algebra $\mathcal{D}$, there exist a topological space X and a mapping $h : D \mapsto P(X)$ which is a Boolean monomorphism and satisfies $h(\Delta(a)) = h(a)^d$, for all $a \in D$.*

We sketch the proof. Let $\mathcal{D}$ be an atomic co-diagonalizable algebra. $At(D)$ denotes the set of all atoms in $\mathcal{D}$. The mapping $h : D \mapsto P(At(D))$ defined by:

$$h(a) = \{x \in At(D) : x \leq a\}$$

is a Boolean monomorphism of the Boolean algebra $(D, \cup, \cap, -, 0, 1)$ into the field of sets $(P(At(D)), \cup, \cap, -, \emptyset, At(D))$. We put $X = At(D)$, and the topology is defined by taking all sets of the form $h(a \cap -\Delta(-a))$, for $a \in D$, as a subbase (actually, it is a base). Accordingly, open sets are joins of sets from the base, and closed sets are meets of sets of the form $h(a \cup \Delta(a))$, for $a \in D$. Therefore, the closure operator C in X is defined by:

$$C(A) = \bigcap \{h(a \cup \Delta(a)) : A \subseteq h(a \cup \Delta(a))\}.$$

The equality $h(\Delta(a)) = h(a)^d$ amounts to the equivalence:

$$x \leq \Delta(a) \text{ iff } x \in C(h(a) - \{x\})$$

for all $x \in At(D)$, $a \in D$.

We prove the "only if" part. Assume $x \leq \Delta(a)$, $x \in At(D)$, $a \in D$, and take an arbitrary $b \in D$ such that $h(a) - \{x\} \subseteq h(b \cup \Delta(b))$. We have $h(x) = \{x\}$, since x is an atom, which yields $h(a) - \{x\} = h(a - x)$, and consequently $a - x \leq b \cup \Delta(b)$, since h is a Boolean monomorphism. We have $x \leq \Delta(a) = \Delta(a - \Delta(a)) \leq \Delta(a - x) \leq \Delta(b \cup \Delta(b)) = \Delta(b)$, by the assumption, (CD3), the monotonicity of Δ, the latter inequality, and a law mentioned above. Then, $x \leq b \cup \Delta(b)$, and consequently $x \in h(b \cup \Delta(b))$, which yields $x \in C(h(a) - \{x\})$.

We prove the "if" part. Assume $x \in C(h(a) - \{x\})$, $x \in At(D)$, $a \in D$. We have $h(a) - \{x\} = h(a - x) \subseteq h((a - x) \cup \Delta(a - x))$, by the monotonicity of h. The latter set is closed, hence $x \in h((a-x) \cup \Delta(a-x))$, and consequently $x \leq (a - x) \cup \Delta(a - x)$. Since $x \cap (a - x) = 0$, then $x \leq \Delta(a - x) \leq \Delta(a)$, by the monotonicity of Δ, which finishes the proof.

Generalization

To prove Prucnal's theorem for arbitrary co-diagonalizable algebras we need the following:

Theorem 2. *For every co-diagonalizable algebra $\mathcal{D}$, there exists a monomorphism of $\mathcal{D}$ into an atomic co-diagonalizable algebra.*

The above theorem has been proven in [3]. Clearly, if f is an embedding of a co-diagonalizable algebra $\mathcal{D}$ into an atomic co-diagonalizable algebra $\mathcal{D}'$, and h is the mapping of $\mathcal{D}'$ into the field of subsets of a topological space X, described in the preceding section, then $h \circ f$ is a mapping of $\mathcal{D}$ into the field of subsets of X, fulfilling the conditions of Prucnal's theorem.

We sketch the proof of Theorem 2. Fix a co-diagonalizable algebra $\mathcal{D} = (D, \cup, \cap, -, 0, 1, \Delta)$. By the Stone representation theorem for Boolean algebras, we can assume D be a field of subsets of a nonempty set U; the operations $\cup$, $\cap$, and $-$ are set-theoretic join, meet and complementation, $0 = \emptyset$ and $1 = U$, while Δ is an operator from D into D.

We choose an arbitrary infinite set N and, for every $A \in D$, define $f(A) = A \times N$. Clearly, f is a Boolean monomorphism from the algebra $\mathcal{D}$ into the field of all subsets of the set $T = U \times N$ such that, for all $A \in D$, $A \neq \emptyset$ iff $f(A)$ is infinite. For $Y, Z \subseteq T$, we define an equivalence relation: $Y \sim Z$ iff the symmetrical difference $Y \div Z$ (i.e. the set $(Y - Z) \cup (Z - Y)$) is finite. Clearly, the relation $\sim$ is a congruence on the field $P(T)$. We define:

$$D' = \{B \subseteq T : (\exists A \in f[D]) A \sim B\}$$

where $f[D]$ denotes the image of D under f. Since $f[D]$ is a field of subsets of T, and $\sim$ is a congruence, then D' is also a field of subsets of T which contains $f[D]$. Further, for any $B \in D'$, there is exactly one set $A \in f[D]$ such that $A \sim B$. For, if $A_1 \sim B$, $A_2 \sim B$, $A_1, A_2 \in f[D]$, then $A_1 \div A_2$ is finite, and consequently, $A_1 = A_2$, since all nonempty sets in $f[D]$ are infinite. By $B^\star$ we denote the unique set $A \in f[D]$ such that $A \sim B$. Clearly, $(\cdot)^\star$ is a Boolean homomorphism from D' onto $f[D]$.

Consider an algebra $\mathcal{D}' = (D', \cup, \cap, -, \emptyset, T, \Delta')$ such that $(D', \cup, \cap, -, \emptyset, T)$ is the field of sets described above, and the operator Δ' is defined as follows: $\Delta'(A) = f(\Delta(f^{-1}(A)))$, for $A \in f[D]$, and $\Delta'(B) = \Delta'(B^\star)$, for $B \notin f[D]$, $B \in D'$. Evidently, this definition is correct, and f is a Boolean monomorphism of $\mathcal{D}$ into $\mathcal{D}'$. Further, $f(\Delta(A)) = \Delta'(f(A))$, for all $A \in D$, hence f is a monomorphism of $\mathcal{D}$ into $\mathcal{D}'$.

We show that $\mathcal{D}'$ is a co-diagonalizable algebra. (CD1) is true, since $\emptyset \in f[D]$, hence $\Delta'(\emptyset) = f(\Delta(\emptyset)) = f(\emptyset) = \emptyset$. For (CD2), we calculate: $\Delta'(A \cup B) = \Delta'((A \cup B)^\star) = \Delta'(A^\star \cup B^\star) = \Delta'(A^\star) \cup \Delta'(B^\star) = \Delta'(A) \cup \Delta'(B)$. (CD3) is also true, by a similar argument (use $(\Delta'(A))^\star = \Delta'(A^\star)$).

Finally, $\mathcal{D}'$ is atomic, since all finite subsets of T belong to D' (they are equivalent to the empty set).

The proof of Theorem 2 is finished. Theorems 1 and 2 yield:

Theorem 3. *For every co-diagonalizable algebra $\mathcal{D}$, there exist a topological space X and a mapping $h : D \mapsto P(X)$ which is a Boolean monomorphism and satisfies $h(\Delta(a)) = h(a)^d$, for all $a \in D$.*

Notice that the derivative operator $(\cdot)^d$ on a topological space need not fulfill (CD3), hence the field of all subsets of a topological space enriched with the derivative operator need not be a co-diagonalizable algebra. It would be interesting to characterize those topological spaces which determine co-diagonalizable algebras.

More on Theorem 2

The method of embedding a co-diagonalizable algebra into an atomic one, applied in the proof of Theorem 2, can be used for many other modal structures. General conditions for that kind of embedding have been formulated in [2].

A *modal structure* is a structure $\mathcal{D} = (D, \cup, \cap, -, 0, 1, (R_s)_{s\in S}, (F_t)_{t\in T})$ such that $(D, \cup, \cap, -, 0, 1)$ is a Boolean algebra, R_s, for $s \in S$, is an n_s-ary relation on D, and F_t, for $t \in T$, is an n_t-ary operation on D. We refer to relations R_s and operations F_t as *modal* relations and operations, respectively. A modal structure $\mathcal{D}$ is said to be *atomic*, if the underlying Boolean algebra is atomic.

Let L be the first-order language for a fixed signature of modal structures. We use r_s and f_t as the relation and operation symbols corresponding to R_s and F_t, respectively. The set TER, of all terms of L, is defined as usual. We define a subset T_0 of TER by the following recursion: (i) all terms $f_t(u_1, \ldots, u_n)$ such that $t \in T$, $n = n_t$, and $u_1, \ldots, u_n$ are arbitrary terms, belong to T_0, (ii) $0, 1 \in T_0$, (iii) if $u, v \in T_0$, then also $(-u)$, $(u \cup v)$ and $(u \cap v)$ belong to T_0. Informally, T_0 consists of those terms from TER whose every variable is in the scope of some modal operation symbol. We also define a set At_0 which consists of all atomic formulas of the form $u = v$ such that $u, v \in T_0$ or $r_s(u_1, \ldots, u_n)$ such that $s \in S$, $n = n_s$ and $u_1, \ldots, u_n$ are arbitrary terms. Finally, the set Fm_0 is defined as the set of all formulas of L which are formed out of atomic formulas from At_0 by means of logical connectives and quantifiers. Clearly, Fm_0 is the set of those formulas of L in which every occurrence of a variable falls into the scope of some modal relation or operation symbol (except for the occurrences in $\forall x$ and $\exists x$).

Let $\mathcal{D}$ and $\mathcal{D}'$ be modal structures of the same signature. A *momomorphism* h from $\mathcal{D}$ into $\mathcal{D}'$ is a Boolean monomorphism from the Boolean algebra underlying $\mathcal{D}$ into that underlying $\mathcal{D}'$ which satisfies the following conditions:

(h1) $R_s(a_1, \ldots, a_n)$ iff $R'_s(h(a_1), \ldots, h(a_n))$, where $n = n_s$,

(h2) $h(F_t(a_1, \ldots, a_n)) = F'_t(h(a_1), \ldots, h(a_n))$, where $n = n_t$,

for all $s \in S$, $t \in T$, $a_1, \ldots, a_n \in D$; here R_s, R'_s stand for the corresponding designated relations in $\mathcal{D}$ and $\mathcal{D}'$, respectively, and similarly for F_t, F'_t. This notion is a standard notion of a monomorphism for relational structures.

A monomorphism h from $\mathcal{D}$ into $\mathcal{D}'$ is said to be *elementary for* Fm_0, if for every formula φ from Fm_0 and all elements $a_1, \ldots, a_n \in D$, there holds the equivalence:

$$\mathcal{D} \models \varphi[a_1, \ldots, a_n] \text{ iff } \mathcal{D}' \models \varphi[h(a_1), \ldots, h(a_n)],$$

where n is the number of free variables in φ, and $\models$ denotes the standard satisfaction relation. We write $\mathcal{D} \prec_0 \mathcal{D}'$, if there exists a monomorphism from $\mathcal{D}$ into $\mathcal{D}'$ which is elementary for Fm_0. Clearly, if $\mathcal{D} \prec_0 \mathcal{D}'$, then $\mathcal{D} \models \varphi$ iff $\mathcal{D}' \models \varphi$, for any sentence φ from Fm_0.

Now, Theorem 2 can be generalized to the following:

Theorem 4. *For every modal structure $\mathcal{D}$, there exists an atomic modal structure $\mathcal{D}'$ such that $\mathcal{D} \prec_0 \mathcal{D}'$.*

It follows from this theorem that, if Γ is a set of sentences from Fm_0, then every modal structure $\mathcal{D}$ which is a model of Γ is embeddable into an atomic modal structure $\mathcal{D}'$ which is also a model of Γ. Thus, Theorem 2 is a consequence of Theorem 4, since (the universal closures of) axioms (CD1)-(CD3) belong to Fm_0. In the same way, analogous embedding theorems can be obtained for other modal structures, e.g. diagonalizable algebras, normal modal algebras, deontic algebras corresponding to logics KD and KD4, various preference algebras (with a modal preference ordering) and so on. This method cannot be applied to modal structures whose at least one axiom does not belong to Fm_0, as e.g. $d(a) \leq a$ corresponding to the logic T.

The proof of Theorem 4 is similar to that of Theorem 2. One identifies $\mathcal{D}$ with a field of sets, supplied with modal relations and operations. The field D' and a Boolean monomorphism f from D into D' are defined as above. Then, modal relations and operations are defined on D', by setting:

(R1) $R'_s(A_1, \dots, A_n)$ iff $R_s(f^{-1}(A_1), \dots, f^{-1}(A_n))$, for $A_1, \dots, A_n \in f[D]$,

(R2) $R'_s(A_1, \dots A_n)$ iff $R'_s(A_1^\star, \dots, A_n^\star)$, for $A_1, \dots, A_n \in D'$,

(F1) $F'_t(A_1, \dots, A_n) = f(F_t(f^{-1}(A_1), \dots, f^{-1}(A_n)))$, for $A_1, \dots, A_n \in f[D]$,

(F2) $F'_t(A_1, \dots, A_n) = F'_t(A_1^\star, \dots, A_n^\star)$, for $A_1, \dots, A_n \in D'$.

Again, $\mathcal{D}'$ is an atomic modal structure. The mapping f is a monomorphism of $\mathcal{D}$ into $\mathcal{D}'$, hence $f[D]$ is the universe of a substructure $f[\mathcal{D}]$ of $\mathcal{D}'$. The following equivalence holds:

$$\mathcal{D}' \models \varphi[A_1, \dots, A_n] \text{ iff } f[\mathcal{D}] \models \varphi[A_1^\star, \dots, A_n^\star],$$

for every formula φ from Fm_0 and all $A_1, \dots, A_n \in D'$. To prove this equivalence, one first proves that, for every term $u \in T_0$,

$$u^{\mathcal{D}'}(A_1, \dots, A_n) = u^{f[\mathcal{D}]}(A_1^\star, \dots, A_n^\star),$$

for all $A_1, \dots, A_n \in D'$, where n is the number of variables occurring in u, and $u^{\mathcal{M}}$ stands for the value of term u in structure $\mathcal{M}$ under the given assignment. For, $(\cdot)^\star$ is a Boolean homomorphism and satisfies:

$$F'_t(A_1, \dots, A_n)^\star = F'_t(A_1, \dots, A_n) = F'_t(A_1^\star, \dots, A_n^\star),$$

since $F'_t(A_1, \dots, A_n) \in f[D]$. Consequently, $(\cdot)^\star$ is a homomorphism from D' onto $f[D]$ with respect to all Boolean and modal operations. Now, the required equality can easily be proven by double induction on the complexity of u and the number of occurrences of modal operation symbols in u. The equivalence is proved by induction on the complexity of φ.

Since f is an isomorphism of $\mathcal{D}$ onto $f[\mathcal{D}]$, then, for every formula φ of L and all $A_1, \ldots, A_n \in D$, there holds:

$$\mathcal{D} \models \varphi[A_1, \ldots, A_n] \text{ iff } f[\mathcal{D}] \models \varphi[f(A_1), \ldots, f(A_n)],$$

which, together with the equivalence from the preceding paragraph and the fact $f(A_i)^\star = f(A_i)$, shows that f is a monomorphism from $\mathcal{D}$ into $\mathcal{D}'$, elementary for Fm_0. The proof of Theorem 4 is finished.

Let κ be an infinite cardinal. If the Boolean algebra underlying $\mathcal{D}$ is a $\kappa-$complete field of sets (that means, it is closed under joins of families $\{A_j\}_{j \in J}$ such that $A_j \in D$, for $j \in J$, and the cardinality of J is not greater than κ), then there exists an atomic modal structure $\mathcal{D}'$ such that $\mathcal{D} \prec_0 \mathcal{D}'$, and the Boolean algebra underlying $\mathcal{D}'$ is a $\kappa-$complete field of sets. This fact can be proved in a similar way as Theorems 2 and 4 except that for the set N one chooses a set of cardinality κ^+ (the successor of κ), and the relation $\sim$ is defined as follows: $A \sim B$, if $A \div B$ is of cardinality not greater than κ. In [2], the assumption is more general: the Boolean algebra underlying $\mathcal{D}$ is $\kappa-$complete, and it is erroneous, since not every $\kappa-$complete Boolean algebra can be represented as a $\kappa-$complete field of sets.

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