REPORTS ON MATHEMATICAL LOGIC
34 (2000), 159–165

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ON THE FINITE EMBEDDABILITY PROPERTY FOR RESIDUATED LATTICES, POCRIMS AND BCK-ALGEBRAS

The following is an extended abstract of a forthcoming paper, part of which was presented at the Workshop on Algebra & Substructural Logic held at JAIST, 10–17 November 1999.

1. Introduction

To see that a finitely axiomatizable logic is decidable—in the sense that it has a decidable set of theorems—it suffices to show that it has the finite model property, and indeed, this has been the method of choice for establishing decidability of propositional logics. A logic is said to have the *finite model property* (FMP) if every formula that fails to be a theorem of the logic can be refuted in a finite model of the logic. The first one to apply the method in a non-trivial way was J.C.C. McKinsey in [6], where he obtained decision procedures for the modal logics S2 and S4.

Although traditionally logics have often been identified with their sets of theorems, work on the algebraization of logic (such as in [1]) has emphasized the importance of the inferences of the logic. We say that a logic has

Received 4 December 2000

the *strong finite model property* (SFMP) if for every finite set of premises Γ, and every formula φ, if φ is not a consequence of Γ, then there is an interpretation in a finite model of the logic that makes all of the formulas of Γ true, but φ false. To see that the set of inferences $\Gamma \vdash \varphi$ (Γ finite) of a finitely axiomatizable logic is decidable it suffices to verify it has the SFMP. Clearly, every logic that has the SFMP also has the FMP. If a logic has the "deduction detachment theorem" the converse holds as well. In the full paper we discuss the FMP and SFMP in general, and compare the two properties for some extensions of fragments of intuitionistic linear logic.

The logics we consider are *algebraizable* in the sense of [1], and we carry out our considerations in the domain of algebra. We say that a quasivariety of algebras has the FMP if every identity that fails to hold in the class can be refuted in a finite member of the class, and that it has the SFMP if every quasi-identity that fails to hold in the class can be refuted in a finite member of the class. It is easy to see that a quasivariety of algebras has the FMP iff it is included in the variety generated by its finite members, and that it has the SFMP if it is included in (and hence equals) the quasivariety generated by its finite members. Furthermore, a deductive system that is algebraizable in the sense of [1] has the FMP or SFMP if and only if its equivalent quasivariety has the FMP or SFMP respectively.

A quasivariety $\mathcal{K}$ has the *finite embeddability property* (FEP) if every finite partial subalgebra of a member of $\mathcal{K}$ can be embedded in a *finite* member of $\mathcal{K}$. It is easy to see that every quasivariety with the FEP has the SFMP; interestingly, the converse also holds (see Section 1).

Okada and Terui [7] established the FMP for a Gentzen style formalization of the multiplicative additive fragment of intuitionistic linear logic (IMALL), and also—using a different construction—for the extension of that system obtained by adding the rule of *weakening* (IMALL^w). The classes of algebras corresponding with these systems are the variety $\mathcal{R}^*$ of bounded lattice-ordered commutative residuated monoids and the variety $\mathcal{R}$ of *integral* bounded lattice-ordered commutative residuated monoids, respectively; the algebras in $\mathcal{R}$ are better known as *residuated lattices*.

The construction Okada and Terui used to show that IMALL has the FMP carries over to the domain of algebra: the variety $\mathcal{R}^*$ has the FMP.

In Section 2 we show that $\mathcal{R}^*$ does not have the SFMP, and hence also does not have the FEP. Okada and Terui's second construction, used to show that IMALL^w has the FMP, also carries over, and enables us to show that the variety $\mathcal{R}$ has the FEP, and hence the SFMP. We outline the proof in Section 3. The same construction allows us to settle in the affirmative two problems that had remained open for some time, *viz.*, whether the quasivarieties of pocrims and of BCK-algebras have the FEP (Theorem 3.5). For details, further results and references, we refer to the full paper [3].

2. Failure of the FEP

The following theorem explains our interest in the FEP. A proof of the only non-trivial implication, (iii) $\Rightarrow$ (i), is given in [4].

Theorem 2.1. *Let $\mathcal{K}$ be a quasivariety of algebras. The following are equivalent.*

(i) $\mathcal{K}$ has the FEP,

(ii) $\mathcal{K}$ has the SFMP,

(iii) $\mathcal{K}$ is generated, as a quasivariety, by its finite members; i.e., $\mathcal{K} = ISPP_U(\mathcal{K}_F)$, where $\mathcal{K}_F$ denotes the class of finite members of $\mathcal{K}$.

Let $\mathcal{R}^*$ denote the class of algebras $\langle A, \wedge, \vee, \cdot, \rightarrow, e, 0, 1\rangle$, such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a lattice with smallest element 0 and largest element 1, $\langle A, \cdot, e\rangle$ is a monoid with identity e, and $\rightarrow$ is the residuation operation defined by $c \leq a \rightarrow b$ if and only if $a \cdot c \leq b$. In such an algebra the monoid operation respects the partial order. A *residuated lattice* is an algebra in $\mathcal{R}^*$ that is *integral*, *i.e.,* an algebra in which e coincides with; $\mathcal{R}$ will denote the variety of all residuated lattices (with e dropped from the type). An example of an algebra that belongs to $\mathcal{R}^*$, but not to $\mathcal{R}$, is

$$\mathbf{Z} = \langle \mathrm{Z} \cup \{\bot, \top\}, \wedge, \vee, +, \rightarrow, 0, \bot, \top\rangle.$$

Here Z is the set of integers with its usual linear order, $\bot < x < \top$ for all $x \in \mathrm{Z}$, $+$ is the usual addition on Z and

$$\top + x = x + \top = \top \quad \text{if } x \neq \bot,$$
$$\bot + x = x + \bot = \bot \quad \text{for all } x.$$

In particular, this structure is residuated. Since $e^{\mathbf{Z}} = 0 \neq \top = 1^{\mathbf{Z}}$, $\mathbf{Z} \notin \mathcal{R}$.

Theorem 2.2. *The variety $\mathcal{R}^*$ does not have the FEP.*

In view of Theorem 2.1, in order to show that $\mathcal{R}^*$ fails to have the FEP it suffices to show that it is not generated, as a quasivariety, by its finite algebras. We do this by exhibiting a quasi-identity that is satisfied by all finite algebras in $\mathcal{R}^*$, but that fails in the (infinite) algebra $\mathbf{Z}$. Let φ be the sentence

$$\forall x \forall y[(0 \leq x \ \& \ x + y \approx 0) \Rightarrow x \approx 0].$$

This quasi-identity does not hold in $\mathbf{Z}$: indeed, we have $0 \leq 1$, $1 + (-1) = 0$ but $1 \neq 0$. But it can be shown to hold in all finite algebras in $\mathcal{R}^*$.

We already observed in the introduction that $\mathcal{R}^*$ does have the FMP; $\mathcal{R}^*$ is thus an example of a variety that has the FMP but does not have the SFMP.

3. The FEP in $\mathcal{R}$ and its subreducts

In this section we will show that $\mathcal{R}$ has the FEP. More strongly, we show that any partial *pocrim* $\boldsymbol{B}$ may be embedded in the pocrim reduct of a (complete) residuated lattice $\boldsymbol{D}$, and the embedding preserves any meets, joins and bounds that happen to be defined in $\boldsymbol{B}$ (Theorem 3.3). If the partial algebra $\boldsymbol{B}$ is finite, then the residuated lattice $\boldsymbol{D}$ is finite as well. For a discussion of pocrims and their BCK-subreducts, see [2].

Let $\boldsymbol{A}$ be a pocrim and let B be a partial subalgebra of $\boldsymbol{A}$. Let $M = \langle M, \cdot, 1, \leq \rangle$ be the partially ordered submonoid of $\langle A, \cdot, 1, \leq \rangle$ generated by B. For each $a \in M$ and $b \in B$, define

$$(a \rightsquigarrow b] = \{c \in M : ac \leq b\} \quad (= \{c \in M : c \leq a \to b\}).$$

The set $(a \rightsquigarrow b]$ is a downward closed subset of M with respect to the order of $\boldsymbol{M}$ inherited from $\boldsymbol{A}$. Note that $(1 \rightsquigarrow a] = (a]$. Set

$$\overline{D} = \{(a \rightsquigarrow b] : a \in M,\ b \in B\},$$

and let D be the closure system generated by $\overline{D}$, that is,

$$D = \{\bigcap \mathcal{X} : \mathcal{X} \subseteq \overline{D}\}.$$

Let C be the closure operator on the set of subsets of M associated with D, i.e., for $X \subseteq M$, let

$$C(X) = \bigcap \{Y \in \overline{D} : X \subseteq Y\}.$$

We shall define an algebra $\boldsymbol{D}$ whose universe is the set D. For $X, Y \subseteq M$ and $a \in M$, set $XY = \{ab : a \in X, b \in Y\}$ and $Xa = X\{a\}$. For $X, Y \subseteq M$ and $X_i \subseteq M$, $i \in I$, define

$$\begin{aligned}
&X \cdot^{\boldsymbol{D}} Y = C(XY) \\
&X \to^{\boldsymbol{D}} Y = \{a \in M : Xa \subseteq Y\} \\
&\textstyle\bigvee^{\boldsymbol{D}}_{i \in I} X_i = C(\bigcup_{i \in I} X_i) \\
&\textstyle\bigwedge^{\boldsymbol{D}}_{i \in I} X_i = \bigcap_{i \in I} X_i \\
&0^{\boldsymbol{D}} = \bigcap \overline{D} \\
&1^{\boldsymbol{D}} = M.
\end{aligned}$$

It can be shown that $\boldsymbol{D}$ is closed under the operation $\to^{\boldsymbol{D}}$ and that $\cdot^{\boldsymbol{D}}$ is an associative and commutative operation with identity $1^{\boldsymbol{D}}$ on $\boldsymbol{D}$. Moreover, we show that $\to^{\boldsymbol{D}}$ is the residuation operation on $\boldsymbol{D}$ with respect to $\cdot^{\boldsymbol{D}}$ and $\subseteq$. We obtain the following result.

Lemma 3.1. *The structure* $\boldsymbol{D} = \langle D, \cdot^{\boldsymbol{D}}, \to^{\boldsymbol{D}}, 1^{\boldsymbol{D}}, \subseteq \rangle$ *is a pocrim. Moreover, the partial order* $\subseteq$ *is a complete lattice order with lattice meet* $\bigwedge^{\boldsymbol{D}}$*, lattice join* $\bigvee^{\boldsymbol{D}}$*, largest element* $1^{\boldsymbol{D}}$ *and smallest element* $0^{\boldsymbol{D}}$.

The map ι from $\boldsymbol{B}$ to $\boldsymbol{D}$ which sends a to $(a]$ is an embedding which preserves all existing operations in $\boldsymbol{B}$. Thus, $\boldsymbol{B}$ is isomorphic to a partial subalgebra of $\boldsymbol{D}$. We have the following result.

Lemma 3.2. *The map ι is an embedding of the partial subalgebra B of $\boldsymbol{A}$ into $\boldsymbol{D}$. Moreover, ι preserves all meets and joins that exist in B. In particular, if 0 is the least element of $\boldsymbol{A}$ and $0 \in B$ then $\iota(0) = 0^{\boldsymbol{D}}$.*

Theorem 3.3.

(i) Every partial pocrim can be embedded in a (complete) residuated lattice,

(ii) Every partial BCK-algebra can be embedded in a (complete) residuated lattice.

We prove that finiteness is preserved in the construction. Let $\langle \mathrm{N}, \leq \rangle$ denote the natural numbers. When $\boldsymbol{B}$ is finite, with $|B| = k$ say, we show that there is a surjective order-reversing map h from $\langle \mathrm{N}, \leq \rangle^k$ to the ordered monoid $\boldsymbol{M}$ and a subset Z of N^k such that for each $(a \rightsquigarrow b] \in \overline{D}$ there is a set $W \subseteq Z$ such that $(a \rightsquigarrow b] = h([W])$. Recall that a partially ordered set $\langle S, \leq \rangle$ is *well-quasi-ordered* if it is well-ordered and if it contains no infinite antichains. It is known that the direct product of well-quasi-ordered sets is well-quasi-ordered (see, for example, [8]). Since the natural numbers are well-quasi-ordered, so is the partially ordered set $\langle \mathrm{N}, \leq \rangle^k$. The set Z can be chosen to be the downward closure of an antichain in $\langle \mathrm{N}, \leq \rangle^k$ and hence is finite. Thus Z has only finitely many subsets, and it follows that $\overline{D}$, and hence D, are finite.

Lemma 3.4. *If B is a finite partial subalgebra of $\boldsymbol{A}$ then the algebra $\boldsymbol{D}$ is finite.*

Theorem 3.5. *Each of the following quasivarieties has the FEP:*

(i) The variety $\mathcal{R}$ of residuated lattices,

(ii) the quasivariety of pocrims,

(iii) the quasivariety of BCK-algebras.

The subvariety $\mathcal{R}_n$ of $\mathcal{R}$ consisting of all n-potent residuated lattices, defined, relative to $\mathcal{R}$, by the identity $x^{n+1} \approx x^n$, also has the FEP, and this can be seen in a more straightforward way (see [3] or [5]). The proof depends however on the presence of the $\vee$-operation, and does not carry over to the case of pocrims.

Problem 3.6. *Does the variety of pocrims satisfying $x^{n+1} \approx x^n$ possess the FEP?*

The same question about n-potent BCK-algebras is also open.

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