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Francesco Paoli

S IS CONSTRUCTIVELY COMPLETE

Introduction

The logic S (Symmetric Propositional Relatedness Logic) was introduced in the late 1970s by Richard Epstein, and thoroughly studied over the subsequent decade by Epstein himself and his collaborators - a clear outline of this work is presented in [2]. Epstein's starting point was a semantical analysis of the concept of subject matter relatedness among propositions, which eventually led to an axiomatization and a nonconstructive completeness proof with respect to that semantics.

We shall provide the axiom system S devised by Epstein with a constructive completeness proof, extending thereby to the full system of propositional logic some of our results on first degree relatedness conditionals (cp. [3]). For this purpose, however, we shall make use of Epstein's semantics, as well as of a syntactic counterpart of a normal form theorem, already proved by Epstein in a semantic guise.

An Axiom System

The system **S** (cp. [2], p. 80) has a language $\mathcal{L}(\mathbf{S})$ containing a denumerable list of variables p_1, p_2, ... and the connectives $\neg$, $\wedge$, $\rightarrow$.

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We use the following abbreviations:

$A \vee B =_{\mathrm{df}} \neg(\neg A \wedge \neg B)$;
$A \leftrightarrow B =_{\mathrm{df}} (A \to B) \wedge (B \to A)$;
$R(A, B) =_{\mathrm{df}} A \to (B \to B)$.

The set WFF of formulas is defined as the smallest set containing the variables and being closed under $\neg$, $\wedge$, $\to$. The axiom schemata and the rules of **S** are the following (A, B are metavariables for elements of WFF):

A1. $R(A, A)$
A2. $R(B, A) \to R(A, B)$
A3. $R(A, \neg B) \leftrightarrow R(A, B)$
A4. $R(A, B \to C) \leftrightarrow R(A, B) \vee R(A, C)$
A5. $R(A, B \wedge C) \leftrightarrow R(A, B \to C)$
A6. $(A \wedge B) \to A$
A7. $A \to (B \to (A \wedge B))$
A8. $(A \wedge B) \to (B \wedge A)$
A9. $A \leftrightarrow \neg\neg A$
A10. $(A \to B) \leftrightarrow \neg(A \wedge \neg B) \wedge R(A, B)$
A11. $A \to (\neg(B \wedge A) \to \neg B)$
A12. $\neg(A \wedge B) \to (\neg(C \wedge \neg B) \to \neg(A \wedge C))$
A13. $\neg((C \to D) \wedge (C \wedge \neg D))$
R1. $\dfrac{A \qquad A \to B}{B}$

Lemma 1. *The following rules are derived rules of* **S***:*

R2. $\dfrac{A \to B \neg B}{\neg A}$ (A6, A10, A11)
R3. $\dfrac{A \wedge B}{A} \quad \dfrac{A \wedge B}{B}$ (A6, A8)
R4. $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ (A6, A9, R2)
R5. $\dfrac{A \quad B}{A \wedge B}$ (A7)

R6. $\dfrac{A \to B \quad B \to C}{A \to C}$ provided that $\vdash R(A, C)$

R7. $\dfrac{A}{B \to A}$ provided that $\vdash R(B, A)$

R8. $\dfrac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)}$ (A6, A9, A10, A11, A12, A13)

R9. $\dfrac{A \to B \quad A \to C}{A \to B \wedge C}$ (A4-5, A6, A10, R2, R5, R6, R8)

R10. $\dfrac{A \wedge (B \wedge C)}{(A \wedge B) \wedge C}$ (A6, R5)

R11. $\dfrac{A \to (B \to C)}{(A \wedge B) \to C}$ provided that $\vdash R(B, C)$ (A1-5, A6, A9, A10, R2, R6, R10)

R12. $\dfrac{A \to BA \to (B \to C)}{A \to C}$ provided that $\vdash R(A, C)$

Proof. Left to the reader, except for R6, R7, and R12, especially important for our deduction theorem below.

Proof of R6. Suppose $A \to B$ and $B \to C$. Then (A10, A6) $\neg(A \wedge \neg B)$ and $\neg(B \wedge \neg C)$. From the latter item, via A8 and R2, we get $\neg(\neg C \wedge B)$. But $\neg(\neg C \wedge B) \to (\neg(A \wedge \neg B) \to \neg(\neg C \wedge A))$ is an instance of A12. Applying R1 twice, we get $\neg(\neg C \wedge A)$, whence, by the same procedure as before, $\neg(A \wedge \neg C)$. But we assumed $\vdash R(A, C)$ at the outset. By R5, $\neg(A \wedge \neg C) \wedge R(A, C)$. Notice that $\neg(A \wedge \neg C) \wedge R(A, C) \to (A \to C)$ is an instance of A10. Thus, by R1, $A \to C$.

Proof of R7. $\neg A \wedge B \to \neg A$ and $B \wedge \neg A \to \neg A \wedge B$ are instances, resp., of A6 and A8. By A1-A5 we get $\vdash R(B \wedge \neg A, \neg A)$ and by R6, then, $B \wedge \neg A \to \neg A$. From our assumption A and $A \to \neg\neg A$ (A9), we derive $\neg\neg A$ by R1. Hence, by R2, $\neg\neg A$ and $B \wedge \neg A \to \neg A yield \neg(B \wedge \neg A)$. But we assumed $\vdash R(B, A)$. By R5, $\neg(B \wedge \neg A) \wedge R(B, A)$. Remark that $\neg(B \wedge \neg A) \wedge R(B, A) \to (B \to A)$ is an instance of A10. By R1, we get $B \to A$.

Proof of R12. Assume $A \to B$ and $A \to (B \to C)$. From A6, A10, R6 we have $(B \to C) \to \neg(B \wedge \neg C)$. From $\vdash R(A, C)$ and A1-A5, we get $\vdash$

$R(A, \neg(B \wedge \neg C))$. Hence, from $A \rightarrow (B \rightarrow C)$ and $(B \rightarrow C) \rightarrow \neg(B \wedge \neg C)$, by R6, we get $A \rightarrow \neg(B \wedge \neg C)$. Now, R9 gives us $A \rightarrow (B \wedge \neg(B \wedge \neg C))$. But $B \rightarrow (\neg(B \wedge \neg C) \rightarrow \neg\neg C)$ is an instance of A11. Since $R(\neg(B \wedge \neg C), \neg\neg C)$ is provable by A1-A5, by R11 we can derive $B \wedge \neg(B \wedge \neg C) \rightarrow \neg\neg C$ and (A9 and again R6) $B \wedge \neg(B \wedge \neg C) \rightarrow C$. By transitivity, permissible since $\vdash R(A, C)$, we get $A \rightarrow C$.

Lemma 2. *If A and B share a variable, then* $\vdash R(A, B)$.

Proof. Double induction on the number of connectives occurring in A, resp. B. As usual, we call such a number the complexity of the corresponding formula.

Base. If $A = B = p$, then $R(A, B) = R(p, p)$ is an instance of A1.

Step. (i) Let us suppose that the theorem is true for A of complexity 1 ($A = p$) and B of complexity $\leq n$. We distinguish the following cases:

(i.i) $B = \neg C$. Let us suppose that $\neg C$ (hence, C) contains p. We have:

1. $R(p, C)$ IH
2. $R(p, C) \rightarrow R(p, \neg C)$ A3
3. $R(p, \neg C)$ 1, 2, R1

(i.ii) $B = C \rightarrow D$. Let us suppose that $C \rightarrow D$ contains p. Then, either C or D contains p. Assume it is C (the other case is similar):

1. $R(p, C)$ IH
2. $R(p, C) \vee R(p, D)$ R4
3. $R(p, C) \vee R(p, D) \rightarrow R(p, C \rightarrow D)$ A4
4. $R(p, C \rightarrow D)$ 2, 3, R1

(i.iii) $B = C \wedge D$. Like in the preceding case, let us suppose that C contains p. Steps 1-4 are the same as before. Then:

5. $R(p, C \rightarrow D) \rightarrow R(p, C \wedge D)$ A5
6. $R(p, C \wedge D)$ 4, 5, R1

(ii) Let us suppose, now, that the theorem holds for A of complexity $\leq n$ and B whatsoever. We explicitly treat the case of negation - the other ones are analogous. Well, assume that $A = \neg C$. If $\neg C$ and B share a variable, then so do C and B. hence:

1. $R(C, B)$ IH
2. $R(C, B) \rightarrow R(B, C)$ A2
3. $R(B, C)$ 1, 2, R1
4. $R(B, C) \rightarrow R(B, \neg C)$ A3
5. $R(B, \neg C)$ 3, 4, R1
6. $R(B, \neg C) \rightarrow R(\neg C, B)$ A2
7. $R(\neg C, B)$ 5, 6, R1

Corollary. *If A and C share a variable, the rule of transitivity:*

R6'. $$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}$$

holds unrestrictedly.

Proof. By Lemma 1 and Lemma 2.

Lemma 3. *Suppose $\vdash R(A, B)$. If B is deducible in S from G, A (in the ordinary, classical sense), then $A \rightarrow B$ is deducible in S from Γ.*

Proof. As usual for deduction theorems, we prove this lemma by induction on the length of the deduction of B from Γ, A.

(i) B is an axiom.

1. B Ax.
2. $R(A, B)$ Theor.
3. $A \rightarrow B$ 1, 2, R7

(ii) B is in Γ.

1. B Hyp.
2. $R(A, B)$ Theor.

3. $A \to B$ 1, 2, R7

(iii) $B = A$

1. $A \to \neg\neg A$ A9
2. $\neg\neg A \to A$ A9
3. $R(A, A)$ A1
4. $A \to A$ 1, 2, 3, R6

(iv) B is obtained by R1 from C and $C \to B$. By IH, there exists a deduction of $A \to C$ and $A \to (C \to B)$ from Γ.

...

n. $A \to C$ IH

n+1 $A \to (C \to B)$ IH

n+2 $R(A, B)$ Theor.

n+3 $A \to B$ n, n+1, n+2, R12

A straightforward consequence of Lemma 2 and Lemma 3 is the following

Corollary. *If A and B share a variable, and B is deducible in S from Γ, A (in the ordinary, classical sense), then $A \to B$ is deducible in S from Γ.*

Now, let us list some theses and some more derived rules of S:

T1. $A \to A$ (A9)

T2. $\neg A \vee A$ (T1, A8, A9, A10)

T3. $A \wedge B \to B$ (R3)

T4. $A \to A \wedge (B \vee \neg B)$ (T2, T4)

R13. $\dfrac{A \to B}{\neg B \to \neg A}$ (R2, A1-5)

T5. $(A \to B) \to (\neg B \to \neg A)$ (R13)

T6. $A \to A \vee B$

$B \to A \vee B$ (R4)

R14. $\dfrac{A \to C \quad B \to C}{A \vee B \to C}$ (R9, R13)

T7. $A \leftrightarrow A \wedge A$ (A6, T1, R9)

T8. $A \vee A \leftrightarrow A$ (T1, T6, R14)

T9. $A \vee B \leftrightarrow B \vee A$ (T6, R14)

T10. $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$ (R8)

T11. $A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C$ (T6, R14)

T12. $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$ (A6, A9, T3, T6, R9, R13, R14)

T13. $(A \vee B) \wedge C \to (A \wedge C) \vee (B \wedge C)$ (A8, R8)

T14. $(A \wedge B) \vee C \leftrightarrow (A \vee C) \wedge (B \vee C)$ (A9, T13)

T15. $A \leftrightarrow (A \wedge p_1 \wedge \ldots \wedge p_n \wedge R(p_1, p_2) \wedge \ldots \wedge R(p_{n-1}, p_n)) \vee (A \wedge \neg p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge R(p_1, p_2) \wedge \ldots \wedge R(p_{n-1}, p_n)) \vee (A \wedge p_1 \wedge \neg p_2 \wedge \ldots \wedge p_n \wedge R(p_1, p_2) \wedge \ldots \wedge R(p_{n-1}, p_n)) \vee \ldots \vee (A \wedge \neg p_1 \wedge \ldots \wedge \neg p_n \wedge \neg R(p_1, p_2) \wedge \ldots \wedge \neg R(p_n, p_{n-1}))$

(right to left: A6, R14; left to right: A8, T2, R5, T9, T10, T11, T13, T14)

$A \leftrightarrow (A \vee p_1 \vee \ldots \vee p_n \vee R(p_1, p_2) \vee \ldots \vee R(p_{n-1}, p_n)) \wedge (A \vee \neg p_1 \vee p_2 \vee \ldots \vee p_n \vee R(p_1, p_2) \vee \ldots \vee R(p_{n-1}, p_n)) \wedge (A \vee p_1 \vee \neg p_2 \vee \ldots \vee p_n \vee R(p_1, p_2) \vee \ldots \vee R(p_{n-1}, p_n)) \wedge \ldots \wedge (A \vee \neg p_1 \vee \ldots \vee \neg p_n \vee \neg R(p_1, p_2) \vee \ldots \vee \neg R(p_n, p_{n-1}))$

(right to left: T6, R9; left to right: A8, A9, T2, R5, T9, T11, T12, T13, T14)

T16. $(A \leftrightarrow B) \to (\neg A \leftrightarrow \neg B)$ (A6, T3, T5, R9)

T17. $((A \leftrightarrow B) \wedge (C \leftrightarrow D)) \to (A \wedge C \leftrightarrow B \wedge D)$ (A6, A7)

T18. $((A \vee B) \wedge (A \to C) \wedge (B \to C)) \to C$ (A6, R14)

To prove such theorems, use Lemma 3 (and its corollary) and the clues provided aside. T15 (rather clumsy in its appearance, indeed) is a generalized relatedness counterpart of such classical tautologies as $A \leftrightarrow (A \wedge p) \vee (A \wedge \neg p)$ and $A \leftrightarrow (A \vee p) \wedge (A \vee \neg p)$. Both conjunctions and disjunctions are associated to the left when brackets are omitted.

A standard unrestricted replacement theorem is not available in S (cp. [2], p. 81). Nonetheless, we are now in a position to prove the following:

Theorem 1. (Restricted replacement) *Let $D[A/B]$ be obtained by simultaneously replacing zero or more occurrences of A by B in D. Moreover, assume that $R(A,C) \leftrightarrow R(B,C)$ for every subformula C of D. Then $\vdash A \leftrightarrow B \Rightarrow \vdash D \rightarrow D[A/B]$.*

Proof. Induction on the complexity of D.

(i) $D = p$. First case: $A \neq p$. In such a case, $D \rightarrow D[A/B]$ is nothing else than $p \rightarrow p$, an instance of T1. Second case: $A = p$. Then $D = p = A$. Therefore, from $\vdash p \leftrightarrow B$ we are allowed to infer $\vdash p \rightarrow p/B$.

(ii) $D = \neg E$. We use the IH and T16.

(iii) $D = E \wedge F$. We use the IH, R5 and T17.

(iv) $D = E \rightarrow F$. We must show that $\vdash A \leftrightarrow B \Rightarrow \vdash (E \rightarrow F) \rightarrow (E[A/B] \rightarrow F[A/B])$ under the assumption $R(A,C) \leftrightarrow R(B,C)$.

First case: A appears neither in E nor in F, or else no occurrence of A in either formula is replaced by B. Then $E[A/B] \rightarrow F[A/B] = E \rightarrow F$, and $(E \rightarrow F) \rightarrow (E[A/B] \rightarrow F[A/B])$ is an instance of T1.

Second case: A appears in both E and F, and some occurrence thereof is actually replaced in both formulas. By A10, A6 and R6', $(E \rightarrow F) \rightarrow \neg(E \wedge \neg F)$; but, using the IH and the same principles as in cases (ii) and (iii), we have that $\neg(E \wedge \neg F) \rightarrow \neg(E[A/B] \wedge \neg F[A/B])$. If we want to apply R6 to the preceding formulas and get $(E \rightarrow F) \rightarrow \neg(E[A/B] \wedge \neg F[A/B])$, we need to know that $\vdash R(E \rightarrow F, \neg(E[A/B] \wedge \neg F[A/B]))$. But $A \rightarrow B$ is provable by hypothesis: hence so is $R(A,B)$. Remember that, by our assumption, $E \rightarrow F$ contains A and some occurrence thereof is actually replaced by B. Then, repeatedly using A1-A5, we have that $R(E \rightarrow F, B)$ is provable, and, by A1-A5 again, so is $R(E \rightarrow F, \neg(E[A/B] \wedge \neg F[A/B]))$. By R6', then, we can deduce $(E \rightarrow F) \rightarrow \neg(E[A/B] \wedge \neg F[A/B])$.

Likewise, by A10, A6 and R6' we get (2) $(E \rightarrow F) \rightarrow R(E,F)$. Moreover, let $\gamma_1(A), \ldots, \gamma_n(A)$ be exactly those subformulas of F where some occurrences of A are to be replaced by B in $F[A/B]$, and let $\delta_1, \ldots, \delta_m$ be the remaining subformulas of F. Then, repeatedly resorting to A2-A5, we achieve:

(3) $R(E, F) \to R(E, \gamma_1(A)) \vee \ldots \vee R(E, \gamma_n(A)) \vee R(E, \delta_1) \vee \ldots \vee R(E, \delta_m)$,

whence, using A2-A5, T9, T11 and R6' (first to "ungroup" formulas, then to "regroup" them, possibly utilizing T8 to cancel redundancies):

(4) $R(E, F) \to R(E, A) \vee R(E, G)$,

where G is a disjunction (or, for that matter, a conjunction) of subformulas of F with no (to-be-replaced) occurrence of A therein. Moreover, by our assumption,

(5) $R(E, A) \to R(E, B)$,

and from (5), by A2-A5 and R6',

(6) $R(E, A) \to R(E, F[A/B])$.

By A2-A5 again,

(7) $R(E, G) \to R(E, F[A/B])$.

Then, collecting together (2)-(7), via R6' and T18 we have that

(8) $(E \to F) \to R(E, F[A/B])$.

Carrying out the previous reasoning with respect to $E \to F[A/B]$, we are allowed to deduce $R(E[A/B], F[A/B])$, whence, by R6,

(9) $(E \to F) \to R(E[A/B], F[A/B])$,

which in turn yields, together with (1), by R9 and A10,

(10) $(E \to F) \to (E[A/B] \to F[A/B])$.

The remaining cases are treated similarly.

As a consequence of the theorem, we have unrestricted admissibility of the replacement rule if A and B contain the same variables $p_1, \ldots, p_n$. In fact, starting from $R(p_1, C) \vee \ldots \vee R(p_n, C)$, by A1-A5, T8, T9, T11, R6' and Theorem 1 we compound both formulas out of their atoms and eventually get $R(A, C) \leftrightarrow R(B, C)$.

Remark that, in the proof of Theorem 1, we used the additional stipulation that $R(A, C) \leftrightarrow R(B, C)$ for every subformula C of D nowhere but in the inductive step $D = E \to F$. Hence, we immediately have the following lemma. Let A be an implicative formula iff it has the form $B \to C$. Then,

Lemma 4. (Truth-functional replacement) *Let C be a formula containing some occurrences of the formula A, and let $C[A/B]$ be obtained by replacing one or more occurrences of A by B in C. Moreover, suppose that the replaced occurrences of A are not subformulas of any implicative subformula of C. Then $\vdash A \leftrightarrow B \Rightarrow \vdash C \leftrightarrow C[A/B]$.*

Normal forms

Let us now introduce some terminology.

A *truth and relatedness* ($T\&R$) atom is a variable in $/L(S)$, or the negation of a variable in $/L(S)$, or a formula having the form $R(p, q)$ (where p and q are variables in $/L(S)$) or the negation of such.

A *$T\&R$ primitive conjunction* is a generalized conjunction of $T\&R$ atoms.

A *$T\&R$ setup* in $p_1, \ldots, p_n$ (we borrow this term from [1]) is a $T\&R$ primitive conjunction $B_1 \wedge \ldots \wedge B_m$ s.t.:

(i) $B_1 \wedge \ldots \wedge B_m$ contains at most the variables $p_1, \ldots, p_n$;

(ii) its $T\&R$ atoms are alphabetically ordered, the alphabetical order being given by $p_1, \neg p_1, p_2, \neg p_2, \ldots, p_n, \neg p_n, R(p_1, p_2), \neg R(p_1, p_2), R(p_1, p_3), \ldots, R(p_n, p_{n-1}), \neg R(p_n, p_{n-1})$;

(iii) there are no repetitions of $T\&R$ atoms (i.e. for $i, j \leq m, i \neq j \Rightarrow B_i \neq B_j$);

(iv) if $i \leq n, j \leq m$ and p_i occurs in $B_1 \wedge \ldots \wedge B_m, R(p_i, p_i) \neq B_j$;

(v) if $i, j \leq n$ and for some $k \leq mB_k = R(p_i, p_j)$, then for every $l \leq mB_l \neq R(p_j, p_i)$.

A *$T\&R$ state description* in $p_1, \ldots, p_n$ is a complete and consistent $T\&R$ setup in $p_1, \ldots, p_n$, i.e. a $T\&R$ setup such that for every $i, j \leq n, i \neq j$, exactly one of $p_i, \neg p_i$ (exactly one of $R(p_i, p_j), \neg R(p_i, p_j)$) occurs as conjunct therein.

A $T\&RP$- (respectively S-, D-) *disjunctive normal form* in $p_1, \dots, p_n$ is a generalized disjunction of $T\&R$ primitive conjunctions (resp. setups in $p_1, \dots, p_n$, state descriptions in $p_1, \dots, p_n$).

A $T\&R$ *perfect tautology* in $p_1, \dots, p_n$ is a "combinatorially complete" $T\&RD-$disjunctive normal form in $p_1, \dots, p_n$, i.e. a generalized disjunction $A_1 \vee \dots \vee A_{2^{n+n(n-1)/2}}$ where, if B_i is arbitrarily chosen from the set $p_i, \neg p_i$ and B_j is arbitrarily chosen from the set $R(p_k, p_l), \neg R(p_k, p_l), k \neq l$, then, for every i, j, k, l and for every such possible choice, there exists an $x \leq 2^{n+n(n-1)/2}$ s.t. (let indexed conjunction be introduced as usual) $A_x = (\wedge_{i \leq n}(B_i)) \wedge (\wedge_{j \leq n(n-1)/2}(B_j))$.

Henceforth, whenever no risk of ambiguity is impending, we shall drop the prefix "$T\&R$", as well as brackets and commas in formulas of the form $R(p, q)$.

As a first result we have:

Lemma 5. *Every perfect tautology is provable in S.*

Proof. Let A be a perfect tautology in $p_1, \dots, p_n$. Then, by T2 we have that, for every $i, j, k \leq n$, $j \neq k$, $\vdash p_i \vee \neg p_i, \vdash Rp_jp_k \vee \neg Rp_jp_k$. Then, in virtue of R5, $\vdash (\wedge i \leq n(p_i \vee \neg p_i)) \wedge (\wedge j, k \leq n(Rp_jp_k \vee \neg Rp_jp_k))$. Therefore, to attain our conclusion, we apply several times A9 and the "normal form" theorems T7-T14, as well as instances of replacement permitted by Theorem 1 and Lemma 4.

A semantic normal form theorem for S was proved by Epstein ([2], p. 79). We now extend his result to a purely syntactic normal form theorem.

Theorem 2. (S-disjunctive normal form) *If $A \in WFF$, there exists an S-disjunctive normal form B, containing exactly the same variables as A, s.t. $\vdash A \leftrightarrow B$.*

Proof. We first show that (1) for every wff there is a P-disjunctive normal form provably equivalent to the former; we shall then proceed to demonstrate that (2) every P-disjunctive normal form can be strengthened to an S-disjunctive normal form still preserving provable equivalence. (1) and (2), conjoined, yield Theorem 2.

(Proof of 1) Induction on the complexity of formulas.

Base. If $A = p$, then $p \leftrightarrow p \wedge (p \vee \neg p) \wedge (Rpp \vee \neg Rpp)$ is an instance of T15 (second version). Applying T13, R6', and Theorem 1 to it, we get $\vdash p \leftrightarrow (p \wedge p \wedge Rpp) \vee (p \wedge p \wedge \neg Rpp) \vee (p \wedge \neg p \wedge Rpp) \vee (p \wedge \neg p \wedge \neg Rpp)$.

Step.

(i) $A = \neg B$. Left to the reader (clue: use A9, T12, and Theorem 1).

(ii) $A = B \wedge C$. Just one more exercise (use A8, T9-T11, T13 and Theorem 1).

(iii) $A = B \to C$. By IH, there exist P-dnfs B^*, C^* s.t. $\vdash B \leftrightarrow B^*, \vdash C \leftrightarrow C^*$. We shall show that there is a P-dnf provably equivalent to $B^* \to C^*$ - whence, as B and B^* (resp. C and C^*) contain by IH the same variables, applying Theorem 1 we are able to infer that such a formula is provably equivalent to $B \to C$.

1) $\vdash B^* \to C^* \leftrightarrow (\neg B^* \vee C^*) \wedge R(B^*, C^*)$ [A10].

2) $\vdash (\neg B^* \vee C^*) \wedge R(B^*, C^*) \leftrightarrow (\neg B^* \vee C^*) \wedge R(B_1 \vee \ldots \vee B_n, C_1 \vee \ldots \vee C_m)$ [Def.].

3) $\vdash (\neg B^* \vee C^*) \wedge R(B_1 \vee \ldots \vee B_n, C_1 \vee \ldots \vee C_m) \leftrightarrow (\neg B^* \vee C^*) \wedge (R(B_1, C_1 \vee \ldots \vee C_m) \vee \ldots \vee R(B_n, C_1 \vee \ldots \vee C_m))$ [A3-A5, Theo. 1].

4) $\vdash (\neg B^* \vee C^*) \wedge (R(B_1, C_1 \vee \ldots \vee C_m) \vee \ldots \vee R(B_n, C_1 \vee \ldots \vee C_m)) \leftrightarrow (\neg B^* \vee C^*) \wedge (R(B_1, C_1) \vee \ldots \vee R(B_1, C_n) \vee \ldots \vee R(B_n, C_1) \vee \ldots \vee R(B_n, C_m))$ [A3-A5, Theo. 1].

Now, let $B_i = b_{i1} \wedge \ldots \wedge b_{ij}, C_k = c_{k1} \wedge \ldots \wedge c_{kl}$; likewise, let D be the formula $(R(b_{11}, c_{11}) \vee \ldots \vee R(b_{1j}, c_{11}) \vee \ldots \vee R(b_{1j}, c_{1k}) \vee \ldots \vee R(b_{ij}, c_{kl}))$. The proof goes on as follows:

5) $\vdash (\neg B^* \vee C^*) \wedge (R(B_1, C_1) \vee \ldots \vee R(B_1, C_n) \vee \ldots \vee R(B_n, C_1) \vee \ldots \vee R(B_n, C_m)) \leftrightarrow (\neg B^* \vee C^*) \wedge D$ [A3-A5, Theo. 1].

6) $\vdash (\neg B^* \vee C^*) \wedge D \leftrightarrow (\neg B^* \wedge D) \vee (C^* \wedge D)$ [T13, Theo. 1].

7) $\vdash (\neg B^* \wedge D) \vee (C^* \wedge D) \leftrightarrow ((\neg B)^* \wedge D) \vee (C^* \wedge D)$ [A9, T12, T13, Theo. 1],

where $(\neg B)^*$ is what you get by repeatedly resorting to distribution after having "thrust" negation into B^* via De Morgan (used twice). Let $B'_1, \dots, B'_m$ be the disjuncts thus obtained.

8) $\vdash ((\neg B)^* \wedge D) \vee (C^* \wedge D) \leftrightarrow (B'_1 \wedge D) \vee \dots \vee (B'_m \wedge D) \vee (C_1 \wedge D) \vee \dots \vee (C_m \wedge D)$ [T13].

By applying T13 to $(B'_1 \wedge D) \vee \dots \vee (B'_m \wedge D) \vee (C_1 \wedge D) \vee \dots \vee (C_m \wedge D)$, we have that $((\neg B)^* \wedge D) \vee (C^* \wedge D)$ is provably equivalent to a P-dnf in the same variables. Then, by 1)-8) and R6', $B^* \to C^*$ is provably equivalent to a P-dnf as well.

bf (Proof of 2). Given a wff A, in virtue of the first part of this theorem, there exists a P-dnf $A^* = B_1 \vee \dots \vee B_n$, with exactly the same variables as A, which is provably equivalent to it. We shall single out an S-dnf A^{**} in the same variables which is provably equivalent to A^* - and this will suffice, as we can apply restricted transitivity.

We "tinker" with A^*, namely with each one of the B_i's, as follows.

(A) First, by A8 and T10, we arrange its atoms in alphabetical ordering; we thus get a B'_i which is (due to Theorem 1) still provably equivalent to B_i.

(B) Next, we do away with redundancies, applying T7. A further recourse to Theorem 1 guarantees that $B'_i = B_i[A \wedge A/A]$ is still equivalent to B_i.

(C) Then, we resort to the theorem:

(T19) $A \to (B \leftrightarrow (A \wedge B))$ (A2, A4, A7, A10, T2, T6, T11, R9)

whence, by A1 and R1, we are allowed to conclude that $\vdash B_i \leftrightarrow Rpp \wedge B_i$. Provided that the variable p occurs in B_i, then, we can erase Rpp therein (the proviso is needed to comply with the requirements of Theorem 1).

(D) In a similar fashion, by A2, we rub out Rqp if B_i contains Rpq.

Let B_i' be, for every B_i, the result of the previous adjustments. By our construction, B_i' is a setup containing the same variables as B_i and provably equivalent to it. Thus, appealing to restricted replacement, we have that A^* is provably equivalent to $A^*[B_i/B_i']$. Carrying out the forementioned procedure as many times as necessary, we get our conclusion.

Semantics

We shall limit ourselves to just a few hints with regard to the general semantics of S; for more details, the reader is referred to the extensive and exhaustive treatment of [2].

Let PV be the set of propositional variables in $/L(S)$ and let $v : PV \to 0, 1$. Moreover let $\mathfrak{R}^{PV} \subset PV^2$ be a reflexive and symmetric relation which is extended to $\mathfrak{R} \subset WFF^2$ by means of:

R1. $\mathfrak{R}(A, B)$ iff $\mathfrak{R}(\neg A, B)$

R2. $\mathfrak{R}(A, B \wedge C)$ iff $\mathfrak{R}(A, B \to C)$

R3. $\mathfrak{R}(A, B)$ iff $\mathfrak{R}(B, A)$

R4. $\mathfrak{R}(A, A)$

R5. $\mathfrak{R}(A, B \wedge C)$ iff $\mathfrak{R}(A, B)$ or $\mathfrak{R}(A, C)$

A *valuation* $V_{\mathfrak{R}} : WFF \to 0, 1$ is inductively defined as:

$V_{\mathfrak{R}}(p) = v(p)$;
$V_{\mathfrak{R}}(\neg A)$, $V_{\mathfrak{R}}(A \wedge B)$ are calculated with the aid of the classical truth tables for $\neg$ and $\wedge$;
$V_{\mathfrak{R}}(A \to B) = 1$ if $\mathfrak{R}(A, B)$ and $(V_{\mathfrak{R}}(A) = 0$ or $V_{\mathfrak{R}}(B) = 1)$; $= 0$ otherwise.

We define:

$V_{\mathfrak{R}} \models A$ (A *true* in $V_{\mathfrak{R}}$) iff $V_{\mathfrak{R}}(A) = 1$;
$\models_S A$ (A S-logically true) iff $V_{\mathfrak{R}} \models A$ for every $V_{\mathfrak{R}}$.

Recall that:

Theorem 3. (Soundness) *If* $\vdash_S A$*, then* $\models_S A$.

Proof. Induction on the length of proofs.

We now have at our disposal the necessary ingredients to "cook" our:

Theorem 4. (Completeness) *If* $\models_S A$*, then* $\vdash_S A$.

Proof. As usual, we shall prove the contrapositive. Thus, suppose not $\vdash A$. Let $A^* = A_1 \vee \ldots \vee A_m$ be the S-dnf in $p_1, \ldots, p_n$ whose existence, and provable equivalence to A, is guaranteed by Theorem 2. Of course, we have that not $\vdash A^*$ and moreover, for $i \leq m$, not $\vdash A_i$ (if it were otherwise, by T6 we should have $\vdash A^*$, whence $\vdash A$).

If a *conjunctive complementary pair* is defined as a formula of the form $p \wedge \neg p (Rpq \wedge \neg Rpq)$, then, generally speaking, some of the A_i's will contain conjunctive complementary pairs (ccps), whereas other ones will not. For the sake of simplicity, let us fix an i and suppose that for $j \leq i$, A_j contains some ccp, whereas for $j > i$ A_j does not. It is then clear that, for every $j \leq i$ and for every $V_{\mathfrak{R}}$, $V_{\mathfrak{R}}(A_j) = 0$.

Let now $k > i$. Then

$A_k = (p_1) \wedge (\neg p_1) \wedge \ldots \wedge (p_n) \wedge (\neg p_n) \wedge (Rp_1p_2) \wedge (\neg Rp_1p_2) \wedge \ldots \wedge (Rp_np_{n-1}) \wedge (\neg Rp_np_{n-1})$,

where bracketed items are possibly missing. We now need to complete A_k in order to make it a state description. We know that, because of our hypothesis on ccps, for each pair $pi, \neg pi$ (resp. $Rp_ip_j, \neg Rp_ip_j$), not both the first and the second element occur in A_k. If neither does (i.e., intuitively speaking, if A_k says nothing about whether it is the case that p_i or about whether p_i and p_j are related to each other), we integrate the missing items by T15 (first version), replacing A_k by $A_{k1} \vee \ldots \vee A_{kh}$ in such a way that exactly one element of each "missing" pair occurs in each disjunct. Lemma 4, then, ensures mutual intersubstitutability of A_k and $A_{k1} \vee \ldots \vee A_{kh}$.

Example. Let $p \wedge \neg r \wedge \neg Rpq \wedge Rpr$ be a setup in p, q, r. Applying T15 we get $(p \wedge \neg r \wedge \neg Rpq \wedge Rpr \wedge q \wedge Rqr) \vee (p \wedge \neg r \wedge \neg Rpq \wedge Rpr \wedge q \wedge \neg Rqr) \vee (p \wedge \neg r \wedge \neg Rpq \wedge Rpr \wedge \neg q \wedge \neg Rqr) \vee (p \wedge \neg r \wedge \neg Rpq \wedge Rpr \wedge \neg q \wedge Rqr)$.

After we've got this thing done (and after we have "tidied up" via A8, T7-T11), we have a disjunction C, intersubstitutable with $A_{i+1} \vee \ldots \vee A_m$ at least in cases provided for by Lemma 4, in which each C_k is now a state description in $p_1, \ldots, p_n$ having the form $B_1 \wedge \ldots \wedge B_n \wedge B_{n+1} \wedge \ldots \wedge B_{n+n(n-1)/2}$, where: for $i \leq n$ B_i is either p_i or $\neg p_i$, for $i > n$ B_i has the form Rp_jp_k $(\neg Rp_jp_k)$.

Now construct the valuation $F_{\mathfrak{R}}$ (F stands for False) as follows: if the kth conjunct of the kth disjunct of C is p $(\neg p)$, set $f(p) = 0$ (1); if it is Rpq $(\neg Rpq)$, set $\mathfrak{R}(p, q) = 0$ (1). It may of course happen that, if the number of disjuncts in C is greater than $n + n(n-1)/2$, our procedure is at some time "blocked", i.e. there is a C_h such that, depending on the values thus far assigned, $F_{\mathfrak{R}}(C_h) = 1$.

Example. Let C be $(p \wedge q \wedge Rpq) \vee (\neg p \wedge q \wedge Rpq) \vee (p \wedge \neg q \wedge Rpq) \vee (\neg p \wedge \neg q \wedge \neg Rpq)$. The number of disjuncts is $4 > 3 = 2 + 2x1/2$. By our construction, $f(p) = f(q) = \mathfrak{R}(p, q) = 0$. Then $F_{\mathfrak{R}}(\neg p \wedge \neg q \wedge \neg Rpq) = 1$.

If this is the case, reassign values considering for instance the $k+1$th conjunct of the kth disjunct, until you get an "unblocked" valuation. That you will never thrust yourself into a blind alley is guaranteed by Lemma 5, according to which perfect tautologies are provable in S (and if C were such, then A^* would be such as well, against our hypothesis).

Summing up: by our construction, $F_{\mathfrak{R}}(C) = 0$; then, in virtue of Theorem 3, $F_{\mathfrak{R}}(A_{i+1} \vee \ldots \vee A_m) = 0$; therefore, since for every $V_{\mathfrak{R}}$, $V_{\mathfrak{R}}(A_1 \vee \ldots \vee A_i) = 0$, we have that $F_{\mathfrak{R}}(A_1 \vee \ldots \vee A_i) = 0$ and thus $F_{\mathfrak{R}}(A^*) = 0$. Again, Theorem 3 ensures that $F_{\mathfrak{R}}(A) = 0$.

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Via dell'Ariento 4
50123 Firenze, Italy.