REPORTS ON MATHEMATICAL LOGIC
36 (2002), 3–53

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FROM THE POSET OF LITERALS TO A TEMPORAL NEGATIVE NORMAL FORM

A b s t r a c t. Linear time temporal logics have proven to yield successful formalisms for a wide range of applications. Nevertheless, the lack of an efficient automated deduction method has prevented temporal logics from playing a more important role in computer science.

Recently, the use of unitary implicates and implicants has guided a lot of work in automated deduction.

In this paper, we present a formal framework in the study of unitary implicants and implicates in Temporal Logic. It is focussed on the temporal propositional logic, FNext±, with an infinite, linear and discrete flow of time. In our opinion, this formal framework allows an outstanding advance in the search for efficient automated deduction methods in temporal logics. The sets of unitary implicates and implicants of a formula in FNext± can be infinite and therefore difficult to handle. We introduce the notion of temporal literal in FNext± and we consider the semantic implication as an order relation over the set of temporal literals. Then, we develop theoretical studies of the poset of temporal literals and the sets of implicants and implicates. The formal results obtained allow us to define an efficient way to extract the maximum amount of information about the set of unitary implicants and implicates of the formula.

To emphasize the interest of these theoretical results, we show how this study allows us to introduce a suitable notion of Temporal

Received 17 May 2000.

Negative Normal Form for Temporal Logic.

The definition of such a normal form constitutes an unsolved problem until now. This is a major contribution of the paper and it will have a significant relevance in the future design of efficient automated theorem provers.

1. Introduction

Nowadays, the usefulness of non-classical logics, particularly temporal logic [4], [3], is not denied. Moreover, it is entirely accepted that the design of an efficient automated theorem prover (ATP) for temporal logic is still an unfinished task.

If automated theorem provers are classified into two categories, clausal and non clausal, we think that the latter are the most suitable candidates to successfully complete this task achieving a general frame. With this option as a starting point, our group has developed a new framework for building automated theorem provers, named TAS. Our TAS methods, unlike resolution, but like tableaux or dissolution methods, are non-clausal (i.e., they do not require transforming the input formula to clausal form) and the inputs are *Negative Normal Forms* equivalent to the negation of the formula the validity of which we want to analyze [13], [21], [8]. Nevertheless, there is a significant difference between dissolution/tableaux methods and TAS: while the former focus on searching an unsatisfiable branch in the branching process (see [14] and [1] respectively), TAS uses more refined techniques to avoid distributions.

A remarkable feature of TAS is the systematic use of information provided by the unitary implicants/implicates of the formula to achieve efficiency, and also to build countermodels (should the analyzed formula not be valid).

The techniques used have shown their usefulness for classical logic ([5]), multivalued logics ([6], [7]) and temporal logics ([8], [9], [10]). The aim of this paper is to establish the theoretical foundations to extend the TAS methodology to temporal propositional logic with an infinite, linear and discrete flow of time, named FNext±. This requires three objectives:

- The adequate choice of the definition of temporal literal in FNext±.
- Structure theorems in the poset of temporal literals.
- Introducing a suitable normal negative form in FNext± and defining a transformation to negative normal form with linear complexity.

The sets of unitary implicates and implicants of a formula can be infinite and therefore difficult to handle. In this work we study the structure of these sets and present the concept of *base* as the smallest finite set that generates them. Working with bases instead of the sets of implicates and implicants, we can treat them more efficiently. For this, we introduce a set of operators having linear time and space complexity.

This paper is organized in the following way. In section 2 we describe the logic we are going to work with. In the following one, we present the set of literals and study the structure of such a set. Such structure will be used to introduce the notion of *closed set of literals* and the notion of *base of a closed set* in section 4. To manage them in an efficient way, we structure those sets as Δ-lists and we also present the way to associate a Δ-list with every formula of the FNext± logic, where the information about implicants and implicates will be contained (sections 5 and 5.2). In section 6 the definition of temporal negative normal form is introduced and we demonstrate that this definition is the most appropriate to obtain a greater amount of information about the implicants and implicates of the well formed formulas in FNext±.

2. The FNext± Logic

The target logic, named FNext±, is the Temporal Propositional Logic with an infinite, linear and discrete flow of time. This logic was first introduced by Dana Scott in [19]. Von Wright also presents this logic in [22] and [23].

In [2], the author develops a resolution method for a new version of this logic.

Although most of the applications of Temporal Logics uses the US Logic [11], we have selected the FNext Logic as a preliminary logic to develop the first step of a new theory of implicants and implicates for Temporal Logic. This choice has been guided by the criteria of simplicity and expressive power.

Our object language contains the connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication), F (sometime in the future), G (always in the future), $\oplus$ (tomorrow), P (sometime in the past), H (always in the past), $\ominus$ (yesterday) and the symbols $\top$ (truth) and $\perp$ (falsity). $\mathcal{V}$ denotes the set of propositional variables $p, q, r, \dots$ (possibly with subscripts) which is assumed to be totally ordered with the lexicographical order, e.g. $p_n \leq q_m$ for all n, m and $p_n \leq p_m$ if and only if $n \leq m$.

We use the standard syntax and semantics:

Definition 2.1. The well-formed formulae (*wffs*) are generated by the construction rules of classical propositional logic together with the following rule:

- If A is a well-formed formula (*wff*), $\oplus A$, FA, GA, $\ominus A$, PA and HA are well-formed formulae.

Definition 2.2 (Hintikka Structure). A *temporal structure* is a tuple $S = (\mathbb{Z}, <, h)$, where $\mathbb{Z}$ is the set of integers, $<$ is the standard ordering on $\mathbb{Z}$, and h is a *temporal interpretation* which is a function $h : FNext\pm \longrightarrow 2^{\mathbb{Z}}$ satisfying:

1. $h(\top) = \mathbb{Z}; \quad h(\perp) = \emptyset$
2. $h(\neg A) = \mathbb{Z} - h(A)$
3. $h(A \vee B) = h(A) \cup h(B)$
4. $h(A \wedge B) = h(A) \cap h(B)$
5. $h(A \rightarrow B) = \big(\mathbb{Z} - h(A)\big) \cup\ h(B)$.
6. $t \in h(\oplus A)$ if and only if we have $t + 1 \in h(A)$
7. $t \in h(FA)$ if and only if there exists t' such that $t < t'$ and $t' \in h(A)$

8. $t \in h(GA)$ if and only if for all t' with $t < t'$ we have $t' \in h(A)$

9. $t \in h(\ominus A)$ if and only if we have $t - 1 \in h(A)$

10. $t \in h(PA)$ if and only if there exists t' such that $t > t'$ and $t' \in h(A)$

11. $t \in h(HA)$ if and only if for all t' with $t > t'$ we have $t' \in h(A)$

Definition 2.3. A formula A is said to be *satisfiable* if there exists a temporal structure $S = (\mathbb{Z}, <, h)$ such that $h(A) \neq \emptyset$ and, in this case, if $t \in h(A)$, then h is said to be a *model of A in t*. If $h(A) = \mathbb{Z}$, then A is said to be *true in the temporal structure* S, and we denote this by $\models_S A$. If A is true in every temporal structure, then A is said to be *valid*, and we denote this property by $\models A$.

Finally, $\equiv$ denotes semantic equality, i.e., $A \equiv B$ if and only if for every temporal structure $S = (\mathbb{Z}, <, h)$ we have that $h(A) = h(B)$.

3. The poset of temporal literals

As we mention in the introduction, we are interested in the study of the information contained in unitary implicants and implicates of the formulae in $FNext\pm$. In temporal logic the problem arises because the past and future temporal connectives increase the complexity of the relation among implicants and implicates. As a preliminary step, we define an order relation in $FNext\pm$.

Definition 3.1. We define a binary relation ,$\trianglelefteq$, in $FNext\pm$ as follows:

$$\text{If } A \text{ and } B \text{ are } \textit{wffs}, \quad A \trianglelefteq B \text{ if and only if } \models A \to B$$

Notice that there exist pairs of *wffs* A and B such that $A \equiv B$, but $A \neq B$. Therefore, $\trianglelefteq$ is not an order relation in FNext±. For example, the formulae $A = (p \vee q) \wedge (p \vee \neg q)$ and $B = p$ are equivalent.

Let $\Psi = FNext\pm/_{\equiv} = \{[A]_{\Psi} \mid A \in FNext\pm\}$, the relation induced by $\trianglelefteq$ on the set $FNext\pm/_{\equiv}$, denoted with the symbol $\trianglelefteq_{\Psi}$, is a partial order relation and the pair $(\Psi, \trianglelefteq_{\Psi})$ is a Boolean Algebra, where:

$$\neg[A]_{\Psi} = [\neg A]_{\Psi} \qquad [A]_{\Psi} \vee [B]_{\Psi} = [A \vee B]_{\Psi} \qquad [A]_{\Psi} \wedge [B]_{\Psi} = [A \wedge B]_{\Psi}$$

Let $FNext\pm^{mon}$ be the set formed with:

- the constants $\top$, $\bot$ and
- the formulae with the schema $A = \gamma_1 \dots \gamma_n \ell_p$ where $\ell_p \in \{p, \neg p \mid p \in \mathcal{V}\}$ and $\gamma_i \in \{F, G, \oplus, P, H, \ominus, \neg\}$ for all $1 \leq i \leq n$.

The superscript *mon* stands for *monary* and denotes that the formulae of $FNext\pm^{mon}$ do not have any binary connective.

We define the set $\Phi = FNext\pm^{mon}/_{\equiv}$. Note that Φ is not a subset of Ψ, but, for all $\ell \in FNext\pm^{mon}$, the following property of strict inclusion is satisfied in $FNext\pm$:

$$[\ell]_\Phi \subset [\ell]_\Psi$$

We define the function $\iota : \Phi \longrightarrow \Psi$ where

$$\iota([\ell]_\Phi) = [\ell]_\Psi$$

and the set of literals, Lit which is the image of ι; i.e.,

$$Lit = \mathrm{Im}(\iota) = \iota(\Phi) \subseteq \Psi$$

therefore, a literal is a class of Ψ which contains an element of $FNext\pm^{mon}$.

Example 1.- In $FNext\pm$ we have the following equalities:

$$\begin{aligned} [\neg P \oplus GF \oplus \oplus \neg G \neg \ominus p]_\Psi &= [HFGG \oplus \oplus \oplus \ominus \neg p]_\Psi = \\ &= [HFG \oplus \oplus \oplus \neg p]_\Psi \\ &= [FG \oplus \oplus \oplus \neg p]_\Psi = [FG\neg p]_\Psi \end{aligned}$$

We are interested in selecting a canonical form for each class, to deal with them in a more efficient way.

The following laws allow us to select the canonical form for each class (in appendix B we prove some of these laws):

- $\neg\neg A \equiv A$; $\neg \oplus A \equiv \oplus \neg A$; $\neg \ominus A \equiv \ominus \neg A$.
- $\neg FA \equiv G\neg A$; $\neg GA \equiv F\neg A$; $\neg PA \equiv H\neg A$; $\neg HA \equiv P\neg A$.
- $\oplus \ominus A \equiv \ominus \oplus A \equiv A$; $PFA \equiv FPA$; $HGA \equiv GHA$.
- $FFA \equiv F \oplus A$; $GGA \equiv G \oplus A$; $PPA \equiv P \ominus A$; $HHA \equiv H \ominus A$.
- $HFA \equiv F \ominus A$; $PGA \equiv G \ominus A$; $GPA \equiv P \oplus A$; $FHA \equiv H \oplus A$.

- If $\gamma \in \{F, G, P, H\}$ then $\oplus\gamma A \equiv \gamma \oplus A$ and $\ominus\gamma A \equiv \gamma \ominus A$.
- If $\gamma \in \{FG, GF, PH, HP, FP, GH\}$ then:

  $F\gamma A \equiv \gamma A;\quad G\gamma A \equiv \gamma A;\quad P\gamma A \equiv \gamma A;\quad H\gamma A \equiv \gamma A.$

  $\gamma \oplus A \equiv \gamma A;\quad \gamma \ominus A \equiv \gamma A.$

These laws allows us to select the canonical form. In appendix A.1, we present an algorithm which, with linear time and space complexity, compute the canonical form of the literals of FNext.

The set of literals can be characterized as follows:

$$Lit = \bigcup_{\ell_p \in \mathcal{V}^{\pm}} Lit\,(\ell_p)$$

where

- $\mathcal{V}^{\pm} = \{p, \neg p \mid p \in \mathcal{V}\}$ (the elements of this sets will be named *Classical Literals*)
- $$\begin{aligned} Lit(\ell_p) \;=\; & \{\top, \bot\} \cup \{FG\ell_p, GF\ell_p, PH\ell_p, HP\ell_p, FP\ell_p, GH\ell_p\} \\ & \cup \{\odot^k \ell_p, F\odot^k \ell_p, G\odot^k \ell_p, P\odot^k \ell_p, H\odot^k \ell_p \mid k \in \mathbb{Z}\} \end{aligned}$$

  where $\odot^k \ell_p$ stands for: $\oplus \overset{k}{\ldots} \oplus\ell_p$ if $k > 0$, ℓ_p if $k = 0$, and $\ominus \overset{k}{\ldots} \ominus\ell_p$ if $k < 0$.

The above definition, allows us to assert that the tuple $(Lit, \wedge, \vee)$, where $\wedge$ and $\vee$ are defined as the restriction of the homonymous operators over Lit, is a partial lattice.

In this partial lattice, the operators $\wedge$ and $\vee$ may be characterized as follows:

Let ℓ_1, ℓ_2 be literals, then:

1. If $\ell_1 \trianglelefteq \ell_2$ then $\ell_1 \wedge \ell_2 = \ell_1$ and $\ell_1 \vee \ell_2 = \ell_2$.
2. If $\ell_1 \in \overline{\ell_2}\downarrow$ then $\ell_1 \wedge \ell_2 = \bot$ and if $\ell_1 \in \overline{\ell_2}\uparrow$ then $\ell_1 \vee \ell_2 = \top$.
3. For all $k \in \mathbb{Z}$ and $\ell_p \in \mathcal{V}^{\pm}$:

   $G \odot^k \ell_p \wedge \odot^k \ell_p = G \odot^{k-1} \ell_p, \quad F \odot^k \ell_p \vee \odot^k \ell_p = F \odot^{k-1} \ell_p,$

   $H \odot^k \ell_p \wedge \odot^k \ell_p = H \odot^{k+1} \ell_p$ and $P \odot^k \ell_p \vee \odot^k \ell_p = P \odot^{k+1} \ell_p$.

4. Let $k_1, k_2 \in \mathbb{Z}$ and $\ell_p \in \mathcal{V}^{\pm}$. If $k_1 \geq k_2$, then

$$H \odot^{k_1} \ell_p \wedge G \odot^{k_2} \ell_p = GH\ell_p \text{ and } P \odot^{k_1} \ell_p \vee F \odot^{k_2} \ell_p = FP\ell_p.$$

We illustrate the functioning of these operators, with an example of the $\wedge$ operator:

EXAMPLE 2.- We can use items 3 and 4 to obtain the following equalities:

$$G \oplus^5 p \wedge \oplus^5 p \wedge \oplus^4 p \wedge \oplus^3 p \wedge H \oplus^4 p = G \oplus^2 p \wedge H \oplus^4 p = GHp$$

The pair $(Lit(\ell_p), \trianglelefteq)$ is a poset but it is not a lattice because there exist pairs of literals which do not have supremum or infimum. For example, the set of upper bounds of $\{p, \oplus p\}$ has two minimal elements, $P \oplus^2 p$ and $F \ominus p$. The ordered set $(Lit(p), \trianglelefteq)$ is depicted in Figure 1. However, if we only consider the future or past fragments,

$$\begin{aligned} Lit^+(\ell_p) &= \{\top, \bot\} \cup \{FG\ell_p, GF\ell_p\} \cup \{\oplus^k \ell_p, F \oplus^k \ell_p, G \oplus^k \ell_p \mid k \in \mathbb{N}\} \\ Lit^-(\ell_p) &= \{\top, \bot\} \cup \{PH\ell_p, HP\ell_p\} \cup \{\ominus^k \ell_p, P \ominus^k \ell_p, H \ominus^k \ell_p \mid k \in \mathbb{N}\} \end{aligned}$$

we have that $(Lit^+(\ell_p), \trianglelefteq)$ and $(Lit^-(\ell_p), \trianglelefteq)$ are lattices. The lattice $(Lit^+(\ell_p), \trianglelefteq)$ is depicted in Figure 2.

Definition 3.2. If $\ell \in Lit$ then $\overline{\ell}$ denotes its opposite literal and is defined as follows:

- $\overline{\bot} = \top$, $\overline{\top} = \bot$,
- If $\ell = \gamma_1 \dots \gamma_n \ell_p$ with $\gamma_i \in \{F, G, P, H, \oplus, \ominus\}$ for all $1 \leq i \leq n$ then $\overline{\ell} = \overline{\gamma_1} \dots \overline{\gamma_n} \overline{\ell_p}$ where:
  $\overline{p} = \neg p$; $\overline{\neg p} = p$; $\overline{\oplus} = \oplus$, $\overline{\ominus} = \ominus$, $\overline{G} = F$, $\overline{H} = P$, $\overline{F} = G$ and $\overline{P} = H$.

Definition 3.3. Let $\ell \in Lit$. We define its *upward* and *downward closures* as

$$\ell \uparrow = \{\ell' \in Lit \mid \ell \trianglelefteq \ell'\} \text{ and } \ell \downarrow = \{\ell' \in Lit \mid \ell' \trianglelefteq \ell\}$$

respectively. In addition, if $\Gamma \subseteq Lit$ then we define

$$\Gamma \uparrow = \bigcup_{\ell \in \Gamma} \ell \uparrow \qquad \text{and} \qquad \Gamma \downarrow = \bigcup_{\ell \in \Gamma} \ell \downarrow$$

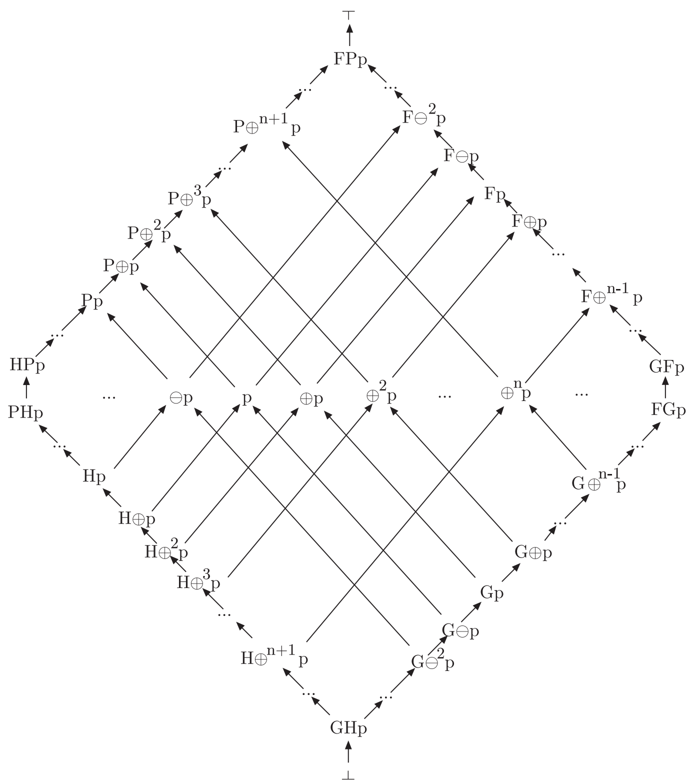


Figure 1: The ordered set $Lit(p)$

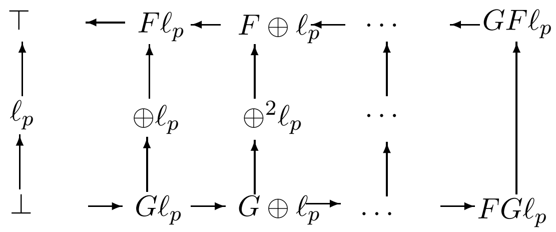


Figure 2: Lattice $Lit(\ell_p)$

Lemma 3.4. *Let $\ell_p \in \mathcal{V}^{\pm}$, $\Gamma \subseteq Lit(\ell_p)$ a finite set, and $\ell_0 \in Lit(\ell_p)$.*

1. $\models \wedge_{\ell\in\Gamma}\ell \to \ell_0$ *if and only if there exists* $\Gamma' \subseteq \Gamma$ *such that*

$$\wedge_{\ell\in\Gamma'}\ell \in Lit(\ell_p) \quad \textit{and} \quad \ell_0 \in (\wedge_{\ell\in\Gamma'}\ell)\uparrow$$

2. $\models \ell_0 \to \vee_{\ell\in\Gamma}\ell$ *if and only if there exists* $\Gamma' \subseteq \Gamma$ *such that*

$$\vee_{\ell\in\Gamma'}\ell \in Lit(\ell_p) \quad \textit{and} \quad \ell_0 \in (\vee_{\ell\in\Gamma'}\ell)\downarrow$$

Proof. We only show the proof of item 1. Item 2 is proved by duality.

The condition is, obviously, sufficient. To prove the necessity of it, we distinguish two cases :

(i) $\ell_0 \in Lit(\ell_p) - \big(\{G \odot^k \ell_p, H \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{GH\ell_p\}\big)$

(ii) $\ell_0 \in \{G \odot^k \ell_p, H \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{GH\ell_p\}$

(i) In this case, we prove that there exists $\ell \in \Gamma$ such that $\ell \trianglelefteq \ell_0$. We may consider the following cases:

$(i.1)$ $\ell_0 = \top$;	$(i.2)$ $\ell_0 = \bot$;	$(i.3)$ $\ell_0 \in \{\odot^k\ell_p \mid k \in \mathbb{Z}\}$;
$(i.4)$ $\ell_0 = FG\ell_p$;	$(i.5)$ $\ell_0 = GF\ell_p$;	$(i.6)$ $\ell_0 \in \{F \odot^k \ell_p \mid k \in \mathbb{Z}\}$;
$(i.7)$ $\ell_0 = PH\ell_p$;	$(i.8)$ $\ell_0 = HP\ell_p$;	$(i.9)$ $\ell_0 \in \{P \odot^k \ell_p \mid k \in \mathbb{Z}\}$.

If $\ell_0 = \top$, then it is obvious, because $\ell \trianglelefteq \top$ for all $\ell \in \Gamma$. If $\ell_0 = \bot$, then $\bigwedge_{\ell\in\Gamma} \ell \equiv \bot$ and, therefore, $\bot \in \Gamma$; which implies $\ell_0 \in \Gamma\uparrow$.

We only prove $(i.4)$ (the other cases can be proved similarly): suppose $\ell_0 = FG\ell_p$. If there does not exist $\ell \in \Gamma$ such that $\ell \trianglelefteq \ell_0$,

$$\Gamma \cap \Big(\{G \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{GH\ell_p, \bot\}\Big) = \emptyset$$

Furthermore, from the finiteness of Γ, there exists k_0 such that

- if $H \odot^k \ell_p \in \Gamma$ then $k < k_0$
- if $\odot^k \ell_p \in \Gamma$ then $k < k_0$

Therefore, for all $t \in \mathbb{Z}$, any interpretation $h : \mathcal{V} \to 2^{\mathbb{Z}}$ such that

$$h(\ell_p) = (-\infty, t + k_0] \cup \{2n \mid n \in \mathbb{Z}\}$$

satisfies $t \in h(\bigwedge_{\ell \in \Gamma} \ell)$ and $t \notin h(\ell_0) = h(FG\ell_p)$. So, $\not\models \bigwedge_{\ell \in \Gamma} \ell \to \ell_0$, contrary to the hypothesis. Therefore, there exists $\ell \in \Gamma$ such that $\ell \trianglelefteq \ell_0$.

(ii) Let $\ell_0 \in \{G \odot^k \ell_p, H \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{GH\ell_p\}$. In this case, we prove that there exists $\Gamma' \subseteq \Gamma$ such that $\bigwedge_{\ell \in \Gamma'} \ell \equiv \ell'$ and $\ell' \trianglelefteq \ell_0$. We may consider the following cases:

$(ii.1)$ $\ell_0 \in \{G \odot^k \ell_p \mid k \in \mathbb{Z}\}$
$(ii.2)$ $\ell_0 \in \{H \odot^k \ell_p \mid k \in \mathbb{Z}\}$
$(ii.3)$ $\ell_0 = GH\ell_p$

We only prove $(ii.3)$ (the other cases can be proved similarly).

if $\ell_0 = GH\ell_p$ then one of the following conditions is fulfilled:

- $GH\ell_p \in \Gamma$ or $\bot \in \Gamma$
- there exist k_1 and k_2 such that $H \odot^{k_1} \ell_p, G \odot^{k_2} \ell_p \in \Gamma$ and $k_1 > k_2$. In this case $H \odot^{k_1} \ell_p \wedge G \odot^{k_2} \ell_p \equiv GH\ell_p$ and $GH\ell_p \trianglelefteq \ell_0$
- there exist k_1 and k_2 such that $H \odot^{k_1} \ell_p, G \odot^{k_2} \ell_p \in \Gamma$, $k_1 \leq k_2$ and $\odot^k \ell_p \in \Gamma$ for all $k_1 \leq k \leq k_2$. In this case, we have

$$H \odot^{k_1} \ell_p \wedge \bigwedge_{k=k_1}^{k_2} \odot^k \ell_p \wedge G \odot^{k_2} \ell_p \equiv GH\ell_p \quad \text{and} \quad GH\ell_p \trianglelefteq \ell_0$$

We assume that these conditions are not true. Then, there exists $k_0 \in \mathbb{Z}$ such that:

- $\odot^{k_0} \ell_p \notin \Gamma$.
- If $G \odot^k \ell_p \in \Gamma$ then $k > k_0$.
- If $H \odot^k \ell_p \in \Gamma$ then $k < k_0$.

Given any $t \in \mathbb{Z}$, any interpretation h such that $h(\ell_p) = \mathbb{Z} - \{t + k_0\}$ satisfies $t \in h(\bigwedge_{\ell \in \Gamma} \ell)$. But $t \notin h(\ell_0) = h(GH\ell_p)$ and, consequently, $\not\models \bigwedge_{\ell \in \Gamma} \ell \to \ell_0$, contrary to the hypothesis. □

We conclude this section with the following definition:

Definition 3.5. Let B be a *wff*

- A literal ℓ is an implicate of B if $\models B \to \ell$.
- A literal ℓ is an implicant of B if $\models \ell \to B$.

4. Closed Sets of Literals

In this section, we consider sets of literals that will allow us to design an efficient treatment of sets of implicants and implicates of the formulae of FNext$\pm$. This efficient treatment is the heart of section 5.

Definition 4.1. A non-empty set $\Sigma \subseteq Lit(\ell_p)$ is said to be *α-closed* if it contains all the literals which are implicates of the conjunction of any two elements of Σ; i.e., for each $\ell_1, \ell_2 \in \Sigma$ and $\ell \in Lit(\ell_p)$, if $\models (\ell_1 \wedge \ell_2) \to \ell$, we have $\ell \in \Sigma$.

Dually, a non-empty set $\Sigma \subseteq Lit(\ell_p)$ is said to be *β-closed* if it contains all the literals which are implicants of the disjunction of any two elements of Σ; i.e., for each $\ell_1, \ell_2 \in \Sigma$ and $\ell \in Lit(\ell_p)$, if $\models \ell \to (\ell_1 \vee \ell_2)$, we have $\ell \in \Sigma$.

We extend the above definition to subsets of Lit as follows:

Definition 4.2. A non-empty set $\Sigma \subseteq Lit$ is said to be *α-closed* if $\Sigma \cap Lit(\ell_p)$ is α-closed for all $\ell_p \in \mathcal{V}^{\pm}$. Dually, a non-empty set $\Sigma \subseteq Lit$ is said to be *β-closed* if $\Sigma \cap Lit(\ell_p)$ is β-closed for all $\ell_p \in \mathcal{V}^{\pm}$.

As a direct consequence of the definition, $\top$ belongs to all α-closed sets and $\bot$ belongs to all β-closed sets.

Example 3.- The following set of literals is α-closed:

$$\begin{aligned} \Sigma = \quad & \{p, \top, FPp\} \cup \{F \odot^k p \mid k \in \mathbb{Z}\} \cup \{P \odot^k p \mid k \in \mathbb{Z}\} \\ \cup \quad & \{FGq, GFq, FPq, \top\} \cup \{G \oplus^k q \mid k \in \mathbb{N}\} \cup \{F \odot^k q \mid k \in \mathbb{Z}\} \\ \cup \quad & \{\oplus^k q \mid k \geqslant 2\} \cup \{P \oplus^k q \mid k \geqslant 3\} \end{aligned}$$

The following theorem, jointly with Lemma 3.4, expresses the good behaviour of the α and β-closed sets. [1]

[1] We use the terms α-closed and β-closed in honour of Smullyan.

Theorem 4.3. *Let $\Sigma \subseteq Lit(\ell_p)$ be a non-empty set of literals, then*

1. *Σ is α-closed if and only if the two following conditions are fulfilled:*
   (a) $\Sigma\uparrow = \Sigma$
   (b) *for all $\ell, \ell' \in \Sigma$ such that $\ell \wedge \ell' \in Lit$ we have $\ell \wedge \ell' \in \Sigma$.*
2. *Σ is β-closed if and only if the two following conditions are fulfilled:*
   (a) $\Sigma\downarrow = \Sigma$
   (b) *for all $\ell, \ell' \in \Sigma$ such that $\ell \vee \ell' \in Lit$ we have $\ell \vee \ell' \in \Sigma$*

Proof. We only show the proof for item 1 (item 2 can be proved similarly). First, we prove the necessary condition: if $\Sigma \subseteq Lit$ is α-closed, then:

(a) By definition, $\Sigma \subseteq \Sigma\uparrow$. Moreover, if $\ell \in \Sigma\uparrow$ there exists $\ell_1 \in \Sigma$ such that $\models \ell_1 \to \ell$. Therefore $\models (\ell_1 \wedge \ell_1) \to \ell$ and, by hypothesis, $\ell \in \Sigma$. Consequently, $\Sigma = \Sigma\uparrow$

(b) If $\ell_1, \ell_2 \in \Sigma$ and $\ell_1 \wedge \ell_2 \equiv \ell$, then $\models (\ell_1 \wedge \ell_2) \to \ell$ and, by hypothesis, $\ell \in \Sigma$.

Conversely, let $\Sigma \subseteq Lit$ satisfying conditions (a) and (b). If $\ell_1, \ell_2 \in \Sigma$ and $\models (\ell_1 \wedge \ell_2) \to \ell_0$, then Lemma 3.4 ensures that either $\ell_0 \in \Sigma\uparrow$, or $\ell_1 \wedge \ell_2 \equiv \ell \in Lit$ and $\ell \trianglelefteq \ell_0$. Therefore, from the hypothesis (a) and (b), we have $\ell_0 \in \Sigma$. □

Corollary 4.4. *The only sets which are both α-closed and β-closed are the sets Lit(lp) for p belonging $\mathcal{V}$.*

Our next objective is to characterize the minimal α-closed set (resp. β-closed set) which contains a given set of literals.

Definition 4.5. Given $\Sigma \subseteq Lit$, we define the α-closure of Σ, denoted by $\langle\Sigma\rangle^\alpha$, as follows:

$$\langle\Sigma\rangle^\alpha = \bigcup_{\ell_p \in \mathcal{V}^\pm} \left\{ \ell_0 \in Lit(\ell_p) \;\middle|\; \begin{array}{c} \text{there exists a finite set } \Gamma \subseteq \Sigma \cap Lit(\ell_p) \\ \text{such that } \models \wedge_{\ell\in\Gamma}\ell \to \ell_0 \end{array} \right\}$$

Dually, we define the β-closure of Σ, denoted by $\langle\Sigma\rangle^\beta$, as follows:

$$\langle\Sigma\rangle^\beta = \bigcup_{\ell_p\in\mathcal{V}^\pm} \left\{ \ell_0 \in Lit(\ell_p) \;\middle|\; \begin{array}{c}\text{there exists a finite set } \Gamma \subseteq \Sigma \cap Lit(\ell_p)\\ \text{such that } \models \ell_0 \to \vee_{\ell\in\Gamma}\ell\}\end{array} \right\}$$

We present the following lemma, which collects some particular closures:

Lemma 4.6. *Let $\ell, \ell_1, \ell_2, \ell_3 \in Lit$, then:*

1. $\langle\emptyset\rangle^\alpha = \{\top\}$, $\langle\{\top\}\rangle^\alpha = \{\top\}$, $\langle\{\bot\}\rangle^\alpha = Lit$
2. $\langle\{\ell\}\rangle^\alpha = \ell\uparrow$
3. *If* $\ell_1 \wedge \ell_2 \equiv \ell_3 \in Lit$, *then* $\langle\{\ell_1, \ell_2\}\rangle^\alpha = \ell_3\uparrow$

Theorem 4.7. *Assume $\Sigma \subseteq Lit$, then*

- $\langle\Sigma\rangle^\alpha$ *is the intersection of all the α-closed sets containing* Σ.
- $\langle\Sigma\rangle^\beta$ *is the intersection of all the β-closed sets containing* Σ.

Proof. If $\Sigma = \emptyset$ the proof is trivial (by $\langle\emptyset\rangle^\alpha = \{\top\}$). If $\Sigma \neq \emptyset$, it suffices to prove that the two following conditions are fulfilled (we only show the proof for the α-closed sets):

i. $\langle\Sigma\rangle^\alpha$ is α-closed; therefore, $\langle\Sigma\rangle^\alpha$ contains the intersection of all the α-closed sets containing Σ.

ii. If Σ' is an α-closed set with $\Sigma \subseteq \Sigma'$, then $\langle\Sigma\rangle^\alpha \subseteq \Sigma'$.

These two conditions are proved as follows:

i. Let us prove that $\langle\Sigma\rangle^\alpha$ is α-closed. From definition 4.2, all the sets $\langle\Sigma\rangle^\alpha \cap Lit(\ell_p)$ are α-closed. Then, given $\ell_1, \ell_2 \in \langle\Sigma\rangle^\alpha \cap Lit(\ell_p)$ and $\ell_0 \in Lit(\ell_p)$ such that $\models (\ell_1 \wedge \ell_2) \to \ell_0$, by the definition of $\langle\Sigma\rangle^\alpha$, there exist two finite sets $\Gamma_1, \Gamma_2 \subseteq \Sigma \cap Lit(\ell_p)$ such that:

$$\models \wedge_{\ell\in\Gamma_1}\ell \to \ell_1, \text{ and } \models \wedge_{\ell\in\Gamma_2}\ell \to \ell_2$$

Therefore, we have $\models (\bigwedge_{\ell\in\Gamma_1\cup\Gamma_2} \ell) \to \ell_0$, and, consequently, $\ell_0 \in \langle\Sigma\rangle^\alpha \cap Lit(\ell_p)$.

ii. Let $\Sigma' \subseteq Lit$ be an α-closed set such that $\Sigma \subseteq \Sigma'$. We will prove that $\langle\Sigma\rangle^\alpha \subseteq \Sigma'$.

If $\ell_0 \in \langle\Sigma\rangle^\alpha$, then there exists a finite set $\Gamma \subseteq Lit(\ell_p)$ such that $\Gamma \subseteq \Sigma \subseteq \Sigma'$ and

$$\models \bigwedge_{\ell\in\Gamma} \ell \to \ell_0$$

Since Σ' is α-closed, Lemma 3.4 ensures one of the following conditions:

- $\ell_0 \in \Gamma'\!\uparrow\, \subseteq \Sigma'\!\uparrow$ and, since Σ' is α-closed, $\ell_0 \in \Sigma'$
- there exists $\Gamma'' \subseteq \Gamma'$ such that $\bigwedge_{\ell\in\Gamma''} \ell \equiv \ell_0$ and, from Σ' α-closed, we conclude $\ell_0 \in \Sigma'$. □

Example 4.- For $\Gamma = \{\ominus p, p, Gp\}$ we have:

$$\begin{aligned}\langle\Gamma\rangle^\alpha \;&=\; \{F \odot^k p \mid k \in \mathbb{Z}\} \cup \{P \odot^k p \mid k \geq 0\} \cup \{\odot^k p \mid k \geq -1\}\\ &\cup \{G \odot^k p \mid k \geq -2\} \cup \{\top, FGp, GFp, FPp\}\end{aligned}$$

This example is illustrated in figure 3.

Example 5.- For $\Gamma = \{H \oplus^2 p, G \ominus^2 p\}$ we have:

$$\langle\Gamma\rangle^\alpha = GHp\uparrow = Lit(p) \smallsetminus \{\bot\}$$

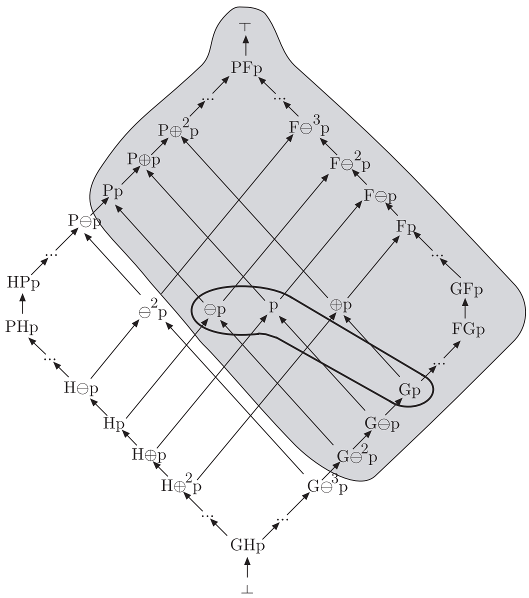
The previous theorem leads directly to the following result:

Corollary 4.8. *$\Sigma \subseteq Lit$ is α-closed if and only if $\Sigma = \langle\Sigma\rangle^\alpha$. Dually, Σ is β-closed if and only if $\Sigma = \langle\Sigma\rangle^\beta$.*

4.1 Unitary Implicants and Implicates

As we mention in the introduction, our goal is to manage implicit information contained in the formulae. So, we are interested in an efficient management of the sets of unitary implicates and implicants, denoted $\mathcal{I}_0(A)$ and $\mathcal{I}_1(A)$ respectively, which are closed sets:

Lemma 4.9. *Given a wff A, the set $\mathcal{I}_0(A)$ is α-closed and the set $\mathcal{I}_1(A)$ is β-closed.*



Note: The shadow set is the α-closure of $\Gamma = \{\ominus p, p, Gp\}$.

Figure 3: α-Closure of Γ

The following lemma shows some interesting properties about implicates and implicants:

Lemma 4.10. *Let A and B wffs .*

1. $\mathcal{I}_0(A) \cap \mathcal{I}_0(B) = \mathcal{I}_0(A \vee B)$
2. $\mathcal{I}_1(A) \cap \mathcal{I}_1(B) = \mathcal{I}_1(A \wedge B)$
3. $\langle \mathcal{I}_0(A) \cup \mathcal{I}_0(B)\rangle^\alpha \subseteq \mathcal{I}_0(A \wedge B)$
4. $\langle \mathcal{I}_1(A) \cup \mathcal{I}_1(B)\rangle^\beta \subseteq \mathcal{I}_1(A \vee B)$

Proof. We only prove items 1. and 3. (items 2. and 4. are duals).

1. $\mathcal{I}_0(A) \cap \mathcal{I}_0(B) = \mathcal{I}_0(A \vee B)$ is a direct consequence of the following property:

$$\models A \to \ell \text{ and } \models B \to \ell \quad \text{if and only if} \quad \models A \vee B \to \ell$$

3. If $\ell_0 \in \langle \mathcal{I}_0(A) \cup \mathcal{I}_0(B)\rangle^\alpha$, there exists $\Gamma \subseteq \mathcal{I}_0(A) \cup \mathcal{I}_0(B)$ finite such that $\models \wedge_{\ell\in\Gamma}\ell \to \ell_0$. Since $\models A \to \bigwedge_{\ell\in\Gamma\cap\mathcal{I}_0(A)} \ell$ and $\models B \to \bigwedge_{\ell\in\Gamma\cap\mathcal{I}_0(B)} \ell$, we have that $\models (A \wedge B) \to \ell_0$. □

As the following example shows, the above result can not be improved; i.e., there exist *wffs* A and B such that $\langle \mathcal{I}_0(A) \cup \mathcal{I}_0(B)\rangle^\alpha \subsetneq \mathcal{I}_0(A \wedge B)$:

EXAMPLE 6.- Let $A = p$ and $B = \neg p \vee \oplus p$, we have:

$$\mathcal{I}_0(A) = \{p, \top\}; \quad \mathcal{I}_0(B) = \{\top\}; \quad \langle \mathcal{I}_0(A) \cup \mathcal{I}_0(B)\rangle^\alpha = \{p, \top\}$$

$$\mathcal{I}_0(A \wedge B) = \langle\{p, \oplus p\}\rangle^\alpha = \{p, \oplus p, Fp, \top\}$$

The sets of unitary implicants and implicates in Fnext$\pm$ can be infinite. To deal with these sets, we need:

- A way to characterize them using finite sets.
- A set of efficient operators to manage them. These operators are based in the set-operators union and intersection.

These goals are covered in the following two sections.

4.2 α-bases and β-bases

In this section we characterize an efficient way of dealing with α-closed and β-closed sets. Therefore, our next step is to look for the minimal sets that generate them.

Definition 4.11. Let $\Sigma \subseteq Lit$ be an α-closed set, we say that a set $\Gamma \subseteq \Sigma$ is α-generating for Σ if and only if $\langle\Gamma\rangle^\alpha = \Sigma$.

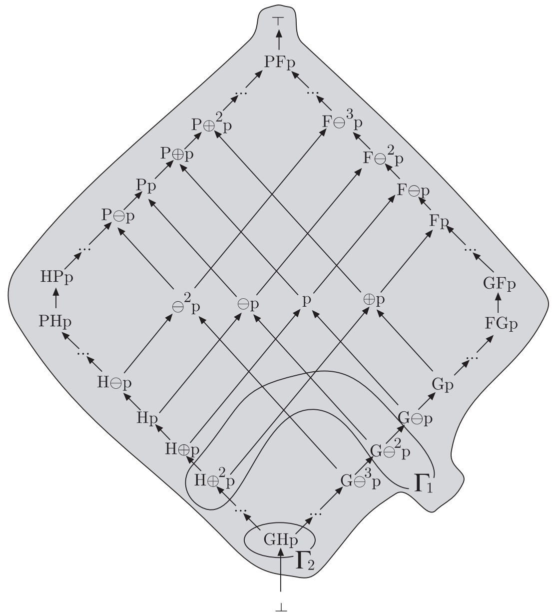
Dually, let $\Sigma \subseteq Lit$ an β-closed set, we say that a set $\Gamma \subseteq \Sigma$ is β-generating for Σ if and only if $\langle\Gamma\rangle^\beta = \Sigma$.

EXAMPLE 7.- $\Gamma_1 = \{H \oplus^2 p, G \ominus^2 p\}$ and $\Gamma_2 = \{GHp\}$ are α-generating sets for $\Sigma = Lit(p) \smallsetminus \{\bot\}$. These sets are depicted in Figure 4.

From now on, if there is no ambiguity, we will refer to these sets as *generating sets*, instead of α-generating or β-generating for Σ.

In the following definition, we introduce the concept of *base* for a closed set. The base is the *smallest* generating set: we shall see that, apart from having the lowest cardinality, the base has a lot of good properties that make it very interesting and easy to manage.

Definition 4.12. Let $\Sigma \subseteq Lit$ be an α-closed set. We say that $\Gamma \subseteq Lit$ is an α-base for Σ if the two following conditions are fulfilled:



Note: Γ_1 and Γ_2 are α-generating sets for the shadow set.

Figure 4: α-generating sets for Σ

- Γ is a non-empty α-generating set for Σ,
- all the elements of Γ are minimal elements of Σ in $(Lit, \trianglelefteq)$.

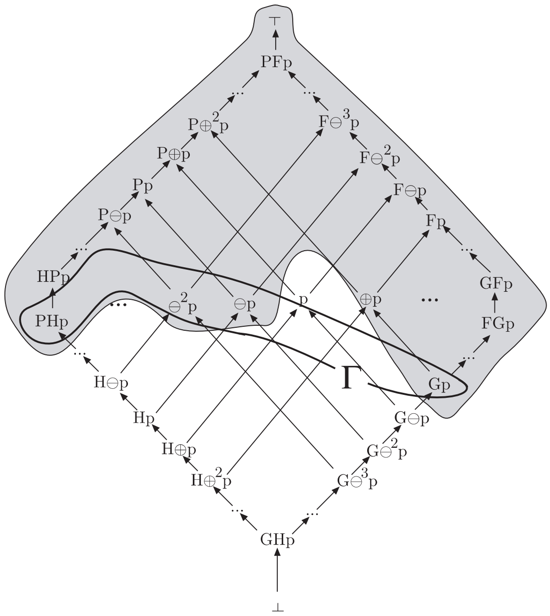
Dually, let $\Sigma \subseteq Lit$ be a β-closed set. We say that $\Gamma \subseteq Lit$ is a β-base for Σ if the two following conditions are fulfilled:

- Γ is a non-empty β-generating set for Σ,
- all the elements of Γ are maximal elements of Σ in $(Lit, \trianglelefteq)$.

EXAMPLE 8.- Let $p \in \mathcal{V}$, then:

- $\Gamma_2 = \{GHp\}$ is an α-base for $\Sigma = Lit(p) \smallsetminus \{\perp\}$ (see Figure 4).
- Let $\Gamma = \{PHp, \ominus^2 p, \ominus p, Gp\}$. Γ is the α-base for the following set (see Figure 5):

$$\{\top, PFp, HPp, PHp, GFp, FGp, \ominus^2 p, \ominus p\} \cup \{\oplus^k p \mid k > 0\} \cup$$
$$\cup \{P \odot^k p, F \odot^k p \mid k \in \mathbb{Z}\} \cup \{G \oplus^k p \mid k \in \mathbb{N}\}$$



Note: Γ is the α-base for the shadow set.

Figure 5: Non-unitary α-base.

As in the case of the generating sets, if there is no ambiguity, we will use the term *base* instead of α-base or β base.

Lemma 4.13. *Let $\Sigma \subseteq Lit$ be an α-closed set and let $\emptyset \neq \Gamma \subseteq Lit$. Then Γ is a base for Σ if and only if the two following conditions are fulfilled:*

- *Γ is a generating set for Σ.*
- *For all $\ell_p \in \mathcal{V}^{\pm}$ and for all $\ell, \ell' \in \Gamma \cap Lit(\ell_p)$ $(\ell \neq \ell')$ we have that $\ell \wedge \ell' \notin Lit(\ell_p)$.*

Dually, given $\Sigma \subseteq Lit$ be a β-closed set and let $\emptyset \neq \Gamma \subseteq Lit$. Then Γ is a base for Σ if and only if the two following conditions are fulfilled:

- *Γ is a generating set for Σ.*
- *For all $\ell_p \in \mathcal{V}^{\pm}$ and for all $\ell, \ell' \in \Gamma \cap Lit(\ell_p)$ $(\ell \neq \ell')$ we have that $\ell \vee \ell' \notin Lit(\ell_p)$.*

As a direct consequence of this lemma, we have the following results which characterize the contents of the bases:

Lemma 4.14. *Let $\Sigma \subseteq Lit$ be an α-closed (or β-closed) set and let Γ be a base for Σ. For all $\ell_p \in \mathcal{V}^{\pm}$ we denote $\Gamma_{\ell_p} = \Gamma \cap Lit(\ell_p)$. Then*

- *If there exists $\ell \in \{\bot, \top\} \cap \Gamma$ then $\Gamma = \{\ell\}$.* [2]
- *If there exists $\ell \in \{GH\ell_p, FP\ell_p\} \cap \Gamma_{\ell_p}$ then $\Gamma_{\ell_p} = \{\ell\}$.*
- *Γ_{ℓ_p} contains at most one element of the set*

$$\{FG\ell_p, GF\ell_p\} \cup \{F \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{G \odot^k \ell_p \mid k \in \mathbb{Z}\}$$

  and at most one element of the set

$$\{PH\ell_p, HP\ell_p\} \cup \{P \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{H \odot^k \ell_p \mid k \in \mathbb{Z}\}$$

Lemma 4.15. *Let $\Sigma \subseteq Lit$. For all $\ell_p \in \mathcal{V}^{\pm}$ we denote $\Gamma_{\ell_p} = \Gamma \cap Lit(\ell_p)$. Then*

i. If Σ is α-closed and Γ is a base for Σ, then the following conditions are fulfilled:

- *If $\{H \odot^k \ell_p, G \odot^{k'} \ell_p\} \subseteq \Gamma_{\ell_p}$ then $k' > k$*
- *If $G \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' < k$*
- *If $F \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' \leqslant k$*
- *If $H \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' > k$.*
- *If $P \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' \geqslant k$.*

ii. Dually, if Σ is β-closed and Γ is a base for Σ, then the following conditions are fulfilled:

- *If $\{P \odot^k \ell_p, F \odot^{k'} \ell_p\} \subseteq \Gamma_{\ell_p}$ then $k' > k$*
- *If $F \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' < k$*
- *If $G \odot^k \ell_p \in \Gamma_{\ell_p}$, any other literal $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ fulfills $k' \leqslant k$*

[2] In particular, if Σ is α-closed and $\top \in \Gamma$, then $\Sigma = \Gamma = \{\top\}$. Besides, if Σ is α-closed and $\bot \in \Gamma$, then $\Sigma = \Gamma = \{\bot\}$.

- *If* $P \odot^k \ell_p \in \Gamma_{\ell_p}$*, any other literal* $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ *fulfills* $k' > k$
- *If* $H \odot^k \ell_p \in \Gamma_{\ell_p}$*, any other literal* $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ *fulfills* $k' \geqslant k$

The following lemma illustrates the importance of the bases.

Lemma 4.16. *Let* $\Sigma \subseteq Lit$ *be* α*-closed and let* Γ *be a base for* Σ*, then:*

$$\langle \Gamma \rangle^{\alpha} = \Gamma \uparrow = \Sigma$$

Dually, let $\Sigma \subseteq Lit$ *be* β*-closed and let* Γ *be a base for* Σ*, then:*

$$\langle \Gamma \rangle^{\beta} = \Gamma \downarrow = \Sigma$$

The following theorem ensures the uniqueness of the bases.

Theorem 4.17. *Let* Σ *be* α*-closed (respectively* β*-closed). If* Γ *is a base for* Σ*, then* Γ *is the only base for* Σ*.*

Moreover, if Γ *is finite, then any other non-empty generating set for* Σ*,* Γ'*, satisfies* $|\Gamma| < |\Gamma'|$ [3]*.*

Proof. Let $\Sigma \subseteq Lit$ be α-closed and Γ_1 and Γ_2 two different bases for Σ then, there exists $\ell \in Lit - \{\perp\}$ such that $\ell \in \Gamma_1$ and $\ell \notin \Gamma_2$. Since Γ_2 is an α-base for Σ, lemma 4.16 ensures that there must exist $\ell' \in \Gamma_2$ such that $\ell \neq \ell'$ and $\ell \in \ell' \uparrow$. Therefore, ℓ is not a minimal element of Σ and Γ_1 is not a base for Σ.

Now, we suppose that the second assertion of this theorem is not true. Then there exists Γ' ($\Gamma \neq \Gamma'$) generating for Σ such that $|\Gamma'| \leq |\Gamma|$.

So, we can define a list of α-generating sets for Σ, $\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots$, such that $\Gamma_0 = \Gamma'$ and $|\Gamma| \geq |\Gamma_0| > |\Gamma_1| > |\Gamma_2| > \dots > |\Gamma_n| > \dots$ as follows:

From the uniqueness of the bases, Γ_i is an α-generating set for Σ, but it is not a base. Therefore, lemma 4.13 ensures that there must exist two literals $\ell, \ell' \in \Gamma_i$ ($\ell \neq \ell'$), such that $\ell \wedge \ell' \equiv \ell''$ for some other literal ℓ''. Now, we define

$$\Gamma_{i+1} = \big(\Gamma_i - \{\ell, \ell'\}\big) \cup \{\ell''\}$$

The sequence $\Gamma_0, \Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots$ must be finite. Furthermore, there exist $m \in \mathbb{N}$ such that Γ_m is the last element in this sequence [4]. Therefore

[3] $|\Gamma|$ denotes the cardinality of Γ.

[4] Although it is not necessary for the proof, it can be proved that Γ_m is a singleton, as a consequence of lemma 4.13.

Γ_m is a base, contrary to the hypothesis. The proof for the β-bases is analogous. $\square$

Lemma 4.13 leads to the definition of two *normalization* operators which transform any finite generating set for a given α-closed or β-closed set into its corresponding base. The definition of these normalization operators is based in the lemmas 4.14 and 4.15:

Definition 4.18. Let $\Gamma \subseteq Lit$ be a finite set, the *0-normalizor* operator, denoted by $\mathcal{N}_0$, and the *1-normalizor* operator, denoted by $\mathcal{N}_1$, perform the following transformations:

- $\mathcal{N}_0(\emptyset) = \{\top\}$ and $\mathcal{N}_1(\emptyset) = \{\bot\}$
- – If $\top \in \Gamma$ then $\mathcal{N}_0(\Gamma) = \mathcal{N}_1(\Gamma) = \top$;
  – If $\bot \in \Gamma$ then $\mathcal{N}_0(\Gamma) = \mathcal{N}_1(\Gamma) = \bot$. [5]
- Let $\Gamma_{\ell_p} = \Gamma \cap Lit(\ell_p)$, then $\mathcal{N}_i(\Gamma) = \bigcup_{\Gamma_{\ell_p}\neq\varnothing} \mathcal{N}_i(\Gamma_{\ell_p})$ for $i = 1, 2$, where:

  – If $GH\ell_p \in \Gamma_{\ell_p}$ then $\mathcal{N}_0(\Gamma_{\ell_p}) = \mathcal{N}_1(\Gamma_{\ell_p}) = GH\ell_p$;
  – If $FP\ell_p \in \Gamma_{\ell_p}$ then $\mathcal{N}_0(\Gamma_{\ell_p}) = \mathcal{N}_1(\Gamma_{\ell_p}) = FP\ell_p$. [6]

  In other cases,

- $\mathcal{N}_0$ carries out the following operations:

  (1) If $\{FG\ell_p, GF\ell_p\} \cup \{F \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{G \odot^k \ell_p \mid k \in \mathbb{Z}\} \cap \Gamma_{\ell_p} \neq \emptyset$ then $\mathcal{N}_0$ replaces this set by the minimum of its elements, named ℓ_1.

  (2) If $\{PH\ell_p, HP\ell_p\} \cup \{P \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{H \odot^k \ell_p \mid k \in \mathbb{Z}\} \cap \Gamma_{\ell_p} \neq \emptyset$ then $\mathcal{N}_0$ replaces this set by the minimum of its elements, named ℓ_2.

  (3) – If $\ell_1 = F \odot^k \ell_p$ and there exists $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ such that $k' < k$ then $\mathcal{N}_0$ removes ℓ_1.

  – If $\ell_2 = P \odot^k \ell_p$ and there exists $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ such that $k' > k$, then $\mathcal{N}_0$ removes ℓ_2.

  (4) If $\ell_1 \in \{G \odot^k \ell_p \mid k \in \mathbb{Z}\}$ and/or $\ell_2 \in \{H \odot^k \ell_p \mid k \in \mathbb{Z}\}$, then:

[5] From lemma 4.14, only one of these two elements belongs to Γ.
[6] From lemma 4.14, only one of these two elements belongs to Γ_{ℓ_p}.

– If $\ell_1 = G \odot^k \ell_p$, $\ell_2 = H \odot^{k'} \ell_p$ and $k' > k$ then $\mathcal{N}_0(\Gamma_{\ell_p}) = GH\ell_p$.

In other cases,

– $\mathcal{N}_0$ applies, as often as possible, the law $G \odot^k \ell_p \wedge \odot^k \ell_p = G \odot^{k-1} \ell_p$, replacing $\{G \odot^k \ell_p, \odot^k \ell_p\}$ by $\ell'_1 = G \odot^{k-1} \ell_p$, and, as often as possible, the law $H \odot^k \ell_p \wedge \odot^k \ell_p = H \odot^{k-1} \ell_p$, replacing $\{H \odot^k \ell_p, \odot^k \ell_p\}$ by $\ell'_2 = H \odot^{k-1} \ell_p$.

Moreover, if $\ell'_1 = G \odot^n \ell_p$, removes the literals $\odot^{k'} \ell_p$ where $k' > n$, and if $\ell'_2 = H \odot^n \ell_p$, removes the literals $\odot^{k'} \ell_p$ where $k' < n$.

- Dually, $\mathcal{N}_1$ carries out the following operations:

  (1′) If $\{FG\ell_p, GF\ell_p\} \cup \{F \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{G \odot^k \ell_p \mid k \in \mathbb{Z}\} \cap \Gamma_{\ell_p} \neq \emptyset$ then $\mathcal{N}_1$ replaces this set by the maximum of its elements, named ℓ_1.

  (2′) If $\{PH\ell_p, HP\ell_p\} \cup \{P \odot^k \ell_p \mid k \in \mathbb{Z}\} \cup \{H \odot^k \ell_p \mid k \in \mathbb{Z}\} \cap \Gamma_{\ell_p} \neq \emptyset$ then $\mathcal{N}_1$ replaces this set by the maximum of its elements, named ℓ_2.

  (3′) – If $\ell_1 = G \odot^k \ell_p$ and there exists $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ such that $k' < k$ then $\mathcal{N}_1$ removes ℓ_1.

  – If $\ell_2 = H \odot^k \ell_p$ and there exists $\odot^{k'} \ell_p \in \Gamma_{\ell_p}$ such that $k' > k$, then $\mathcal{N}_1$ removes ℓ_2.

  (4′) If $\ell_1 \in \{F \odot^k \ell_p \mid k \in \mathbb{Z}\}$ and/or $\ell_2 \in \{P \odot^k \ell_p \mid k \in \mathbb{Z}\}$, then:

  – If $\ell_1 = F \odot^k \ell_p$, $\ell_2 = P \odot^{k'} \ell_p$ and $k' > k$ then $\mathcal{N}_1(\Gamma_{\ell_p}) = FP\ell_p$.

  In other cases,

  – $\mathcal{N}_1$ applies, as often as possible, the law , $F \odot^k \ell_p \vee \odot^k \ell_p = F \odot^{k-1} \ell_p$ replacing $\{F \odot^k \ell_p, \odot^k \ell_p\}$ by $\ell'_1 = F \odot^{k-1} \ell_p$, and, as often as possible, the law $P \odot^k \ell_p \vee \odot^k \ell_p = P \odot^{k-1} \ell_p$, replacing $\{P \odot^k \ell_p, \odot^k \ell_p\}$ by $\ell'_2 = P \odot^{k-1} \ell_p$.

  Moreover, if $\ell'_1 = F \odot^n \ell_p$, removes the literals $\odot^{k'} \ell_p$ where $k' > n$, and if $\ell'_2 = P \odot^n \ell_p$, removes the literals $\odot^{k'} \ell_p$ where $k' < n$.

EXAMPLE 9.- Let $\Gamma = \{F \oplus^3 p, \oplus p, G \oplus^5 p, \oplus^3 p, H \oplus^2 p\}$.

- The operator $\mathcal{N}_0$ yields the following α-base for $\langle \Gamma \rangle^\alpha$:

  Since neither $\top$, $\bot$, FPp nor GHp are in Γ, (from (1) in definition 4.18) $\mathcal{N}_0$ replaces $\{F \oplus^3 p, G \oplus^5 p\}$ by their minimum $\ell_1 = G \oplus^5 p$ and fixes $\ell_2 = H \oplus^2 p$.

Then (from (4) in definition 4.18), removes the literals $\odot^k p$ with $k > 5$ and $k < 2$, rendering:

$$\mathcal{N}_0(\Gamma) = \{H \oplus^2 p, \oplus^3 p, G \oplus^5 p\}$$

- The operator $\mathcal{N}_1$ yields the following β-base for $\langle\Gamma\rangle^\beta$:

  Since neither $\top$, $\bot$, FPp nor GHp are in Γ (from (1') in definition 4.18), $\mathcal{N}_1$ replaces $\{F \oplus^3 p, G \oplus^5 p\}$ by their maximum $\ell_1 = F \oplus^3 p$ and fixes $\ell_2 = H \oplus^2 p$.

  Then (from (3') in definition 4.18), since $\ell_2 = H \oplus^2 p$ and $\oplus^3 p \in \Gamma_{\ell_p}$, $H \oplus^2 p$ is removed.

  Finally (from (4') in definition 4.18), since $F \oplus^3 p \vee \oplus^3 p \equiv F \oplus^2 p$, replaces $\{F \oplus^3 p, \oplus^3 p\}$ by $F \oplus^2 p$.

  Therefore, the β-base for $\langle\Gamma\rangle^\beta$ is

$$\mathcal{N}_1(\Gamma) = \{F \oplus^2 p, \oplus p\}$$

EXAMPLE 10.- Let $\Gamma = \{G \oplus p, G \oplus^5 p, \oplus p, \oplus^3 p\} \cup \{\oplus s, \oplus^2 s, \oplus^3 s, FPs\}$. The operator $\mathcal{N}_0$ yields the following α-base for $\langle\Gamma\rangle^\alpha$:

$$\mathcal{N}_0(\Gamma) = \{Gp, FPs\}$$

and the operator $\mathcal{N}_1$ yields the following β-base for $\langle\Gamma\rangle^\beta$:

$$\mathcal{N}_1(\Gamma) = \{G \oplus^5 p, \oplus p, \oplus^3 p, FPs\}$$

4.3 Union and intersection of closed sets

In this section we present two suitable set operators of union and intersection over closed sets.

As a direct consequence of the definition of closed sets, we obtain the following lemma:

Lemma 4.19. *Let $\Sigma, \Sigma' \subseteq Lit$. Then, if Σ and Σ' are α-closed (respectively β-closed) then $\Sigma \cap \Sigma'$ is α-closed (resp. β-closed).*

The following example shows that this property is not true for the union of closed sets.

EXAMPLE 11.- Let $\Sigma = \oplus p \uparrow$ and $\Sigma' = G \oplus p \uparrow$. These sets are α-closed, but $\Sigma \cup \Sigma'$ is not α-closed, because $\oplus p, G \oplus p \in \Sigma \cup \Sigma'$, $\oplus p \wedge G \oplus p \equiv Gp$ and $Gp \notin \Sigma \cup \Sigma'$.

For any two given α-closed (resp. β-closed) sets, Σ and Σ', we are interested in the sets $\Sigma \cap \Sigma'$ and $\langle \Sigma \cup \Sigma' \rangle^\alpha$ (resp. $\Sigma \cap \Sigma'$ and $\langle \Sigma \cup \Sigma' \rangle^\beta$). More concretely, we want to characterize the bases for these two sets. For this purpose, we define new binary operators over finite sets of literals:

Definition 4.20. Let $\Gamma, \Gamma' \subseteq Lit$ be two finite set of literals. We define $\Gamma \Cup_0 \Gamma'$ and $\Gamma \Cup_1 \Gamma'$, called *0-union* and *1-union* of Γ and Γ' respectively, as follows:

$$\Gamma \Cup_0 \Gamma' \stackrel{\text{def}}{=} \mathcal{N}_0(\Gamma \cup \Gamma'); \qquad \Gamma \Cup_1 \Gamma' \stackrel{\text{def}}{=} \mathcal{N}_1(\Gamma \cup \Gamma')$$

As a direct consequence of this definition, we obtain the following lemma:

Lemma 4.21. *Let $\Sigma, \Sigma' \subseteq Lit$ be two α-closed (resp. β-closed) sets and Γ,Γ' the bases for Σ and Σ' respectively. Then $\Gamma \Cup_0 \Gamma'$ (resp. $\Gamma \Cup_1 \Gamma'$) is the base for $\langle \Sigma \cup \Sigma' \rangle^\alpha$ (resp. $\langle \Sigma \cup \Sigma' \rangle^\beta$).*

As we shall see in the next section, these two union operators, $\Cup_0$ and $\Cup_1$, have a linear cost when they are applied to ordered bases.

Definition 4.22. Given $\ell_1, \ell_2 \in Lit$, $\mathcal{B}_\alpha(\ell_1, \ell_2)$ denotes the α-base for $\ell_1 \uparrow \cap \, \ell_2 \uparrow$, and $\mathcal{B}_\beta(\ell_1, \ell_2)$ denotes the β-base for $\ell_1 \downarrow \cap \, \ell_2 \downarrow$.

Definition 4.23. For all $\ell_1, \ell_2 \in Lit$, we define

$$\begin{aligned}
\mathcal{B}_\beta^-(\ell_1, \ell_2) &= \mathcal{B}_\beta(\ell_1, \ell_2) \cap \Big(\{PHp, HPp\} \cup \{P \odot^k p, H \odot^k p \mid k \in \mathbb{Z}\}\Big) \\
\mathcal{B}_\beta^+(\ell_1, \ell_2) &= \mathcal{B}_\beta(\ell_1, \ell_2) \cap \Big(\{FGp, GFp\} \cup \{F \odot^k p, G \odot^k p \mid k \in \mathbb{Z}\}\Big) \\
\mathcal{B}_\alpha^-(\ell_1, \ell_2) &= \mathcal{B}_\alpha(\ell_1, \ell_2) \cap \Big(\{PHp, HPp\} \cup \{P \odot^k p, H \odot^k p \mid k \in \mathbb{Z}\}\Big) \\
\mathcal{B}_\alpha^+(\ell_1, \ell_2) &= \mathcal{B}_\alpha(\ell_1, \ell_2) \cap \Big(\{FGp, GFp\} \cup \{F \odot^k p, G \odot^k p \mid k \in \mathbb{Z}\}\Big)
\end{aligned}$$

EXAMPLE 12.- Let $\ell_1 = p$ and $\ell_2 = Gp$, then:

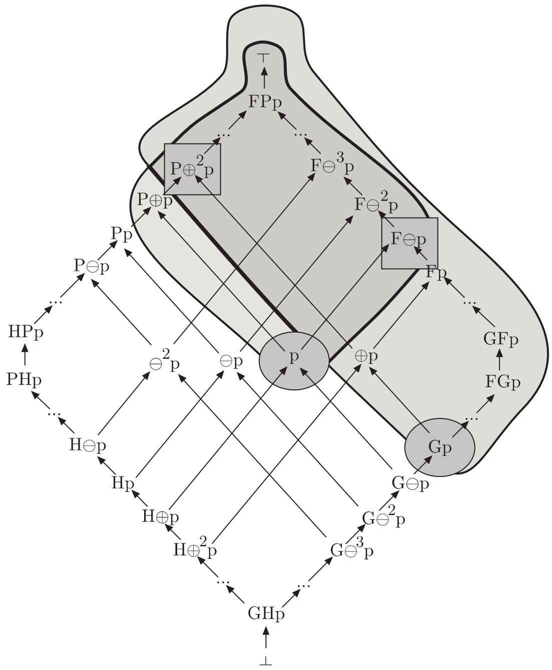


Figure 6: Minimal elements of the set of upper bounds.

$$\mathcal{B}_\alpha(\ell_1,\ell_2)=\{P\oplus^2 p, F\ominus p\};\quad \mathcal{B}^-_\alpha(\ell_1,\ell_2)=\{P\oplus^2 p\};\quad \mathcal{B}^+_\alpha(\ell_1,\ell_2)=\{F\ominus p\}$$

This situation is illustrated in figure 6.

Lemma 4.24. *Let* $\ell_1,\ell_2 \in Lit$.

i. $\ell \in \mathcal{B}_\alpha(\ell_1,\ell_2)$ *if and only if it is a minimal element of the set of upper bounds of* $\{\ell_1,\ell_2\}$.

ii. $\ell \in \mathcal{B}_\beta(\ell_1,\ell_2)$ *if and only if it is a maximal element of the set of lower bounds of* $\{\ell_1,\ell_2\}$.

Proof. If $\ell_1 \in Lit(\ell_p), \ell_2 \in Lit(\ell_{p'})$ and $p \neq p'$, then $\mathcal{B}_\alpha(\ell_1,\ell_2)=\{\top\}$ and $\mathcal{B}_\beta(\ell_1,\ell_2)=\{\bot\}$. Hence, the result arises trivially.

If $\ell_1,\ell_2 \in Lit(\ell_p)$, we prove the result by detailing all the possible situations of item 2 (see figure 7). Item 1 can be proved similarly:

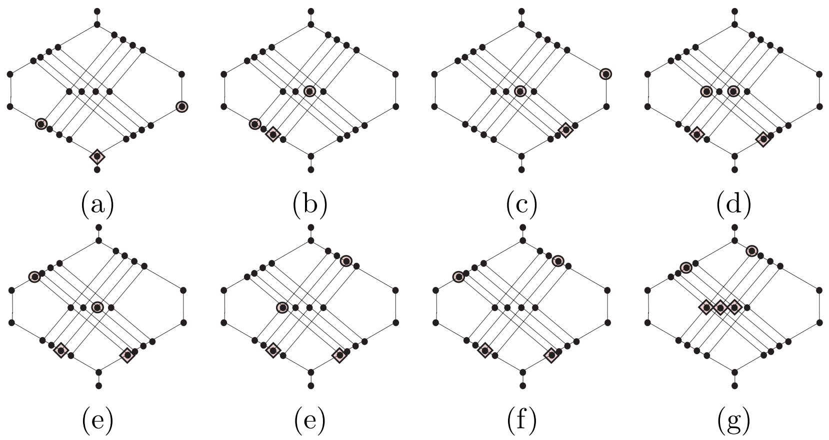


Figure 7: Cases for $\mathcal{B}_\beta(\ell_1, \ell_2)$

i. If $\ell_1 \wedge \ell_2 \in Lit(\ell_p)$ then $\mathcal{B}_\alpha(\ell_1, \ell_2) = \{\ell_1 \wedge \ell_2\}$

ii. In other cases:

(a) If $\ell_1 \trianglelefteq HP\ell_p$ and $\ell_2 \trianglelefteq GF\ell_p$ then $\mathcal{B}_\beta(\ell_1, \ell_2) = \{GH\ell_p\}$

(b) If $\ell \trianglelefteq HP\ell_p$ then $\mathcal{B}_\beta(\ell, \odot^k \ell_p) = \{H \odot^{k+1} \ell_p\}$

(c) If $\ell \trianglelefteq GF\ell_p$ then $\mathcal{B}_\beta(\ell, \odot^k \ell_p) = \{G \odot^{k-1} \ell_p\}$

(d) If $k_2 > k_1$ then $\mathcal{B}_\beta(\odot^{k_1} \ell_p, \odot^{k_2} \ell_p) = \{H \odot^{k_2+1} \ell_p, G \odot^{k_1-1} \ell_p\}$

(e) If $k_2 \geq k_1$ then $\mathcal{B}_\beta(P \odot^{k_1} \ell_p, \odot^{k_2} \ell_p) = \{H \odot^{k_2+1} \ell_p, G \odot^{k_1-2} \ell_p\}$ and $\mathcal{B}_\beta(\odot^{k_1} \ell_p, F \odot^{k_2} \ell_p) = \{H \odot^{k_2+2} \ell_p, G \odot^{k_1-1} \ell_p\}$

(f) If $k_2 > k_1 - 1$ then $\mathcal{B}_\beta(P \odot^{k_1} \ell_p, F \odot^{k_2} \ell_p) = \{H \odot^{k_2+2} \ell_p, G \odot^{k_1-2} \ell_p\}$

(g) If $k_2 < k_1$ then $\mathcal{B}_\beta(P \odot^{k_1} \ell_p, F \odot^{k_2} \ell_p) = \{\odot^k \ell_p \mid k_2 < k < k_1\}$

□

Definition 4.25. Let $\Gamma, \Gamma' \subseteq Lit$ be two finite sets, we define $\Gamma \Cap_0 \Gamma'$ and $\Gamma \Cap_1 \Gamma'$, called *0-intersection* and *1-intersection* of Γ and Γ' respectively, as follows:

$$\Gamma \Cap_0 \Gamma' \stackrel{\text{def}}{=} \mathcal{N}_0\Big(\bigcup_{\substack{\ell \in \Gamma \\ \ell' \in \Gamma'}} \mathcal{B}_\alpha(\ell, \ell')\Big); \qquad \Gamma \Cap_1 \Gamma' \stackrel{\text{def}}{=} \mathcal{N}_1\Big(\bigcup_{\substack{\ell \in \Gamma \\ \ell' \in \Gamma'}} \mathcal{B}_\beta(\ell, \ell')\Big)$$

Theorem 4.26. *Let $\Sigma, \Sigma' \subseteq Lit$ be two finite α-closed (resp. β-closed) sets and Γ,Γ' the bases for Σ and Σ' respectively, then $\Gamma \Cap \Gamma'$ (resp. $\Gamma \Cap \Gamma'$) is the base for $\Sigma \cap \Sigma'$.*

Proof. Since Σ and Σ' are α-closed sets, we have $\Sigma = \Gamma\uparrow$ and $\Sigma' = \Gamma'\uparrow$.
Consequently,
$\Sigma \cap \Sigma' = \Gamma\uparrow \cap \Sigma'\uparrow = (\cup_{\ell\in\Gamma}\ell\uparrow) \cap (\cup_{\ell\in\Gamma'}\ell\uparrow) = \bigcup_{\substack{\ell\in\Gamma\\ \ell'\in\Gamma'}}(\ell\uparrow \cap \ell'\uparrow)$, which is α-closed.

Since for all $\ell \in \Gamma$ and $\ell' \in \Gamma'$ the base for $\ell\uparrow \cap \ell'\uparrow$ is $\mathcal{B}_\alpha(\ell,\ell')$, Lemma 4.21 ensures that $\Sigma \cap \Sigma' = \mathcal{N}_0\Big(\bigcup_{\substack{\ell\in\Gamma\\ \ell'\in\Gamma'}}\mathcal{B}_\alpha(\ell,\ell')\Big) = \Gamma \Cap \Gamma'$ □

> **Remark** 13: The operators $\Cup, \uplus, \Cap$ and $\Cap$ are commutative and associative (which allows us to write $\Cup_{i=1}^n \Gamma_i$, $\uplus_{i=1}^n \Gamma_i$, $\Cap_{i=1}^n \Gamma_i$ and $\Cap_{i=1}^n \Gamma_i$).

In the following section we consider the bases for the closed sets as ordered lists. The order on the closed sets ensures a linear complexity for both, union and intersection operators.

5. Δ-lists of literals

As we mention in the introduction, we are looking for an efficient treatment of implicants and implicates sets. In this section we introduce a new structure, called Δ-list, that allows us an efficient management of unitary implicants and implicates.

These kind of list was introduced for first time in [5] for the case of Propositional Classical Logic (PCL) and have been used exhaustively in the design of Automated Theorem Provers[7, 9]. The sets of implicates and implicants are stored in lists because we want to improve the efficient of the operators defined over them. In all of these works, the Δ-lists does not need to have any structure, because the sets of implicants and implicates of all formula of PCL are finite sets. Besides that, the relation $\trianglelefteq$ in PCl is trivial and does not provide any significant information.

The concepts of generating set and base are uninteresting in PCL; nevertheless, in FNext±, these concepts are unavoidable if we want to manage efficiently the sets of implicants and implicates. The Δ-lists in PCL are sets of implicants and implicates, unlike in FNext$\pm$, the Δ-lists are the bases for these sets.

Definition 5.1. Let λ be a finite list of literals in $Lit(\ell_p)$ and let $\Gamma_\lambda = \{\ell \mid \ell \in \lambda\}$ be the set of elements in λ. Then λ is said to be a $\Delta_0^{\ell_p}$-list (respectively a $\Delta_1^{\ell_p}$-list) if the following conditions are fulfilled:

i. $\top \notin \Gamma_\lambda$ and $\bot \notin \Gamma_\lambda$.

ii. Γ_λ, is a base for $\langle\Gamma_\lambda\rangle^\alpha$ (respectively a base for $\langle\Gamma_\lambda\rangle^\beta$).

iii. Let $\mathcal{D} = \{PH\ell_p, HP\ell_p\} \cup \{H \odot^k \ell_p, P \odot^k \ell_p \mid k \in \mathbb{Z}\}$ and ℓ be the *first* element of Γ_λ. One of the following condition are fulfilled:

$$\Gamma_\lambda \cap \mathcal{D} = \emptyset \quad \text{or} \quad \Gamma_\lambda \cap \mathcal{D} = \{\ell\}$$

iv. Let $\mathcal{D} = \{FG\ell_p, GF\ell_p\} \cup \{G \odot^k \ell_p, F \odot^k \ell_p \mid k \in \mathbb{Z}\}$ and ℓ be the *last* element of Γ_λ. One of the following condition are fulfilled:

$$\Gamma_\lambda \cap \mathcal{D} = \emptyset \quad \text{or} \quad \Gamma_\lambda \cap \mathcal{D} = \{\ell\}$$

v. Their literals of the type $\odot^k \ell_{p_i}$ are located following a decreasing order in k.

Definition 5.2. A finite list of literals, λ, is said to be a Δ_0-list (respectively a Δ_1-list) if the following conditions are fulfilled:

- λ is $\bot$, $\top$ or the concatenation of the lists $\lambda = \lambda_{\ell_{p_1}} \langle\rangle \lambda_{\ell_{p_2}} \langle\rangle \cdots \langle\rangle \lambda_{\ell_{p_n}}$ where:
  i. The sublists $\lambda_{\ell_{pi}}$ are $\Delta_0^{\ell_{p_i}}$-lists (respectively $\Delta_1^{\ell_{p_i}}$-lists).
  ii. The sublists $\lambda_{\ell_{pi}}$ are located following the order in $\mathcal{V}^\pm$ of their subindex $\ell_{p_1}, \ell_{p_2}, \cdots, \ell_{p_n}$.

We extend the normalization operators, $\mathcal{N}_0$ and $\mathcal{N}_1$ to lists of literals, adding to the transformation introduced in definition 4.18, the ordering given in definition 5.2.

Example 14.- Given the list of literals

$$\lambda = [F \oplus^3 p, F \oplus^2 q, \oplus p, \oplus^2 q, \oplus^5 \overline{q}, G \oplus^3 r, G \oplus^5 p, G \oplus^5 \overline{q}, q, \oplus^3 p]$$

the operator $\mathcal{N}_0$ generates the Δ_0-list:

$$\mathcal{N}_0(\lambda) = [G \oplus^5 p, \oplus^3 p, \oplus p, F \oplus^2 q, \oplus^2 q, q, G \oplus^4 \overline{q}, G \oplus^3 r]$$

and the operator $\mathcal{N}_1$ generates the Δ_1-list:

$$\mathcal{N}_1(\lambda) = [F \oplus^2 p, \oplus p, F \oplus q, q, G \oplus^5 \overline{q}, \oplus^5 \overline{q}, G \oplus^3 r]$$

5.1 Set-operators over Δ-lists

All the operators over the bases we have introduced above have a linear time and space complexity when they are applied to Δ-lists (ordered bases). This assertion is trivial in the case of the union operators. The intersection operators require a more detailed explanation.

The structure of the bases (Lemmas 4.14 and 4.15) allows us to ensure that the two intersection operators ($\Cap_0$ and $\Cap_1$) have a linear complexity when they are applied to Δ-lists. Before formalizing it, we will illustrate this assertion with the following example.

Example 15.-

Let $\lambda_1 = [P \ominus^2 p, \ominus p, \oplus p, G \oplus^2 p]$, $\lambda_2 = [p, \oplus p, F \oplus^2 p]$ be two Δ-lists.

First, we compute $\mathcal{B}_\beta(\ell, \ell')$ for all $\ell \in \lambda_1, \ell' \in \lambda_2$:

$\mathcal{B}_\beta(P \ominus^2 p, p) = \{\boxed{\mathbf{H \oplus p}}, G \ominus^4 p\}$	$\mathcal{B}_\beta(\ominus p, p) = \{H \oplus p, G \ominus^2 p\}$
$\mathcal{B}_\beta(P \ominus^2 p, \oplus p) = \{H \oplus^2 p, G \ominus^4 p\}$	$\mathcal{B}_\beta(\ominus p, \oplus p) = \{H \oplus^2 p, G \ominus^2 p\}$
$\mathcal{B}_\beta(P \ominus^2 p, F \oplus^2 p) = \{H \oplus^4 p, G \ominus^4 p\}$	$\mathcal{B}_\beta(\ominus p, F \oplus^2 p) = \{H \oplus^4 p, G \ominus^2 p\}$
$\mathcal{B}_\beta(\oplus p, p) = \{H \oplus^2 p, G \ominus p\}$	$\mathcal{B}_\beta(G \oplus^2 p, p) = \{G \ominus p\}$
$\mathcal{B}_\beta(\oplus p, \oplus p) = \{\boxed{\mathbf{\oplus p}}\}$	$\mathcal{B}_\beta(G \oplus^2 p, \oplus p) = \{Gp\}$
$\mathcal{B}_\beta(\oplus p, F \oplus^2 p) = \{H \oplus^4 p, Gp\}$	$\mathcal{B}_\beta(G \oplus^2 p, F \oplus^2 p) = \{\boxed{\mathbf{G \oplus^2 p}}\}$

The union of all these sets renders the set $\Gamma = \{H \oplus p, G \ominus^4 p, G \ominus^2 p, H \oplus^2 p, H \oplus^4 p, G \ominus p, \oplus p, Gp, G \oplus^2 p\}$, therefore $\lambda_1 \Cap_1 \lambda_2 = \mathcal{N}_1(\Gamma)$.

Since neither $\top$, $\bot$, FPp nor GHp are in Γ (from (1') in definition 4.18), $\mathcal{N}_1$ replaces the set $\{G \ominus^4 p, G \ominus^2 p, G \ominus p, Gp, G \oplus^2 p\}$ by their maximum

$\ell_1 = G \oplus^2 p$, and replaces $\{H \oplus p, H \oplus^2 p, H \oplus^4 p\}$ by their maximum $\ell_1 = H \oplus p$.

Therefore, the final result is:

$$\lambda_1 \Cap \lambda_2 = \{H \oplus p, \oplus p, G \oplus^2 p\}$$

As we have shown, the computation of all the sets $\mathcal{B}_\beta(\ell, \ell')$ where $\ell \in \lambda, \ell' \in \lambda'$ is superfluous. As a direct consequence of the relation $\unlhd$ over Lit, in most cases, it is sufficient to compute only those sets $\mathcal{B}_\beta(\ell, \ell')$ which correspond with the first and the last element of each Δ-list.

This assertion is based in the following lemma, which ensures a linear time and space complexity for the intersection of two Δ-lists. In appendix A.2, we show an algorithm which compute the 0-intersection of two Δ_0-lists.

Lemma 5.3. *Let $\ell_p \in \mathcal{V}^{\pm}$ and let $\lambda = [\ell_1, \ell_2, \ldots, \ell_n]$, $\lambda' = [\ell'_1, \ell'_2, \ldots, \ell'_m]$ be two Δ^{ℓ_p}-lists.*

- *If λ and λ' are two $\Delta_0^{\ell_p}$-lists, then $\lambda \Cap_0 \lambda'$ is:*
  - *i. If $\lambda = FP\ell_p$ or $\lambda' = FP\ell_p$, then $\lambda \Cap_0 \lambda' = FP\ell_p$*
  - *ii. If $\lambda = GH\ell_p$ then $\lambda \Cap_0 \lambda' = \lambda'$ and, if $\lambda' = GH\ell_p$ then $\lambda \Cap_0 \lambda' = \lambda$*
  - *iii. In other case*

$$\lambda \Cap_0 \lambda' \;=\; \mathcal{N}_0\big(\texttt{Prefix}_0(\lambda, \lambda')\langle\,\rangle\texttt{Body}_0(\lambda, \lambda')\langle\,\rangle\texttt{Sufix}_0(\lambda, \lambda')\big)$$

where:

$$\texttt{Prefix}_0(\lambda, \lambda') = \begin{cases} \ell_1 \uparrow \cap \lambda' & \textit{if } \ell_1 \in \{H \odot^k \ell_p | k \in \mathbb{Z}\} \textit{ and } \ell'_1 \in \ell_1 \uparrow \\ \lambda \cap \ell'_1 \uparrow & \textit{if } \ell'_1 \in \{H \odot^k \ell_p | k \in \mathbb{Z}\} \textit{ and } \ell_1 \in \ell'_1 \uparrow \\ \mathcal{B}^-_\alpha(\ell_1, \ell'_1) & \textit{in other case} \end{cases}$$

$$\texttt{Body}_0(\lambda, \lambda') = \lambda \cap \lambda' \cap \{\odot^k \ell_p \mid k \in \mathbb{Z}\}$$

$$\texttt{Sufix}_0(\lambda, \lambda') = \begin{cases} \ell_n \uparrow \cap \lambda' & \textit{if } \ell_n \in \{G \odot^k \ell_p | k \in \mathbb{Z}\} \textit{ and } \ell'_m \in \ell_n \uparrow \\ \lambda \cap \ell'_m \downarrow & \textit{if } \ell'_m \in \{G \odot^k \ell_p | k \in \mathbb{Z}\} \textit{ and } \ell_n \in \ell'_m \uparrow \\ \mathcal{B}^+_\alpha(\ell_n, \ell'_m) & \textit{in other case} \end{cases}$$

- *If λ and λ' are two $\Delta_1^{\ell_p}$-lists, then $\lambda \Cap_1 \lambda'$ is:*

i. *If* $\lambda = GH\ell_p$ *or* $\lambda' = GH\ell_p$*, then* $\lambda \Cap \lambda' = GH\ell_p$

ii. *If* $\lambda = FP\ell_p$ *then* $\lambda \Cap \lambda' = \lambda'$ *and, if* $\lambda' = FP\ell_p$ *then* $\lambda \Cap \lambda' = \lambda$

iii. *In other case*

$$\lambda \Cap \lambda' \quad = \quad \mathcal{N}_1\big(\mathtt{Prefix}_1(\lambda,\lambda')\langle\,\rangle\mathtt{Body}_1(\lambda,\lambda')\langle\,\rangle\mathtt{Sufix}_1(\lambda,\lambda')\big)$$

where:

$$\mathtt{Prefix}_1(\lambda,\lambda') = \begin{cases} \ell_1\downarrow\cap\lambda' & \textit{if } \ell_1 \in \{P\odot^k \ell_p | k\in\mathbb{Z}\} \textit{ and } \ell'_1 \in \ell_1\downarrow \\ \lambda\cap\ell'_1\downarrow & \textit{if } \ell'_1 \in \{P\odot^k \ell_p | k\in\mathbb{Z}\} \textit{ and } \ell_1 \in \ell'_1\downarrow \\ \mathcal{B}^-_\beta(\ell_1,\ell'_1) & \textit{in other case} \end{cases}$$

$$\mathtt{Body}_1(\lambda,\lambda') = \lambda\cap\lambda'\cap\{\odot^k\ell_p \mid k\in\mathbb{Z}\}$$

$$\mathtt{Sufix}_1(\lambda,\lambda') = \begin{cases} \ell_n\downarrow\cap\lambda' & \textit{if } \ell_n \in \{F\odot^k \ell_p | k\in\mathbb{Z}\} \textit{ and } \ell'_m \in \ell_n\downarrow \\ \lambda\cap\ell'_m\downarrow & \textit{if } \ell'_m \in F\odot^k \ell_p | k\in\mathbb{Z}\} \textit{ and } \ell_n \in \ell'_m\downarrow \\ \mathcal{B}^+_\beta(\ell_n,\ell'_m) & \textit{in other case} \end{cases}$$

Example 16.- For the Δ_1-lists in Example 15,

$$\lambda_1 = [P\ominus^2 p, \ominus p, \oplus p, G\oplus^2 p]; \qquad \lambda_2 = [p, \oplus p, F\oplus^2 p],$$

we have:

$$\mathtt{Prefix}_1(\lambda_1,\lambda_2) = [H\ominus p]$$
$$\mathtt{Body}_1(\lambda_1,\lambda_2) = [\oplus p]$$
$$\mathtt{Sufix}_1(\lambda_1,\lambda_2) = [G\oplus^2 p]$$

Therefore: $\lambda_1 \Cap \lambda_2 = [H\ominus p, \oplus p, G\oplus^2 p]$

Example 17.- For the Δ_1-lists,

$$\lambda_1 = [P\oplus^2 r, \oplus^4 r, \oplus^5 r, FGr]; \qquad \lambda_2 = [H\ominus^3 r, \ominus^2 r, r, \oplus^4 r, \oplus^7 r],$$

we have:

$$\mathtt{Prefix}_1(\lambda_1,\lambda_2) = [H\ominus^3 r, \ominus^2 r, r]$$
$$\mathtt{Body}_1(\lambda_1,\lambda_2) = [\oplus^4 r]$$
$$\mathtt{Sufix}_1(\lambda_1,\lambda_2) = [G\oplus^6 r]$$

Therefore: $\lambda_1 \Cap \lambda_2 = [H\ominus^3 r, \ominus^2 r, r, \oplus^4 r, G\oplus^6 r]$

The following lemma generalize the last one to Δ-list.

Lemma 5.4. *Let $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}^{\pm}$. If $\lambda_1 = \langle\,\rangle_{\ell_p \in \mathcal{V}_1} \lambda_1(\ell_p)$, $\lambda_2 = \langle\,\rangle_{\ell_p \in \mathcal{V}_2} \lambda_2(\ell_p)$ are two Δ-lists where $\lambda_i(\ell_p) = \lambda_i \cap Lit(\ell_p)$ then*

$$\lambda_1 \sqcap_0 \lambda_2 = \langle\,\rangle_{\ell_p \in \mathcal{V}_1 \cap \mathcal{V}_2} \Big(\lambda_1(\ell_p) \sqcap_0 \lambda_2(\ell_p)\Big)$$

$$\lambda_1 \sqcap_1 \lambda_2 = \langle\,\rangle_{\ell_p \in \mathcal{V}_1 \cap \mathcal{V}_2} \Big(\lambda_1(\ell_p) \sqcap_1 \lambda_2(\ell_p)\Big)$$

EXAMPLE 18.- Given two Δ_1-lists:
$\lambda_1 = [P\ominus^2 p, \ominus p, \oplus p, G\oplus^2 p, PF\overline{p}, P\oplus^2 r, \oplus^4 r, \oplus^5 r, FGr, H\oplus s, \oplus^5 s, GFs]$
$\lambda_2 = [p, \oplus p, F\oplus^2 p, HPq, \oplus q, FGq, H\ominus^3 r, \ominus^2 r, r, \oplus^4 r, \oplus^7 r]$

$$\begin{aligned}
\lambda_1 \sqcap_1 \lambda_2 &= [P\ominus^2 p, \ominus p, \oplus p, G\oplus^2 p] \sqcap_1 [p, \oplus p, F\oplus^2 p]\langle\,\rangle \\
&\quad\; [P\oplus^2 r, \oplus^4 r, \oplus^5 r, FGr] \sqcap_1 [H\ominus^3 r, \ominus^2 r, r, \oplus^4 r, \oplus^7 r] \\
&= [H\oplus p, \oplus p, G\oplus^2 p, H\ominus^3 r, \ominus^2 r, r, \oplus^4 r, G\oplus^6 r]
\end{aligned}$$

5.2 Other operators over Δ-lists

In this section we introduce new operators that are used in the manipulation of the Δ-lists. We begin with this preliminary definition:

Definition 5.5. Let $\lambda = [\ell_1, \ldots, \ell_n]$ be a Δ-list. We define the opposite of this Δ-list, denoted $\overline{\lambda}$, as follows: $[\overline{\ell_1}, \ldots, \overline{\ell_n}]$

As a direct consequence, if λ is a Δ_0-list then $\overline{\lambda}$ is a Δ_1-list and vice versa.

Lemma 5.6. *Let A be wff. If λ_0 is the base for $\mathcal{I}_0(A)$ and λ_1 is the base for the set $\mathcal{I}_1(\neg A)$, then*

$$\overline{\lambda_0} = \lambda_1 \quad \textit{and} \quad \overline{\lambda_1} = \lambda_0$$

Proof. It is a direct consequence of the following property:

$$\models A \to \ell \quad \text{if and only if} \quad \models \neg\ell \to \neg A$$

□

Now, we introduce the new operators that will be used later on.

Definition 5.7. For each temporal connective $\star \in \{\oplus, F, G, \ominus, P, H\}$ and each Δ-list, λ, we define two operators, called $\mathtt{Add}^0_\star$ and $\mathtt{Add}^1_\star$ as follows:

$$\mathtt{Add}^0_\star(\lambda) = \mathcal{N}_0(\{\ell \in Lit \mid \ell \equiv \star\ell' \text{ with } \ell' \in \lambda\})$$
$$\mathtt{Add}^1_\star(\lambda) = \mathcal{N}_1(\{\ell \in Lit \mid \ell \equiv \star\ell' \text{ with } \ell' \in \lambda\})$$

EXAMPLE 19.- The list $\lambda = [\oplus p, \oplus^3 p, FGp, \oplus^3 q, G\oplus^4 q]$ is a Δ_0-list and a Δ_1-list, and we have:

$$\begin{aligned}
\mathtt{Add}^0_F(\lambda) &= \mathcal{N}_0([F\oplus p, F\oplus^3 p, FGp, F\oplus^3 q, FGq]) = [FGp, FGq]\\
\mathtt{Add}^0_H(\lambda) &= \mathcal{N}_0([H\oplus p, H\oplus^3 p, FGp, H\oplus^3 q, HGq])\\
&= [H\oplus^3 p, FGp, HGq]\\
\mathtt{Add}^1_F(\lambda) &= \mathcal{N}_1([F\oplus p, F\oplus^3 p, FGp, F\oplus^3 q, FGq]) = [F\oplus p, F\oplus^3 q]\\
\mathtt{Add}^1_P(\lambda) &= \mathcal{N}_1([P\oplus p, P\oplus^3 p, FGp, P\oplus^3 q, G\oplus^3 q])\\
&= [P\oplus^3 p, FGp, P\oplus^3 q, G\oplus^3 q]
\end{aligned}$$

The following theorem, illustrates the importance of these new operators.

Theorem 5.8. *Let A be a wff.*

i. If λ is the Δ_0-list which is the base for $\mathcal{I}_0(A)$, then:

(a) if $\star \in \{\oplus, F, \ominus, P\}$ then $\mathtt{Add}^0_\star(\lambda)$ is the Δ_0-list for $\mathcal{I}_0(\star A)$

(b) $\langle \mathtt{Add}^0_G(\lambda)\rangle^\alpha \subseteq \mathcal{I}_0(GA)$ and $\langle \mathtt{Add}^0_H(\lambda)\rangle^\alpha \subseteq \mathcal{I}_0(HA)$

ii. Dually, if λ is the Δ_1-list which is the base for $\mathcal{I}_1(A)$, then

(a) if $\star \in \{\oplus, G, \ominus, H\}$ then $\mathtt{Add}^1_\star(\lambda)$ is the Δ_1-list for $\mathcal{I}_1(\star A)$

(b) $\langle \mathtt{Add}^1_F(\lambda)\rangle^\alpha \subseteq \mathcal{I}_1(FA)$ and $\langle \mathtt{Add}^1_P(\lambda)\rangle^\alpha \subseteq \mathcal{I}_1(PA)$

Proof. We only prove the item i (item iia can be proved similarly).

1a: – The proof for $\odot = \oplus$ or $\ominus$ is based on the following property:

$$\models A \to \ell \quad \text{if and only if} \quad \models \odot A \to \odot\ell$$

– $\langle \mathtt{Add}^0_F(\lambda)\rangle^\alpha \subseteq \mathcal{I}_0(FA)$:

For all $\ell \in \langle \mathtt{Add}^0_F(\lambda)\rangle^\alpha$ there exists $\ell' \in \lambda \subseteq \mathcal{I}_0(A)$ such that $\models F\ell' \rightarrow \ell$. On the other hand, if $\models A \rightarrow \ell'$ we have $\models FA \rightarrow F\ell'$ and, consequently, $\models FA \rightarrow \ell$.

Now, we prove the other inclusion $\langle \mathtt{Add}^0_F(\lambda)\rangle^\alpha \supseteq \mathcal{I}_0(FA)$.

As a previous result, we prove that if $\ell \in \mathcal{I}_0(FA)$ then $H\ell \in \mathcal{I}_0(A)$:

From the hypothesis ($\ell \in \mathcal{I}_0(FA)$), for any interpretation $h : \mathcal{V} \longrightarrow 2^{\mathbb{Z}}$ we have $h(FA) \subseteq h(\ell)$. Therefore, any $t \in h(A)$ satisfies $(-\infty, t) \subseteq h(FA) \subseteq h(\ell)$ and, consequently, $t \in h(H\ell)$.

On the other hand, since λ is a Δ_0-list for $\mathcal{I}_0(A)$ there must exists a literal $\ell' \in \lambda$ such that $\models \ell' \rightarrow H\ell$. Therefore, $\models F\ell' \rightarrow FH\ell$ and, since $\models FH\ell \rightarrow \ell$ we may ensure that $\ell \in \langle \mathtt{Add}^0_F(\lambda)\rangle^\alpha$

– The proof for $\langle \mathtt{Add}^0_P(\lambda)\rangle^\alpha = \mathcal{I}_0(PA)$ can be obtained from $\langle \mathtt{Add}^0_P(\lambda)\rangle^\alpha = \mathcal{I}_0(PA)$ and mirror law.

1b: If $\ell \in \langle \mathtt{Add}^0_G(\lambda)\rangle^\alpha$, then there exists $\ell' \in \lambda$ such that $\models G\ell' \rightarrow \ell$ and $\models A \rightarrow \ell'$. Therefore, $\models GA \rightarrow G\ell'$ and $\ell \in \mathcal{I}_0(GA)$.

If $\ell \in \langle \mathtt{Add}^0_H(\lambda)\rangle^\alpha$, then there exists $\ell' \in \lambda$ such that $\models H\ell' \rightarrow \ell$ and $\models A \rightarrow \ell'$. Therefore, $\models HA \rightarrow H\ell'$ and $\ell \in \mathcal{I}_0(HA)$. □

As the following example shows, the above result can not be improved; i.e., there exist *wffs* in FNext± such that $\langle \mathtt{Add}^0_G(\lambda)\rangle^\alpha \subsetneq \mathcal{I}_0(GA)$:

EXAMPLE 20.- Let $A = (p \vee q) \wedge F\neg p$, we have that $\mathcal{I}_0(A) = \langle\{F\overline{p}\}\rangle^\alpha = \{F\overline{p}, \top\}$ and $\lambda = F\overline{p}$, but;

$$\begin{aligned} \langle \mathtt{Add}^0_G(\{F\overline{p}\})\rangle^\alpha &= \{F \oplus^k \overline{p} \mid k \in \mathbb{N}\} \cup \{GF\overline{p}, \top\} \\ \mathcal{I}_0(GA) &= \langle\{GF\overline{p}, GFq\}\rangle^\alpha = \\ &= \{F \oplus^k \overline{p}, F \oplus^k q \mid k \in \mathbb{N}\} \cup \{GF\overline{p}, GFq, \top\} \end{aligned}$$

6. Temporal Negative Normal Form

As we mentioned in the introduction, in this last section we will define a negative normal form equivalent to a formula A in FNext±. First, let's see a motivation of the forthcoming formal definition:

Theorem 5.8 ensures that if A be *wff*, λ is the base for $\mathcal{I}_0(A)$ and λ' is the base for the set $\mathcal{I}_1(A)$, then:

$$\mathcal{I}_0(GA) \supseteq \langle \mathrm{Add}^0_G(\lambda)\rangle^{\alpha}; \quad \mathcal{I}_1(FA) \supseteq \langle \mathrm{Add}^1_F(\lambda')\rangle^{\beta}$$

$$\mathcal{I}_0(HA) \supseteq \langle \mathrm{Add}^0_H(\lambda)\rangle^{\alpha}; \quad \mathcal{I}_1(PA) \supseteq \langle \mathrm{Add}^1_P(\lambda')\rangle^{\beta}$$

Nevertheless, these inclusions become equalities when A is a literal. Therefore, we are interested in expressing A with as few as possible occurrences of F and G and with the least range for them.

On the other hand, Lemma 4.10 ensures that if A and B are *wffs*, then:

$$\mathcal{I}_0(A) \cup \mathcal{I}_0(B) \subseteq \mathcal{I}_0(A \wedge B); \quad \mathcal{I}_1(A) \cup \mathcal{I}_1(B) \subseteq \mathcal{I}_1(A \vee B)$$

To reduce the information loss caused by the above inclusions, we must enlarge the range of the conjunctions and the disjunctions.

These two aims guided the following definition:

Definition 6.1. The notion of temporal negative normal form, denoted *nnf*$_t$, is recursively defined as follows:

i. Any literal is a *nnf*$_t$

ii. If $A_1, \dots, A_n$ are *nnf*$_t$*s*, then $\vee_{i=1}^{i=n} A_i$ and $\wedge_{i=1}^{i=n} A_i$ are *nnf*$_t$*s*

iii. $F(\wedge_{i\in\{1,\cdots n\}} A_i)$, $P(\wedge_{i\in\{1,\cdots n\}} A_i)$, $GF(\wedge_{i\in\{1,\cdots n\}} A_i)$, $HP(\wedge_{i\in\{1,\cdots n\}} A_i)$, $FP(\wedge_{i\in\{1,\cdots n\}} A_i)$, $G(\vee_{i\in\{1,\cdots n\}} A_i)$, $FG(\vee_{i\in\{1,\cdots n\}} A_i)$, $GH(\vee_{i\in\{1,\cdots n\}} A_i)$, $H(\vee_{i\in\{1,\cdots n\}} A_i)$ and $PH(\vee_{i\in\{1,\cdots n\}} A_i)$ are *nnf*$_t$ if the three following conditions are fulfilled:

(a) $A_1, \dots, A_n$ are *nnf*$_t$*s*

(b) There does not exists an A_i of the form γB with

$$\gamma \in \{FP, GH, FG, GF, PH, HP\}$$

(c) If for all $i \in \{1, \cdots n\}$, we have $A_i = \gamma_i B_i$ where $\gamma_i \in \{F, G, P, H\}$ then there exist $i_1, i_2 \in \{1, \cdots n\}$ such that $\gamma_{i_1} \in \{F, H\}$ and $\gamma_{i_2} \in \{G, P\}$

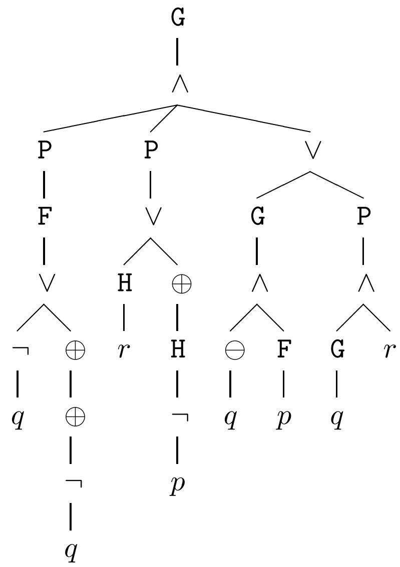
Note that, from this definition, a formula of $FNext\pm$ is in nnf_t if the monary connectives have the minimum scope.

From now on, we will use formula or syntactic tree without distinction, to refer to a nnf_t.

EXAMPLE 21.-

$$G\Big(PF\big(\neg q \vee \oplus \oplus \neg q\big) \wedge P\big(Hr \vee \oplus H \neg q\big) \wedge \big(G(\ominus q \wedge Fp) \vee P(Gq \wedge r)\big)\Big)$$

is a formula of FNext±. Its syntactic tree is the following:



This formula is not in nnf_t because, for example, it has an occurrence of the $\wedge$ whose parent is G (the root).

Theorem 6.2. *Let A be a wff. It is possible to compute a nnf_t equivalent to A with linear time and space complexity.*

Proof. By recursively applying the transformations induced by the double negation, the de Morgan laws and the following equivalences (in appendix B, we supply a proof of these equivalences):

i. $\oplus(A \star B) \equiv \oplus A \star \oplus B$ and $\ominus(A \star B) \equiv \ominus A \star \ominus B$ if $\star \in \{\wedge, \vee\}$

ii. $F(\bigvee_{i \in I} A_i) \equiv \bigvee_{i \in I} FA_i$; $G(\bigwedge_{i \in I} A_i) \equiv \bigwedge_{i \in I} GA_i$

$P(\bigvee_{i\in I} A_i) \equiv \bigvee_{i\in I} PA_i$ and $H(\bigwedge_{i\in I} A_i) \equiv \bigwedge_{i\in I} HA_i$

iii. If $J = \{i \in I \mid A_i = \gamma_i B_i$ and $\gamma_i \in \{GH, FP, FG, GF, HP, PH\}\}$, then

3.1 $F(\bigwedge_{i\in I} A_i) \equiv (\bigwedge_{i\in J} A_i) \wedge F(\bigwedge_{i\in I\setminus J} A_i)$.

3.2 $G(\bigvee_{i\in I} A_i) \equiv (\bigvee_{i\in J} A_i) \vee G(\bigvee_{i\in I\setminus J} A_i)$.

3.3 $P(\bigwedge_{i\in I} A_i) \equiv (\bigwedge_{i\in J} A_i) \wedge P(\bigwedge_{i\in I\setminus J} A_i)$.

3.4 $H(\bigvee_{i\in I} A_i) \equiv (\bigvee_{i\in J} A_i) \vee H(\bigvee_{i\in I\setminus J} A_i)$.

iv. $F(\bigwedge_{i\in I} A_i) \equiv \bigwedge_{i\in I} FA_i$, $G(\bigvee_{i\in I} A_i) \equiv \bigvee_{i\in I} GA_i$, $P(\bigwedge_{i\in I} A_i) \equiv \bigwedge_{i\in I} PA_i$ and $H(\bigvee_{i\in I} A_i) \equiv \bigvee_{i\in I} HA_i$ if one of the following conditions are fulfilled:

- $\forall i \in I,\ A_i = \gamma_i B_i$ with $\gamma_i \in \{F, H\}$
- $\forall i \in I,\ A_i = \gamma_i B_i$ with $\gamma_i \in \{G, P\}$

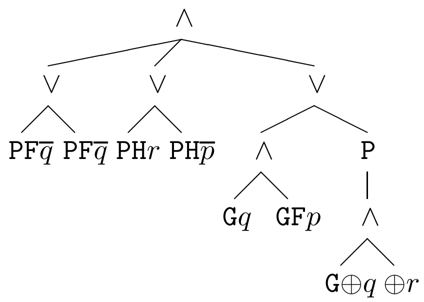
□

Appendix A.3 shows an algorithm with linear complexity which renders the Negative Normal Form for any formula of FNext±.

Example 22.- The Temporal Negative Normal Form of the formula

$$G\Big(PF\big(\neg q \vee \oplus \oplus \neg q\big) \wedge P\big(Hr \vee \oplus H \neg q\big) \wedge \big(G(\ominus q \wedge Fp) \vee P(Gq \wedge r)\big)\Big)$$

presented in example 21 is the following:



In appendix A.3, we show step by step the transformation of the original formula to its nnf_t .

7. Δ-list associated with a *nnf*$_t$

We associate with any *nnf*$_t$, A, two lists of literals, denoted $\Delta_0(A)$ and $\Delta_1(A)$, which are the α and β bases for the sets of implicates and implicants respectively.

Definition 7.1. Given a *nnf*$_t$, A, the Δ-lists associated with A, $\Delta_0(A)$ and $\Delta_1(A)$, are recursively defined as follows:

i. Classical connectives:

$$\Delta_0(\ell) = \{\ell\}; \qquad \Delta_1(\ell) = \{\ell\}; \qquad \ell \in Lit$$

$$\Delta_0(\bigvee_{i=1}^{n} A_i) = (\bigcap_{i=1}^{n}\!{}_{0}\, \Delta_0(A_i)); \qquad \Delta_1(\bigvee_{i=1}^{n} A_i) = \bigcup_{i=1}^{n}\!{}_{1}\, \Delta_1(A_i)$$

$$\Delta_0(\bigwedge_{i=1}^{n} A_i) = \bigcup_{i=1}^{n}\!{}_{0}\, \Delta_0(A_i); \qquad \Delta_1(\bigwedge_{i=1}^{n} A_i) = (\bigcap_{i=1}^{n}\!{}_{1}\, \Delta_1(A_i))$$

ii. Temporal Connectives: If $\star \in \{F, G, P, H\}$, then

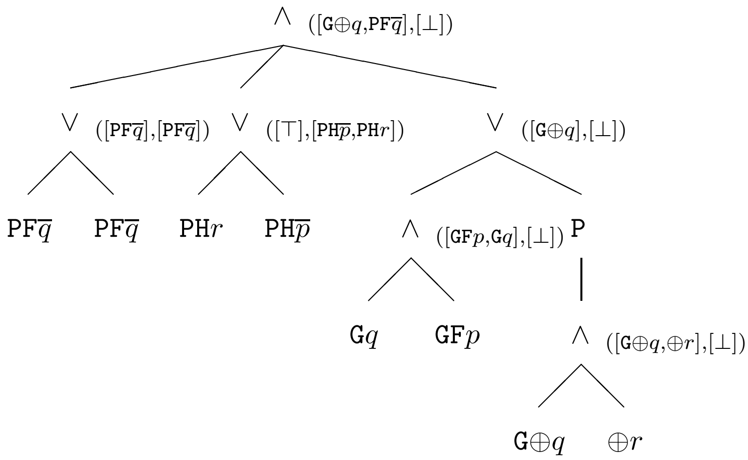
$$\Delta_0(*A) = \mathtt{Add}^0_*(\Delta_0(A))$$

$$\Delta_1(*A) = \mathtt{Add}^1_*(\Delta_1(A))$$

The information about implicates and implicants of the subformula can be used to reduce the size of the formula in order to reduce the cost for proof methods. To incorporate this information to the tree of the formulae, we label each non leaf node B of A with the pair $\big(\Delta_0(B), \Delta_1(B)\big)$.

Example 23.-

After labelling the tree of the *nnf*$_t$ of example 22, we obtain the following tree:



A. Algorithms

In this appendix, we give a description of the algorithms referred to on the body of the paper. These algorithms are presented using a functional programming style. We will use the standard functional operators defined over lists: `nil` denotes the empty list, :: adds an element at the head of the list and $\langle\rangle$ is the concatenation operator.

A.1 Temporal Literals

In this section, we show the algorithm cited in section 3 which, in linear time, reduce any literal to its canonical representative (its *temporal literal*).

Algorithm 1. Let $\mathcal{C}_{ch} = \{FG, GF, HP, PH, GH, FP\}$ and $\gamma_1 \dots \gamma_k p \in FNext\pm^{mon}$, its canonical representative is computed as follows:

$\texttt{Canonical}(\gamma_1 \dots \gamma_k p)\texttt{=Cnnc}(\gamma_1 \dots \gamma_k, \odot^0 p)$

$\texttt{Canonical}(\top) = \top$

$\texttt{Canonical}(\bot) = \bot$

$\texttt{Cnnc}(\texttt{nil}, \ell) = \ell$

$\texttt{Cnnc}(\gamma_1 \dots \gamma_{k-1}\gamma_k, \ell)\texttt{=Cnnc}(\gamma_1 \dots \gamma_{k-1}, \texttt{ Add}_{\gamma_k}(\ell))$

$\texttt{Add}_{\oplus}(\odot^n \ell_p) = \odot^{n+1}\ell_p$
$\texttt{Add}_{\oplus}(\gamma \odot^n \ell_p) = \gamma \odot^{n+1} \ell_p$ where $\gamma \in \{F, G, P, H\}$
$\texttt{Add}_{\oplus}(\gamma\gamma'\ell_p) = \gamma\gamma'\ell_p$ where $\gamma\gamma' \in \mathcal{C}_{ch}$

$\texttt{Add}_{\ominus}(\odot^n \ell_p) = \odot^{n-1}\ell_p$
$\texttt{Add}_{\ominus}(\gamma \odot^n \ell_p) = \gamma \odot^{n-1} \ell_p$ where $\gamma \in \{F, G, P, H\}$
$\texttt{Add}_{\ominus}(\gamma\gamma'\ell_p) = \gamma\gamma'\ell_p$ where $\gamma\gamma' \in \mathcal{C}_{ch}$

$\texttt{Add}_{\neg}(\ell) = \overline{\ell}$

$\texttt{Add}_{\gamma}(\odot^n \ell_p) = \gamma \odot^n \ell_p$ where $\gamma \in \{F, G, P, H\}$
$\texttt{Add}_{P}(F \odot^n \ell_p) = FP\ell_p$
$\texttt{Add}_{H}(G \odot^n \ell_p) = GH\ell_p$
$\texttt{Add}_{\gamma}(\gamma' \odot^n \ell_p) = \gamma' \odot^{n+1} \ell_p$ where $\gamma \in \{F, G\}$ and $\gamma\gamma' \notin \mathcal{C}_{ch}$
$\texttt{Add}_{\gamma}(\gamma' \odot^n \ell_p) = \gamma' \odot^{n-1} \ell_p$ where $\gamma \in \{P, H\}$ and $\gamma\gamma' \notin \mathcal{C}_{ch}$
$\texttt{Add}_{\gamma}(\gamma' \odot^n \ell_p) = \gamma\gamma'\ell_p$ where $\gamma\gamma' \in \mathcal{C}_{ch}$
$\texttt{Add}_{\gamma}(\gamma'\gamma''\ell_p) = \gamma'\gamma''\ell_p$ where $\gamma'\gamma'' \in \mathcal{C}_{ch}$

EXAMPLE 24.- In this example, we apply the above algorithm to the formula $\phi = \neg P \oplus GF \oplus\oplus \neg G \neg \ominus p \in FNext\pm^{mon}$. Specifically, the chain of temporal connectives is traverse from right to left until the chain FG is found. Then, then number of $\neg$ connectives in ϕ is counted, yielding ϕ as the canonical representative of $FG\overline{p}$.

$$
\begin{aligned}
&\texttt{Canonical}(\neg P \oplus GF \oplus \oplus\neg G\neg \ominus p) = \texttt{Cnnc}(\neg P \oplus GF \oplus \oplus\neg G\neg\ominus, \odot^0 p) = \\
&\quad = \texttt{Cnnc}(\neg P \oplus GF \oplus \oplus\neg G\neg, \odot^{-1} p) = \texttt{Cnnc}(\neg P \oplus GF \oplus \oplus\neg G, \odot^{-1}\overline{p}) = \\
&\quad = \texttt{Cnnc}(\neg P \oplus GF \oplus \oplus\neg, G \odot^{-1} \overline{p}) = \texttt{Cnnc}(\neg P \oplus GF \oplus \oplus, F \odot^{-1} p) = \\
&\quad = \texttt{Cnnc}(\neg P \oplus GF\oplus, F \odot^{0} p) = \texttt{Cnnc}(\neg P \oplus GF, F \odot^{1} p) = \\
&\quad = \texttt{Cnnc}(\neg P \oplus G, F \odot^{2} p) = \texttt{Cnnc}(\neg P\oplus, GFp) = \\
&\quad = \texttt{Cnnc}(\neg P, GFp) = \texttt{Cnnc}(\neg, GFp) = \texttt{Cnnc}(\texttt{nil}, FG\overline{p}) = FG\overline{p}
\end{aligned}
$$

A.2 0-Intersection of Δ_0-lists

In this section, we describe the algorithm based on lemma 5.3 which obtains the 0-intersection of two Δ_0-lists making only one traverse of them.

Algorithm 2. Let λ_1, λ_2 be two Δ_0-lists, $\lambda_1 \Cap \lambda_2$ is computed in the following way:

$\top \Cap \lambda = \top \qquad \lambda \Cap \top = \top$

$\bot \Cap \lambda = \lambda \qquad \lambda \Cap \bot = \lambda$

In other case $\qquad \lambda_1 \Cap \lambda_2 = \mathtt{Inter}_0(\mathtt{nil}, \lambda_1, \lambda_2)$

$\mathtt{Inter}_0(\lambda_0, \mathtt{nil}, \lambda_1) = \lambda_0$

$\mathtt{Inter}_0(\lambda_0, \lambda_1, \mathtt{nil}) = \lambda_0$

If $\ell_1 \ntrianglelefteq \ell_2$, $\ell_2 \ntrianglelefteq \ell_1$ and $\mathcal{B}^+_\alpha(\ell_1, \ell_2) = \{\ell\}$

$\mathtt{Inter}_0(\lambda_0, \ell_1, \ell_2) = \lambda_0\langle\rangle\ell$

If ℓ_2 is previous to ℓ_1 (previous to in the sense of the order presented in definition 5.1),

$\mathtt{Inter}_0(\lambda_0, \ell_1 :: \lambda_1, \ell_2 :: \lambda_2) = \mathtt{Inter}_0(\lambda_0, \ell_2 :: \lambda_2, \ell_1 :: \lambda_1)$

If $\ell_1 \ntrianglelefteq \ell_2$, $\ell_2 \ntrianglelefteq \ell_1$ and $\mathcal{B}^-_\alpha(\ell_1, \ell_2) = \{\ell\}$

$$\mathtt{Inter}_0(\mathtt{nil}, \ell_1\ell'_1\langle\rangle\lambda_1, \ell_2 :: \lambda_2) = \begin{cases} \mathtt{Inter}_0(\ell_2, \lambda_1, \lambda_2) & \text{if } \ell_2 = \ell'_1 \\ \mathtt{Inter}_0(\ell, \ell'_1 :: \lambda_1, \ell'_2 :: \lambda_2) & \text{if } \ell_2 \text{ is previous to } \ell'_1 \\ \mathtt{Inter}_0(\mathtt{nil}, \ell_1 :: \lambda_1, \ell_2\ell'_2\langle\rangle\lambda_2) & \text{if } \ell_2 \text{ is previous to } \ell'_1 \end{cases}$$

$\mathtt{Inter}_0(\lambda_0, G \odot^{n-1} \ell_p, H \odot^{n+1} \ell_p :: \lambda_2) = \mathtt{Inter}_0(\lambda_0\langle\rangle \odot^n \ell_p, G \odot^{n-1} \ell_p, \lambda_2)$

If $n_1 < n_2 + 2$

$\mathtt{Inter}_0(\lambda_0, G \odot^{n_1} \ell_p, H \odot^{n_2} \ell_p :: \lambda_2) = \mathtt{Inter}_0(\lambda_0\langle\rangle \odot^{n_1} \ell_p, G \odot^{n_1+1} \ell_p, H \odot^{n_2} \ell_p :: \lambda_2)$

$\mathtt{Inter}_0(\lambda_0, H \odot^{n_2} \ell_p :: \lambda_1, G \odot^{n_1} \ell_p) = \mathtt{Inter}_0(\lambda_0\langle\rangle \odot^{n_1} \ell_p, H \odot^{n_2} \ell_p :: \lambda_1, G \odot^{n_1+1} \ell_p)$

In other case

$$\mathtt{Inter}_0(\lambda_0, \ell_1 :: \lambda_1, \ell_2 :: \lambda_2) = \begin{cases} \mathtt{Inter}_0(\lambda_0\langle\rangle\ell_1, \lambda_1, \lambda_2) & \text{if } \ell_1 = \ell_2 \\ \mathtt{Inter}_0(\lambda_0\langle\rangle\ell_1, \lambda_1, \ell_2 :: \lambda_2) & \text{if } \ell_1 \neq \ell_2 \text{ and } \ell_2 \trianglelefteq \ell_1 \\ \mathtt{Inter}_0(\lambda_0\langle\rangle\ell_2, \ell_1 :: \lambda_1, \lambda_2) & \text{if } \ell_1 \neq \ell_2 \text{ and } \ell_1 \trianglelefteq \ell_2 \\ \mathtt{Inter}_0(\lambda_0, \lambda_1, \ell_2 :: \lambda_2) & \text{in other case} \end{cases}$$

Example 25.-

Let $\lambda_1 = [P \ominus^5 p, \ominus^2 p, \oplus p, G \oplus^3 p]$ and $\lambda_2 = [\oplus p, \oplus^6 p, \oplus^7 p, F \oplus^8 p]$, $\lambda_1 \Cap \lambda_2$ is computed as follows:

$$[P \ominus^5 p, \ominus^2 p, \oplus p, G \oplus^3 p] \Cap [\oplus p, \oplus^6 p, \oplus^7 p, F \oplus^8 p]$$
$$= \texttt{Inter}_0(\texttt{nil}\ ,\ [\texttt{P} \ominus^5 \texttt{p}, \ominus^2\texttt{p}, \oplus\texttt{p}, \texttt{G} \oplus^3 \texttt{p}]\ ,\ [\oplus\texttt{p}, \oplus^6\texttt{p}, \oplus^7\texttt{p}, \texttt{F} \oplus^8 \texttt{p}])$$
$$= \texttt{Inter}_0(\texttt{nil}\ ,\ [\texttt{P} \ominus^5 \texttt{p}, \oplus\texttt{p}, \texttt{G} \oplus^3 \texttt{p}]\ ,\ [\oplus\texttt{p}, \oplus^6\texttt{p}, \oplus^7\texttt{p}, \texttt{F} \oplus^8 \texttt{p}])$$
$$= \texttt{Inter}_0([\oplus p]\ ,\ [G \oplus^3 p]\ ,\ [\oplus^6 p, \oplus^7 p, F \oplus^8 p])$$
$$= \texttt{Inter}_0([\oplus p, \oplus^6 p]\ ,\ [G \oplus^3 p]\ ,\ [\oplus^7 p, F \oplus^8 p])$$
$$= \texttt{Inter}_0([\oplus p, \oplus^6 p, \oplus^7 p]\ ,\ [G \oplus^3 p]\ ,\ [F \oplus^8 p])$$
$$= \texttt{Inter}_0([\oplus p, \oplus^6 p, \oplus^7 p, F \oplus^8 p]\ ,\ [G \oplus^3 p]\ ,\ \texttt{nil})$$
$$= [\oplus p, \oplus^6 p, \oplus^7 p, F \oplus^8 p]$$

EXAMPLE 26.- The 0-intersection of the Δ-lists $\lambda_1 = [H \oplus^2 p, \oplus^5 p, F \oplus^7 p]$ and $\lambda_2 = [\ominus^3 p, G \ominus p]$ is computed as follows:

$$[H \oplus^2 p, \oplus^5 p, F \oplus^7 p] \Cap [\ominus^3 p, G \ominus p]$$
$$= \texttt{Inter}_0(\texttt{nil}\ ,\ [\texttt{H} \oplus^2 \texttt{p}, \oplus^5\texttt{p}, \texttt{F} \oplus^7 \texttt{p}]\ ,\ [\ominus^3\texttt{p}, \texttt{G} \ominus \texttt{p}])$$
$$= \texttt{Inter}_0(\ominus^3 p\ ,\ [H \oplus^2 p, \oplus^5 p, F \oplus^7 p]\ ,\ [G \ominus p])$$
$$= \texttt{Inter}_0([\ominus^3 p, \oplus p]\ ,\ [H \oplus^2 p, \oplus^5 p, F \oplus^7 p]\ ,\ [Gp])$$
$$= \texttt{Inter}_0([\ominus^3 p, \oplus p, \oplus^2 p]\ ,\ [\oplus^5 p, F \oplus^7 p]\ ,\ [G \oplus p])$$
$$= \texttt{Inter}_0([\ominus^3 p, \oplus p, \oplus^2 p, \oplus^5 p]\ ,\ [F \oplus^7 p]\ ,\ [G \oplus p])$$
$$= \texttt{Inter}_0([\ominus^3 p, \oplus p, \oplus^2 p, \oplus^5 p, F \oplus^7 p]\ ,\ \texttt{nil}\ ,\ [\texttt{G} \oplus \texttt{p}])$$
$$= [\ominus^3 p, \oplus p, \oplus^2 p, \oplus^5 p, F \oplus^7 p]$$

A.3 Temporal Negative Normal Form

In this section we present the algorithm which computes the *Temporal Negative Normal Form* corresponding to any well formed formula of $FNext\pm$. This algorithm is based on Theorem 6.2.

Algorithm 3. Let ϕ be a well formed formula of $FNext\pm$, this algorithm computes its Temporal Negative Normal Form, denoted $\texttt{Fnnt}(\phi)$, using the algorithms `Canonical` and `Cnnc` presented in appendix A.1. As we shown before, `Cnnc` transforms any chain of monary connectives to a chain of the set

$$\begin{aligned}\mathcal{M} = \ & \{\varepsilon, \odot^n, F\odot^n, P\odot^n, G\odot^n, H\odot^n, FP, F\ G, GF, PH, HP, GH\} \cup \\ \cup \ & \{\odot^n\neg, F\odot^n\neg, P\odot^n\neg, G\odot^n\neg, H\odot^n\neg, \\ & FP\neg, FG\neg, GF\neg, PH\neg, HP\neg, GH\neg\}\end{aligned}$$

0. If $\phi \in FNext\pm^{mon}$ then $\texttt{Fnnt}(\phi) = \texttt{Canonical}(\phi)$
1. 1.A. $\texttt{Fnnt}(\bigwedge_{i\in J_1}\phi) = \bigwedge_{i\in J_1}(\texttt{Fnnt}(\phi_i))$

   1.B. $\texttt{Fnnt}(\bigvee_{i\in J_1}\phi) = \bigvee_{i\in J_1}(\texttt{Fnnt}(\phi_i))$
2. Let A_i with $i \in J_1$ be an element of $\mathcal{V}$ or a well formed formula such that its root is $\wedge$ or $\vee$ [7], and let γ and γ_i, with $i \in J_1$, be chains of monary connectives.

   If we have that $\{\gamma\} \cup \{\gamma_i \mid i \in J_1\} \not\subseteq \mathcal{M}$ then [8]:

   2.A. $\texttt{Fnnt}(\gamma(\bigwedge_{i\in J_1}\gamma_i A_i)) = \texttt{Fnnt}(\texttt{Cnnc}(\gamma, \odot^0(\bigwedge_{i\in J_1}\texttt{Cnnc}(\gamma_i, \odot^0 A_i))))$

   2.B. $\texttt{Fnnt}(\gamma(\bigvee_{i\in J_1}\gamma_i A_i)) = \texttt{Fnnt}(\texttt{Cnnc}(\gamma, \odot^0(\bigvee_{i\in J_1}\texttt{Cnnc}(\gamma_i, \odot^0 A_i))))$
3. If $\gamma \in \{\varepsilon, \odot^n, F\odot^n, P\odot^n, G\odot^n, H\odot^n, FP, FG, GF, PH, HP, GH\}$ then:

   3.A. $\texttt{Fnnt}(\gamma\neg(\bigwedge_{i\in J_1}A_i)) = \texttt{Fnnt}(\gamma(\bigvee_{i\in J_1}\neg A_i))$

   3.B. $\texttt{Fnnt}(\gamma\neg(\bigvee_{i\in J_1}A_i)) = \texttt{Fnnt}(\gamma(\bigwedge_{i\in J_1}\neg A_i))$
4. If $\gamma \in \{\varepsilon, F, G, P, H\}$, then

   4.A. $\texttt{Fnnt}(\gamma\odot^n(\bigwedge_{i\in J_1}A_i)) = \texttt{Fnnt}(\gamma(\bigwedge_{i\in J_1}\odot^n A_i))$

   4.B. $\texttt{Fnnt}(\gamma\odot^n(\bigvee_{i\in J_1}A_i)) = \texttt{Fnnt}(\gamma(\bigvee_{i\in J_1}\odot^n A_i))$
5. 5.A. If $\gamma \in \{G, H, FG, PH, GH\}$, then
   $\texttt{Fnnt}(\gamma(\bigwedge_{i\in J_1}A_i)) = \bigwedge_{i\in J_1}(\texttt{Fnnt}(\gamma A_i))$

   5.B. If $\gamma \in \{F, P, GF, HP, FP\}$, then
   $\texttt{Fnnt}(\gamma(\bigvee_{i\in J_1}A_i)) = \bigvee_{i\in J_1}(\texttt{Fnnt}(\gamma A_i))$
6. If $\gamma_i \in \{F, H, FG, GF, FP, PH, HP, GH\}$ for all $i \in J_1$, then:

   6.A. If $\gamma \in \{F, P, GF, HP, FP\}$, then
   $\texttt{Fnnt}(\gamma(\bigwedge_{i\in J_1}\gamma_i A_i)) = \bigwedge_{i\in J_1}(\texttt{Fnnt}(\gamma\gamma_i A_i))$

[7] The root of the formula means the root of its syntactic tree.

[8] The above situation ensures that, at least, one of the chains does not belong M

6.B. If $\gamma \in \{G, H, FG, PH, GH\}$, then
$$\texttt{Fnnt}(\gamma(\textstyle\bigvee_{i\in J_1} \gamma_i A_i)) = \textstyle\bigvee_{i\in J_1}(\texttt{Fnnt}(\gamma\gamma_i A_i))$$

7. If $\gamma_i \in \{G, P, FG, GF, FP, PH, HP, GH\}$ for all $i \in J_1$, then:

7.A. If $\gamma \in \{F, P, GF, HP, FP\}$, then
$$\texttt{Fnnt}(\gamma(\textstyle\bigwedge_{i\in J_1} \gamma_i A_i)) = \textstyle\bigwedge_{i\in J_1}(\texttt{Fnnt}(\gamma\gamma_i A_i))$$

7.B. If $\gamma \in \{G, H, FG, PH, GH\}$, then
$$\texttt{Fnnt}(\gamma(\textstyle\bigvee_{i\in J_1} \gamma_i A_i)) = \textstyle\bigvee_{i\in J_1}(\texttt{Fnnt}(\gamma\gamma_i A_i))$$

8. Let $J_2 = \{i \in J_1 \mid \phi_i = \gamma_i\psi_i \text{ con } \gamma_i \in \{FG, GF, FP, PH, HP, GH\}\}$.

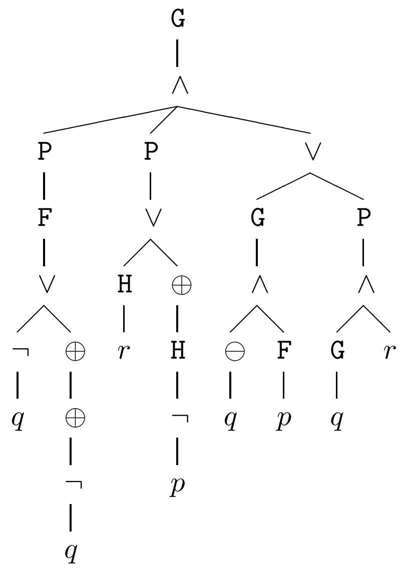
8.A. If $\gamma \in \{F, P, GF, FP, HP\}$, then
$$\texttt{Fnnt}(\gamma(\textstyle\bigwedge_{i\in J_1} A_i)) = (\textstyle\bigwedge_{i\in J_2} \texttt{Fnnt}(A_i)) \wedge (\gamma \textstyle\bigwedge_{i\in J_1 \smallsetminus J_2}(\texttt{Fnnt}(A_i)))$$

8.B. If $\gamma \in \{G, H, FG, GH, PH\}$, then
$$\texttt{Fnnt}(\gamma(\textstyle\bigvee_{i\in J_1} A_i)) = (\textstyle\bigvee_{i\in J_2} \texttt{Fnnt}(A_i)) \vee (\gamma \textstyle\bigvee_{i\in J_1 \smallsetminus J_2}(\texttt{Fnnt}(A_i)))$$

EXAMPLE 27.- In this example we show, step by step, the transformation of the *wff* in example 21 to its corresponding nnf_t. The formula is

$$G\Big(PF\big(\neg q \vee \oplus\oplus\neg q\big) \wedge P\big(Hr \vee \oplus H\neg q\big) \wedge \big(G(\ominus q \wedge Fp) \vee P(Gq \wedge r)\big)\Big)$$

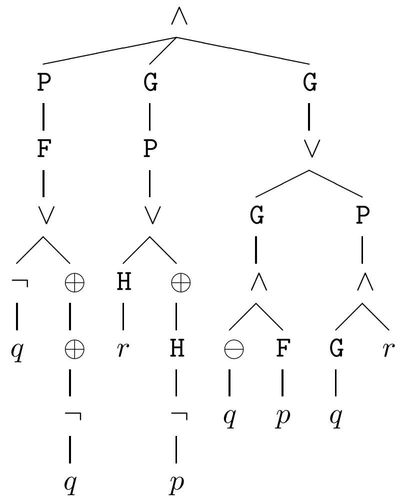
Its syntactic tree is the following:



We apply item 5.A.

$$\texttt{Fnnt}(G(PFA \wedge PB \wedge C)) = \texttt{Fnnt}(PFA) \wedge \texttt{Fnnt}(GPB) \wedge \texttt{Fnnt}(GC)$$

and the resultant tree is the following:



Now, we transform this tree in the following way:

- apply item 6.B. to the first branch:

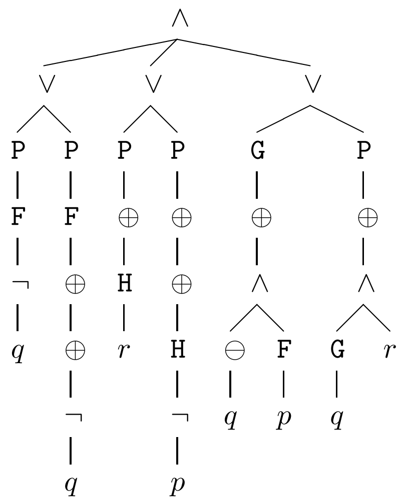
$$\texttt{Fnnt}(PF(A \vee B)) = \texttt{Fnnt}(PFA) \vee \texttt{Fnnt}(PFB)$$

- apply items 2.B., 4.B. and 5.B. to the second branch:

$$\texttt{Fnnt}(GP(A \vee B)) = \texttt{Fnnt}(P \oplus (A \vee B)) = \texttt{Fnnt}(P \oplus A) \vee \texttt{Fnnt}(P \oplus B)$$

- apply items 7.B. and 2.B. to the third branch:

$$\begin{aligned} \texttt{Fnnt}(G(GA \vee PB)) &= \texttt{Fnnt}(GGA) \vee \texttt{Fnnt}(GPB) \\ &= \texttt{Fnnt}(G \oplus A) \vee \texttt{Fnnt}(P \oplus B) \end{aligned}$$



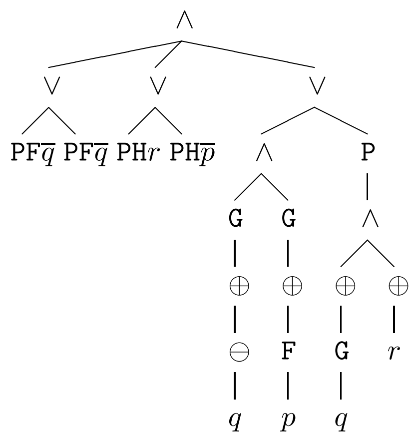
We get the new tree using the following transformations:

- apply item 0 to the first branch: $\texttt{Fnnt}(PF \oplus \oplus\neg q) = PF\overline{q}$
- apply item 0 to the second branch: $\texttt{Fnnt}(P \oplus Hr) = PHr$
- apply item 0 to the third branch: $\texttt{Fnnt}(P \oplus \oplus H\neg p) = PH\overline{p}$
- apply items 4.A. and 5.A. to the fifth branch:

$$\texttt{Fnnt}(G \oplus (A \wedge B)) = \texttt{Fnnt}(G \oplus A) \wedge \texttt{Fnnt}(G \oplus B)$$

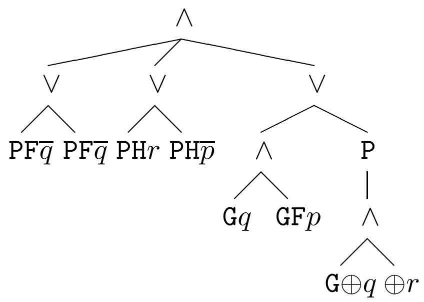
- apply items 4.A. and 8.A. to the sixth branch:

$$\texttt{Fnnt}(P \oplus (A \wedge B)) = P(\texttt{Fnnt}(\oplus A) \wedge \texttt{Fnnt}(\oplus B))$$



We conclude applying item 0 to fourth, fifth and sixth branches:

- $\texttt{Fnnt}(G \oplus \ominus q) = Gq$
- $\texttt{Fnnt}(G \oplus Fp) = GFp$
- $\texttt{Fnnt}(\oplus Gq) = G \oplus q$



This tree corresponds to a nnf_t.

B. Proof of equivalence laws

In this appendix we prove the equivalence laws presented in section 3 and Theorem 6.2. First, we present a preliminary lemma:

Lemma B.1. *Let $A \in FNext\pm$ and $\gamma \in \{FP, FG, GF, HP, PH, GH\}$. If h is an interpretation, then $h(\gamma A) = \varnothing$ or $h(\gamma A) = \mathbb{Z}$.*

Proof.

- If there exists $t_1 \in h(FPA)$, then there exists $t_2 > t_1$ and $t_3 < t_2$ so that $t_3 \in h(A)$. Therefore, for all $t \in \mathbb{Z}$, we have that

$$\max\{t_2, t+1\} > t \quad \text{and} \quad \max\{t_2, t+1\} \in h(PA)$$

  and then $t \in h(FPA)$. We conclude that $h(FPA) = \mathbb{Z}$

- If there exists $t_1 \in h(FGA)$, then there exists $t_2 > t_1$ so that $(t_2, \infty) \subseteq h(A)$. Therefore, for all $t_3 \in \mathbb{Z}$, we have that

$$(\max\{t_2, t_3\}, \infty) \subset h(A)$$

  and then $t_3 \in h(FGA)$. We conclude that $h(FGA) = \mathbb{Z}$

The other cases can be proved analogously. □

Now, we present the proof for the most significant equivalence laws of FNext± introduced in section 3 (see section 3).

Lemma B.2. *Let $A \in FNext\pm$ and $\Gamma \in \{FP, FG, GF, HP, PH, GH\}$. We have the following equivalences:*
$\gamma \oplus A \equiv \gamma A$, $\gamma \ominus A \equiv \gamma A$, $F\gamma A \equiv \gamma A$, $G\gamma A \equiv \gamma A$, $P\gamma A \equiv \gamma A$, *and* $H\gamma A \equiv \gamma A$

Proof. To be brief, we only prove the most significant laws (the proof of the other cases can be obtained analogously or by duality).

- $FP \oplus A \equiv FPA$

  From lemma B.1, for all interpretation h, $t \in h(FP \oplus A)$ if and only if there exists $t' \in h(\oplus A)$; i.e. $t' + 1 \in h(A)$. Therefore, we may conclude that $t \in h(FP \oplus A)$ if and only if $t \in h(FPA)$.

- $FFPA \equiv FPA$

  For all interpretation h, $t \in h(FFPA)$ if and only if there exists $t' > t$ so that $t' \in h(FPA)$. By lemma B.1 we have that it is true if and only if $h(FPA) = \mathbb{Z}$ and, therefore, $t \in h(FPA)$. □

Lemma B.3. *Let $A, B \in FNext\pm$. If $\gamma \in \{FP, FG, GF, PH, HP, GH\}$ then*

$$F(\gamma A \wedge B) \equiv \gamma A \wedge FB \qquad P(\gamma A \wedge B) \equiv \gamma A \wedge PB$$

$$G(\gamma A \vee B) \equiv \gamma A \vee GB \qquad H(\gamma A \vee B) \equiv \gamma A \vee HB$$

Proof. We prove only the first equivalence because the other equivalences are obtained by duality or by the mirror law of this one.

Let h be an interpretation. From lemma B.1, one of the following situations are satisfied:

- $h(\gamma A) = \varnothing$. In this case we have that

$$\begin{aligned}
h(F(\gamma A \wedge B)) &= \{t \in \mathbb{Z} \mid (t, \infty) \cap h(\gamma A \wedge B) \neq \varnothing\} = \\
&= \{t \in \mathbb{Z} \mid (t, \infty) \cap h(\gamma A) \cap h(B) \neq \varnothing\} = \\
&= \{t \in \mathbb{Z} \mid (t, \infty) \cap \varnothing \cap h(B) \neq \varnothing\} = \\
&= \varnothing \\
h(\gamma A \wedge FB) &= h(\gamma A) \cap h(FB) = \varnothing \cap h(FB) = \varnothing
\end{aligned}$$

- $h(\gamma A) = \mathbb{Z}$. In this case we have that

$$\begin{aligned} h(F(\gamma A \wedge B)) &= \{t \in \mathbb{Z} \mid (t,\infty) \cap h(\gamma A \wedge B) \neq \varnothing\} = \\ &= \{t \in \mathbb{Z} \mid (t,\infty) \cap h(\gamma A) \cap h(B) \neq \varnothing\} = \\ &= \{t \in \mathbb{Z} \mid (t,\infty) \cap \mathbb{Z} \cap h(B) \neq \varnothing\} = \\ &= h(FB) \\ h(\gamma A \wedge FB) &= h(\gamma A) \cap h(FB) = \mathbb{Z} \cap h(FB) = h(FB) \end{aligned}$$

Therefore, we conclude that

$$F(\gamma A \wedge B) \equiv \gamma A \wedge FB$$

□

Acknowledgments

The authors wishes to thank the reviewers for their helpful comments and suggested improvements.

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