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Éva JUNGABEL

ON SOME HOMOMORPHISM-HOMOGENEOUS POINT-LINE GEOMETRIES

A b s t r a c t. A relational structure is homomorphism-homogeneous if every homomorphism between finite substructures extends to an endomorphism of the structure. A point-line geometry is a non-empty set of elements called points, together with a collection of subsets, called lines, in a way that every line contains at least two points and any pair of points is contained in at most one line. A line which contains more than two points is called a regular line. Point-line geometries can alternatively be formalised as relational structures. We establish a correspondence between the point-line geometries investigated in this paper and the first-order structures with a single ternary relation L satisfying certain axioms (i.e. that the class of point-line geometries corresponds to a subclass of 3-uniform hypergraphs). We characterise the homomorphism-homogeneous point-line geometries with two regular non-intersecting lines. Homomorphism-homogeneous point-line geometries containing two regular intersecting lines have already been classified by Mašulović.

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1. Introduction

In this article, a structure is a set together with an indexed set of relations on it. A structure is called *homogeneous* if any isomorphism between finite substructures extends to an automorphism. (This is sometimes referred to as ultrahomogeneous structure.)

In 1953 R. Fraïssé [8] defined the age of a relational structure M to be the class $\mathrm{Age}(M)$ of all finite structures embeddable in M as induced substructure. In terms of this notion, he gave a necessary and sufficient condition for the existence of homogenous relational structures with a given age.

In several classes of combinatorial structures, the homogeneous structures are classified: e.g. countable partial orders in [19] by Schmerl, countable graphs in [15] by Lachlan and Woodrow, finite graphs in [9] by Gardiner, countable digraphs in [3] by Cherlin, finite and countable tournaments in [14] by Lachlan, finite or infinite linear spaces in [5] by Devillers and Doyen. In particular the semilinear spaces which have been classified in [4] by Devillers are of special interest to us because semilinear spaces are point-line geometries. A linear space S is a non-empty set of elements called points, provided with a collection of subsets called lines such that any pair of points is contained in exactly one line and every line contains at least two points. A semilinear space S is a non-empty set of elements called points, provided with a collection of subsets called lines such that any pair of points is contained in at most one line and every line contains at least two points. We say that S is d-ultrahomogeneous if every isomorphism from S_1 to S_2 of cardinality at most d can be extended to an automorphism of S. In [4] the authors among other things showed that if S is a 6-ultrahomogeneous non-connected semilinear space, then S is ultrahomogeneous and the connected components of S are isomorphic ultrahomogeneous linear spaces. Any finite connected 6-ultrahomogeneous semilinear space is ultrahomogeneous and it is contained in one of the classes listed in [4].

P. Cameron and J. Nešetřil [2] introduced the following variant of homogeneity: a structure is called *homomorphism-homogeneous* if every homomorphism between finite substructures extends to an endomorphism of the structure.

The first step to understand homomorphism-homogeneous objects is to see a few examples and the characterisation of homomorphism-homogeneous objects in various classes of structures. A structure belongs to the class **MH** if every monomorphism of a finite induced substructure of S into S extends to a homomorphism from S to S. Properties of the homomorphism-homogeneous graphs are investigated by Rusinov and Schweitzer in [18], where the equivalence to the **MH** class is shown. A characterisation of all homomorphism-homogeneous partial orders of arbitrary cardinality was given in [16] by Mašulović. Furthermore, complete classifications are known for finite tournaments with loops [12], finite and countably infinite homomorphism-homogeneous lattices [7], finite homomorphism-homogeneous permutations [6] and homomorphism-homogeneous monounary algebras of arbitrary cardinalities [13] are classified as well. The classification of finite **MH**-homogeneous L-colored graphs where L is a chain is provided by Hartman, Hubička and Mašulović [10]. They also showed that the classes **MH** and **HH** of L-colored graph classes coincide. In the general case the classes **MH** and **HH** do not coincide. Finite homomorphism-homogeneous binary relational structures having two relations that are both symmetric and irreflexive are classified by Hartman and Mašulović [11].

The next natural step would be to characterise homomorphism-homogeneous graphs and hypergraphs. Like in the case of homogeneous structures, where 3-uniform homogeneous hypergraphs [1] are far from being classified, in our case the idea of characterising the homomorphism-homogeneous hypergraphs seems to be more complicated. Moreover, for finite graphs this seems to be also a complicated task at least on a computational level: Rusinov and Schweizer showed in [18] that to decide whether a finite graph is homomorphism-homogeneous or not is coNP-complete.

A point-line geometry is an ordered pair $(\mathcal{X}, \mathcal{L})$, where $\mathcal{X}$ is a non-empty set of elements called points, $\mathcal{L} \subseteq \mathcal{P}(\mathcal{X})$ is a collection of subsets called lines such that every line contains at least two points and every pair of distinct points is contained in at most one line. A regular line is a line which contains more than two points. We will show that the category of point-line geometries is equivalent to a certain subclass of 3-uniform hypergraphs and therefore a first step in characterising homomorphism-homogeneous hypergraphs could be the investigation of point-line geometries. The characterisation of finite homomorphism-homogeneous point-line geometries containing two regular intersecting lines is given by Mašulović [17].

In our paper we continue this idea of describing finite homomorphism-homogeneous point-line geometries containing at least two regular non-intersecting lines. We will discuss the local behavior of finite homomorphism-homogeneous k-stripes, which will allow us to prove our main result, a complete characterisation of finite homomorphism-homogeneous 2-stripes (Theorem 3.11).

2. Preliminaries

A point-line geometry is an ordered pair $(\mathcal{X}, \mathcal{L})$, where $\mathcal{X}$ is a non-empty set of elements, called points, and $\mathcal{L} \subseteq \mathcal{P}(\mathcal{X})$ is a collection of subsets, called lines, such that every line contains at least two points and every pair of distinct points is contained in at most one line. We only consider finite point-line geometries.

A *subgeometry* or *substructure* $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ of the point-line geometry $(\mathcal{X}, \mathcal{L})$ is a point-line geometry, where $\emptyset \neq \mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{L}_\mathcal{Y} = \{l \cap \mathcal{Y} \mid l \in \mathcal{L} \wedge |l \cap \mathcal{Y}| \geq 2\}$. If $\emptyset \neq \mathcal{Y} \subseteq \mathcal{X}$, then the point-line geometry $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ induced on $\mathcal{Y}$ is a subgeometry of $(\mathcal{X}, \mathcal{L})$.

A line which contains more than two points is called a *regular line*. A line which contains exactly two points is called *singular*. Regular lines will be denoted by lower case letters $a, b, c, \dots$ and singular lines will mostly be denoted as AB, where A and B are the points contained in it. An isolated point is a point which belongs to no line of the geometry. The points A and B are collinear if there exists a line $l \in \mathcal{L}$ such that $A, B \in l$. In this case we write $A \sim B$.

A *stripe* is a point-line geometry in which every point lies on precisely one regular line. A *substripe* $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ of the stripe $(\mathcal{X}, \mathcal{L})$ is a stripe which is a subgeometry of $(\mathcal{X}, \mathcal{L})$. A stripe that contains exactly k regular lines is called a k-stripe. We are going to discuss the local behavior of homomorphism-homogeneous k-stripes, which will allow us to prove our main result, a complete characterisation of homomorphism-homogeneous 2-stripes.

We will consider 2-stripes as substripes of a homomorphism-homogeneous k-stripe and we will apply this result to characterise all homomorphism-homogeneous 2-stripes.

For a line $l \in \mathcal{L}$ and for a point $A \in \mathcal{X}$ which does not lie on the line l, let $\mathcal{N}'_l(A)$ be the set of points $A' \in l$ such that $A' \sim A$. In the case when l is a regular line, we omit apostrophe and just write $\mathcal{N}_l(A)$. In the case when there are only two regular lines involved we sometimes omit l and simply write $\mathcal{N}(A)$.

A *singularity of the first type* of $(\mathcal{X}, \mathcal{L})$ in the 2-stripe induced by $(\mathcal{X}, \mathcal{L})$ on the pair (a, b) of regular lines is a point $B \in a$ such that $\mathcal{N}_b(B) = \emptyset$. A *singularity of the second type* of $(\mathcal{X}, \mathcal{L})$ in the 2-stripe induced by $(\mathcal{X}, \mathcal{L})$ on the pair (a, b) of regular lines is a pair points (B, C), where $B \in a$, $C \in b$, such that $\mathcal{N}_b(B) = \{C\}$ and $\mathcal{N}_a(C) = \{B\}$. A point is called singular if itself is a singularity of the first type or it is one of the members of a pair forming a singularity of the second type.

A mapping $f : \mathcal{X} \to \mathcal{Y}$ is a *homomorphism* from a point-line geometry $(\mathcal{X}, \mathcal{L})$ to a point-line geometry $(\mathcal{Y}, \mathcal{K})$ if for every $l \in \mathcal{L}$, either $|f(l)| = 1$ or there is a line $k \in \mathcal{K}$ such that $f(l) \subseteq k$. An *endomorphism* of a point-line geometry is a homomorphism from the point-line geometry into itself. The geometry is homomorphism-homogeneous if every homomorphism between finitely induced subgeometry of the geometry can be extended to an endomorphism of the geometry $(\mathcal{X}, \mathcal{L})$. A homomorphism f from an arbitrary induced subgeometry of a point-line geometry $(\mathcal{X}, \mathcal{L})$ into $(\mathcal{X}, \mathcal{L})$ will be referred to as a *local homomorphism* of $(\mathcal{X}, \mathcal{L})$.

We say that a local homomorphism $f : \mathcal{S} \to \mathcal{X}$, $\mathcal{S} \subsetneq \mathcal{X}$, can be extended to one point if there exists a local homomorphism $f^* : \mathcal{S} \cup \{P\} \to \mathcal{X}$ for any $P \in \mathcal{X} \setminus \mathcal{S}$ and $f^*|_{\mathcal{S}} \equiv f$. We will denote by f^* a one-point extension of the local homomorphism of f.

Lemma 2.1. *The finite point-line geometry $(\mathcal{X}, \mathcal{L})$ is homomorphism-homogeneous if and only if every local homomorphism $f : \mathcal{S} \to \mathcal{X}$ can be extended to one point.*

The point-line geometries in our paper are not first-order structures. In order to relate them to the results mentioned in the introduction, we will define point-line geometries as first-order structures and show that the homomorphisms and substructures of this structure are exactly the same as the homomorphisms and subgeometries according to our definitions. This way we can consider the class of point-line geometries to be a subclass of 3-uniform hypergraphs.

Let $\mathcal{X}$ be a non-empty set of elements. There is a natural correspondence between the point-line geometries and the first-order structures with a single ternary relation L satisfying the following axioms:

1. Reflexivity: For all $A \in \mathcal{X}$ $(A, A, A) \in L$
2. Symmetry: if $(A, B, C) \in L$ then $(B, C, A) \in L$ and $(C, B, A) \in L$. (Therefore also $(A, C, B) \in L$, $(B, A, C) \in L$, and $(C, A, B) \in L$.)
3. Transitivity: if $A \neq B$ and $(A, B, C) \in L$ and $(A, B, D) \in L$ then $(B, C, D) \in L$. (Therefore also $(A, C, D) \in L$.).

We will construct a first-order structure $(\mathcal{X}, L)$ from a point-line geometry $(\mathcal{X}, \mathcal{L})$ and vice versa. Further, we will show that the homomorphisms and substructures of the structure $(\mathcal{X}, L)$ are exactly the same as the homomorphisms and subgeometries of the corresponding point-line geometry $(\mathcal{X}, \mathcal{L})$.

Theorem 2.2. *Let $\mathcal{X}$ be a non-empty set.*

1. *For every point-line geometry $(\mathcal{X}, \mathcal{L})$ let us define $L_{\mathcal{X}}$ as the ternary relation such that (A, B, C) is in $L_{\mathcal{X}}$ if and only if the points A, B and C lie on some line $l \in \mathcal{L}$ or $A = B = C$. Then this relation satisfies the axioms of reflexivity 1, symmetry 2 and transitivity 3.*
2. *Let $(\mathcal{X}, L)$ be a first-order structure with ternary relation L defined with axioms 1, 2 and 3. We define a set $\mathcal{L} \subseteq P(\mathcal{X})$ such that $l \in \mathcal{L}$ if and only if $|l| \geq 2$, the points of l are related by L and for any two points $A, B \in l$ there is no point $C \in \mathcal{X} \setminus l$ such that $(A, B, C) \in L$. Then $(\mathcal{X}, \mathcal{L})$ is a point-line geometry.*

Furthermore, the notion of homomorphisms and substructures (resp. subgeometries) are stable under the above transformations.

Proof.

1. It is easy to see that the ternary relation L satisfies the axioms 1., 2. and 3.

   If $\mathcal{Z} \subseteq \mathcal{X}$ and $(\mathcal{Z}, \mathcal{L}_{\mathcal{Z}})$ is the induced subgeometry of $(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$, then for all $A, B, C \in \mathcal{Z}$ it holds that $(A, B, C) \in L_{\mathcal{Z}} = L_{\mathcal{X}} \cap \mathcal{Z}^3$ if and only if $A = B = C$ or A, B, C are contained in a line $l \in \mathcal{L}_{\mathcal{Z}}$. If $A = B = C$, then this is obvious. If A, B, C are not all equal, then $(A, B, C) \in L_{\mathcal{X}}$ if and only if $A, B, C \in l$ for some $l \in \mathcal{L}_{\mathcal{X}}$. It follows directly from the definition of subgeometry that this holds if and only if $A, B, C \in l$ for some $l \in \mathcal{L}_{\mathcal{Z}}$.

Let $(\mathcal{X}, \mathcal{L}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ be two point-line geometries, where $\mathcal{Y}$ is a non-empty finite set. Let $f : \mathcal{X} \to \mathcal{Y}$ be a homomorphism from the point-line geometry $(\mathcal{X}, \mathcal{L}_\mathcal{X})$ to the point-line geometry $(\mathcal{Y}, \mathcal{L}_\mathcal{Y})$ and let $A, B, C \in \mathcal{X}$. If $(A, B, C) \in L_\mathcal{X}$ then the points A, B and C are on some line $l_x \in \mathcal{L}_\mathcal{X}$ or this is just one point $D = A = B = C$, so there exists a line $l_y \in \mathcal{L}_\mathcal{Y}$ or point $D' \in \mathcal{Y}$ such that $f(l_x) \subseteq l_y$ or $f(D) = D'$, so $(f(A), f(B), f(C)) \in L_\mathcal{Y}$.

Now, let $f : \mathcal{X} \to \mathcal{Y}$ be a homomorphism from the structure $(\mathcal{X}, L_\mathcal{X})$ to the structure $(\mathcal{Y}, L_\mathcal{Y})$. Let $l_x \in \mathcal{L}_\mathcal{X}$. We want to show that $|f(l_x)| = 1$ or $f(l_x) \subseteq l_y$ for some $l_y \in \mathcal{L}_\mathcal{Y}$. Assume that $|f(l_x)| \geq 2$ and let $A, B \in l_x$ so that $f(A) \neq f(B)$. Since $(A, A, B) \in L_\mathcal{X}$, it follows that $(f(A), f(A), f(B)) \in L_\mathcal{Y}$. Since $f(A) \neq f(B)$ it follows that there exists a unique line l_y such that $f(A), f(B) \in l_y$. Now if $C \in l_x$, then $(A, B, C) \in L_\mathcal{X}$, and thus $(f(A), f(B), f(C)) \in L_\mathcal{Y}$. This is only possible if $f(C) \in l_y$ and therefore $f(l_x) \subseteq l_y$.

2. It follows from the definition that every line contains at least two points, because $|l| \geq 2$ for all $l \in \mathcal{L}$. Moreover every pair of distinct points A and B is contained in at most one line l, because the relation L is transitive, symmetric and there is no point $C \in \mathcal{X} \setminus l$ such that $(A, B, C) \in L$. So, $(\mathcal{L}, \mathcal{X})$ is a point-line geometry.

   The fact that the substructures and the homomorphisms coincide follows from the argument given in the proof of the previous part.

$\square$

3. 2-stripes

In this section we will give some general results for k-stripes and characterise the homomorphism-homogeneous 2-stripes.

We shall distinguish three kinds of 2-stripes, *thin*, *full* and *mixed* 2-stripes in the following way: a 2-stripe is thin if all the points are singular, a 2-stripe is full if it has no singular points, and it is called a mixed 2-stripe if it is neither thin nor full. We start by the thin types.

3.1. Thin 2-stripe

In this section we describe thin 2-stripes.

Lemma 3.1. *Let $(\mathcal{X}, \mathcal{L})$ be a homomorphism-homogeneous k-stripe and $a, b \in \mathcal{L}$ two regular lines such that the pair (a, b) induces a thin 2-stripe. Then either $|\mathcal{N}(A)| = 0$ holds for every $A \in a \cup b$ or $|\mathcal{N}(A)| = 1$ holds for every $A \in a \cup b$. (Fig. 1)*

Proof. Suppose to the contrary that there exist points $A_1, A_2 \in a$ such that $|\mathcal{N}_b(A_1)| = 0$ and $|\mathcal{N}_b(A_2)| = 1$. Similar argument works if $B_1, B_2 \in b$ with $|\mathcal{N}_a(B_1)| = 0$ and $|\mathcal{N}_a(B_2)| = 1$.

Let $B_1 \in b$ such that $B_1 \sim A_2$ and $B_2 \neq B_3 \in b$. Since the point A_2 is singular, we have $A_2 \nsim B_2$ and $A_2 \nsim B_3$. Thus the map

$$f : \begin{pmatrix} B_2 & B_3 & A_2 \\ B_2 & B_3 & A_1 \end{pmatrix}$$

is a local homomorphism. Then f extends to a local homomorphism f^* defined also on B_1. Since $B_1 \in b$ we have $f^*(B_1) \in b$ and since $B_1 \sim A_2$, we have $f^*(B_1) \sim A_1$. So, $f^*(B_1)$ is a point on the line b which is collinear with the point A_1, which is impossible.

$\square$

Proposition 3.2. *Let $(\mathcal{X}, \mathcal{L})$ be a thin 2-stripe. This point-line geometry is homomorphism-homogeneous if and only if $|\mathcal{N}(X)| = 0$ for all $X \in \mathcal{X}$, or $|\mathcal{N}(X)| = 1$ for all $X \in \mathcal{X}$. (Fig. 1)*

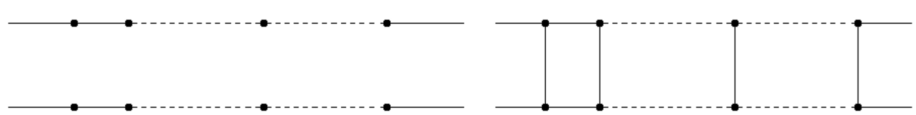

Figure 1: The homomorphism-homogeneous thin 2-stripes.

Proof. If the point-line geometry is homomorphism-homogeneous then from Lemma 3.1 we have that $|\mathcal{N}(X)| = 0$ for all $X \in \mathcal{X}$, or $|\mathcal{N}(X)| = 1$ for all $X \in \mathcal{X}$.

Now, let $(\mathcal{X}, \mathcal{L})$ be a thin 2-stripe and $f : \mathcal{S} \to \mathcal{X}$ be a local homomorphism from some $\mathcal{S} \subsetneq \mathcal{X}$.

If we have $|\mathcal{N}(X)| = 0$ for all $X \in \mathcal{X}$ the proof is trivial. Indeed, we might assume $X \in a \setminus \mathcal{S}$, then if $a \cap \mathcal{S} \neq \emptyset$, let $f^*(X) = f(A)$ for some $A \in a \cap \mathcal{S}$, otherwise $f^*(X) = X$.

Now, let $|\mathcal{N}(X)| = 1$ for all $X \in \mathcal{X}$. One can see that every point of this point-line geometry lies on exactly one regular and singular line. Also one can see that for every line l of this point-line geometry and for every point $X \notin l$ there exists a unique point $X' \in l$ which is collinear with the point X.

Let $A \in \mathcal{X} \setminus \mathcal{S}$ be an arbitrary point, a be the regular line which contains the point A and AB be the singular line which contains the point A.

If the point A is not collinear with any point from $\mathcal{S}$, then we put $f^*(A) = A$. If there is precisely one line c such that $A \in c$ and $c \cap \mathcal{S} \neq \emptyset$, then we choose $A^* \in f(c \cap \mathcal{S})$ arbitrarily and put $f^*(A) = A^*$. If $a \cap \mathcal{S} \neq \emptyset$ and $AB \cap \mathcal{S} \neq \emptyset$, then let a' be a line which contains $f(a \cap \mathcal{S})$. If $f(B) \notin a'$ then let $f^*(A)$ be some element of $\mathcal{N}'_{a'}(f(B))$. If $f(B) \in a'$ then we put $f^*(A) = f(B)$. It is easy to verify that these extensions are local homomorphisms, proving that f extends to a homomorphism of $\mathcal{X}$, since $\mathcal{X}$ is finite. $\square$

3.2. Full 2-stripes

Recall that a 2-stripe is full if it has no singular points.

Proposition 3.3. *Let $(\mathcal{X}, \mathcal{L})$ be a homomorphism-homogeneous k-stripe and let a and b be two regular lines such that the pair (a, b) induces a full 2-stripe. Then $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ is a totally ordered set (Fig. 3).*

Proof. We first prove that for every point X there exists a point $X' \neq X$ on the same regular line such that $\mathcal{N}(X) \cap \mathcal{N}(X') \neq \emptyset$. If $|\mathcal{N}(X)| = 1$ then it is obvious, because the point X is not a singular point. So, let us assume that $|\mathcal{N}(X)| \geq 2$ and $C \neq D \in \mathcal{N}(X)$. Moreover, let A and B be two distinct points on the same regular line as X. Suppose to the contrary that for every point $X' \neq X$ lying on the regular line containing X we have $\mathcal{N}(X) \cap \mathcal{N}(X') = \emptyset$. Let $H \in b \setminus \mathcal{N}(X)$. Such a point exists otherwise

$\mathcal{N}(X) \cap \mathcal{N}(X') \neq \emptyset$ is trivially satisfied since X' is not singular. Thus the map

$$f : \begin{pmatrix} A & B & C & D \\ A & B & C & H \end{pmatrix}$$

is a local homomorphism since the point D is not collinear with the points A and B by our assumption and the points C and H are collinear. By Lemma 2.1 f extends to a local homomorphism f^* defined also on X. Since $f(A) = A$ and $f(B) = B$ we have that $f^*(X)$ is on the same regular line as A, B and X and since $X \sim C$, $X \sim D$, we have $f^*(X) \sim C$, $f^*(X) \sim H$. Then $f^*(X)$ cannot be X since $f^*(X) \sim H$. In this case C is a common point of $\mathcal{N}(X)$ and $\mathcal{N}(f^*(X))$, a contradiction.

Let $A_1 \in a$ be a point such that $|\mathcal{N}_b(A_1)| \geq 2$ and $B_1 \neq B_2 \in \mathcal{N}_b(A_1)$. We claim that for every point $A_2 \in a$ we have $\mathcal{N}_b(A_1) \cap \mathcal{N}_b(A_2) \neq \emptyset$. Suppose the opposite, i.e. there exists a point $A_2 \in a$ such that $\mathcal{N}_b(A_1) \cap \mathcal{N}_b(A_2) = \emptyset$. From the previous observation we know that there exist points $A_3 \in a$ and $B_3 \in b$ such that $A_3 \sim B_3$ and $A_2 \sim B_3$ (Fig. 2). Thus the map

$$f : \begin{pmatrix} B_1 & B_2 & A_2 & A_3 \\ B_1 & B_2 & A_2 & A_1 \end{pmatrix}$$

is a local homomorphism. By Lemma 2.1 f extends to a local homomorphism f^* defined also on B_3. Since $B_3 \in b$ we have $f^*(B_3) \in b$ and because of $B_3 \sim A_2$ and $B_3 \sim A_3$, we have $f^*(B_3) \sim A_2$ and $f^*(B_3) \sim A_1$. So, $f^*(B_3)$ is a point on the line b which is collinear with points A_1 and A_2, which is impossible.

Hence, for every $X, X' \in \mathcal{X}$ such that $|\mathcal{N}_b(X)| \geq 2$ and $|\mathcal{N}_b(X')| = 1$ we have $\mathcal{N}_b(X') \subseteq \mathcal{N}_b(X)$.

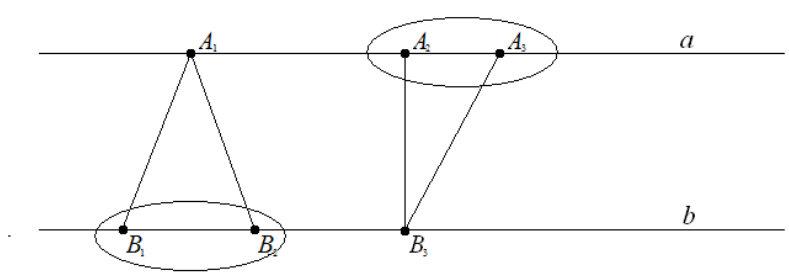

Figure 2: Illustration of the first part of the proof of Proposition 3.3.

Now, let $A_1, A_2 \in a$ be two distinct points such that $|\mathcal{N}_b(A_1)| \geq 2$, $|\mathcal{N}_b(A_2)| \geq 2$ and $\mathcal{N}_b(A_1) \nsubseteq \mathcal{N}_b(A_2)$ and $\mathcal{N}_b(A_2) \nsubseteq \mathcal{N}_b(A_1)$. We know that $\mathcal{N}_b(A_1) \cap \mathcal{N}_b(A_2) \neq \emptyset$. Let B_1 and B_2 be two distinct points such that $B_1 \in \mathcal{N}_b(A_1) \backslash \mathcal{N}_b(A_2)$, $B_2 \in \mathcal{N}_b(A_2) \backslash \mathcal{N}_b(A_1)$ and let $B_3 \in \mathcal{N}_b(A_1) \cap \mathcal{N}_b(A_2)$. Then the map

$$f : \begin{pmatrix} B_1 & B_2 & A_1 & A_2 \\ A_2 & B_2 & A_1 & B_1 \end{pmatrix}$$

is a local homomorphism. By Lemma 2.1 f extends to a local homomorphism f^* defined also on B_3. Since $B_3 \in b$ thus its image $f^*(B_3)$ lies on the singular line A_2B_2. Further we have $B_3 \sim A_1$ and $B_3 \sim A_2$ so $f^*(B_3) \sim A_1$ and $f^*(B_3) \sim B_1$. Thus $f^*(B_3)$ is a point on the line A_2B_2 which is collinear with points A_1 and B_1, a contradiction.

Now, in order to finish the proof let $A_1, A_2 \in a$ and $B_1, B_2 \in b$ be points such that $\mathcal{N}_b(A_1) = \{B_1\}$ and $\mathcal{N}_b(A_2) = \{B_2\}$. Since B_1 and B_2 are not singular points, we have that $|\mathcal{N}_b(B_1)| \geq 2$ and $|\mathcal{N}_b(B_2)| \geq 2$. We have already seen that $\mathcal{N}_a(B_1) \subseteq \mathcal{N}_a(B_2)$ or $\mathcal{N}_a(B_2) \subseteq \mathcal{N}_a(B_1)$. Without loss of generality we may assume $\mathcal{N}_a(B_1) \subseteq \mathcal{N}_a(B_2)$. Since $A_1 \in \mathcal{N}_a(B_1)$ it must be the case that $A_1 \in \mathcal{N}_a(B_2)$ and we have $|\mathcal{N}_b(A_1)| = |\mathcal{N}_b(A_2)| = 1$, so $B_1 = B_2$. □

Remark 3.4. If $(\mathcal{X}, \mathcal{L})$ is full, then the maximal element of the total order $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ is equal to b. Moreover, it is easy to show that in every finite full stripe such that $(\{N_b(A) \mid A \in a\}, \subseteq)$ is a total order, also $(\{N_a(B) \mid B \in b\}, \subseteq)$ has to be totally ordered.

Proposition 3.5. *A full 2-stripe $(\mathcal{X}, \mathcal{L})$ with two regular lines a, b is homomorphism-homogeneous if and only if $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ is a totally ordered set (Fig. 3).*

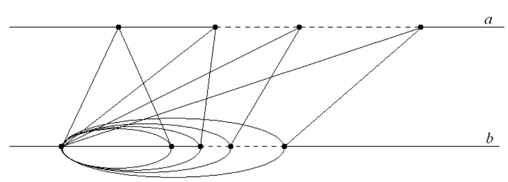

Figure 3: $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ is a totally ordered set.

Proof. It follows from Proposition 3.3 and Remark 3.4 that $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ and $(\{\mathcal{N}_a(B) \mid B \in b\}, \subseteq)$ are totally ordered sets.

For the other direction let $A \in a$ be a point such that $\mathcal{N}_b(A) = b$ and $B \in b$ be a point such that $\mathcal{N}_a(B) = a$. Such points A and B exist by the condition since the point-line geometry is finite. Let $f : \mathcal{S} \to \mathcal{X}$ be a partial homomorphism of $(\mathcal{X}, \mathcal{L})$ for some $\mathcal{S} \subsetneq \mathcal{X}$ and let $X \in \mathcal{X} \setminus \mathcal{S}$. By symmetry we may assume $X \in a$.

If $A \in \mathcal{S}$, then let $f^*(X) = f(A)$. As A was originally connected to every point of b singular lines containing A are preserved by f^*. Also, $f^*((a \cap \mathcal{S}) \cup \{X\}) = f(a \cap \mathcal{S})$. Thus we have that regular line is taken to line as well by f^*.

If $A \notin \mathcal{S}$ and if $f(a \cap \mathcal{S}) \subseteq a$ or $f(a \cap \mathcal{S}) \subseteq b$, then define $f^*(X) = A$ or B, respectively. The map f^* defined this way is obviously a homomorphism. If $f(a \cap \mathcal{S}) = CD$ for some singular line CD, where $C \in a$ and $D \in b$, then one of the points C or D is collinear with all points from the set $f(\mathcal{N}_b(X))$. If $|f(\mathcal{N}_b(X))| \leq 1$, then it is obvious while if $|f(\mathcal{N}_b(X))| \geq 2$, then $f(\mathcal{N}_b(X))$ is contained in a line l. If l is a regular line, then it is also obvious. If l is a singular line EF, where $E \in a$ and $F \in b$, then $C \sim F$ or $E \sim D$ unless $D \in \mathcal{N}_b(C) \setminus \mathcal{N}_b(E)$ and $F \in \mathcal{N}_b(E) \setminus \mathcal{N}_b(C)$. However in this case $\mathcal{N}_b(C)$ and $\mathcal{N}_b(E)$ would be incomparible contradicting the assumption that $(\{\mathcal{N}_b(A) | A \in a\}, \subseteq)$ is a totally ordered set. We put $f^*(X) = C$ or $f^*(X) = D$ depending on that which point of C or D is collinear with all points from the set $f(\mathcal{N}_b(X))$. Also, the map f^* is a homomorphism since the regular line a is taken to a line by f^*. $\square$

3.3. Mixed 2-stripes

Now we will consider mixed 2-stripes which are stripes that contain both singular and nonsingular points.

Lemma 3.6. *If a k-stripe $(\mathcal{X}, \mathcal{L})$ is homomorphism-homogeneous, then the mixed 2-stripe induced by the pair of regular lines cannot have two singular points on the same regular line.*

Proof. Assume on the contrary that there are two singular points P_1 and P_2 on the same regular line. Let a and b be two regular lines which induce the mixed 2-stripe. There exists a point $A_1 \in a \cup b$ such that

$|\mathcal{N}(A_1)| \geq 2$. Without loss of generality assume $A_1 \in a$. Let $B_1 \neq B_2 \in b$ such that $B_1 \sim A_1$ and $B_2 \sim A_1$. We distinguish two cases:

1. One of the singularities is of the first type. Let it be P_2.

   If $P_1, P_2 \in a$ (Fig. 4), then the map $f : \begin{pmatrix} P_1 & P_2 & B_1 \\ B_1 & B_2 & P_2 \end{pmatrix}$ is a homomorphism which cannot be extended to the point A_1 because the point $f^*(A_1)$ has to be on the line b collinear with the point P_2 which is impossible. Otherwise let $P_1, P_2 \in b$ (Fig. 5) and let $g : \begin{pmatrix} P_1 & P_2 & A_1 \\ A_1 & A_2 & P_2 \end{pmatrix}$, where $A_1 \neq A_2 \in a$ is an arbitrary point. Then g is a homomorphism which cannot be extended to the point B_1 because the point $f^*(B_1)$ has to be on the line a collinear with the point P_2 which is impossible.

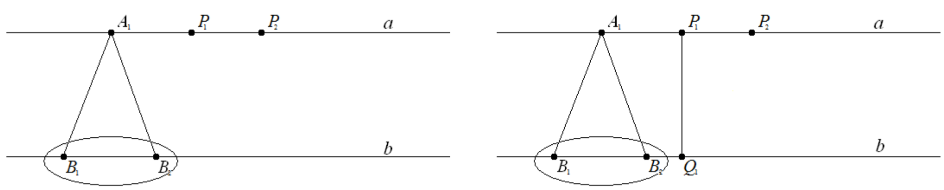

Figure 4: When $P_1, P_2 \in a$

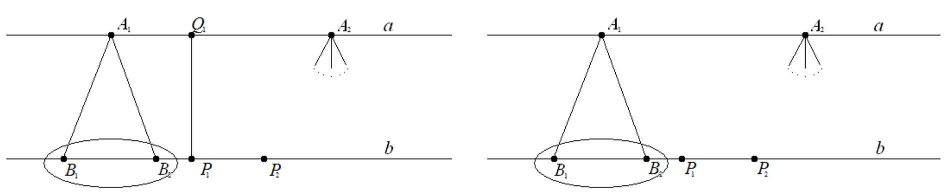

Figure 5: When $P_1, P_2 \in b$

2. The 2-stripe has two singularities (P_1, Q_1) and (P_2, Q_2) of the second type (Fig. 6). Then the map $f : \begin{pmatrix} P_1 & P_2 & B_1 & B_2 \\ P_1 & P_2 & B_1 & Q_1 \end{pmatrix}$ is clearly

a local homomorphism. It cannot be extended to the point A_1 since its image should be on the line a collinear with the points B_1 and Q_1, which is impossible.

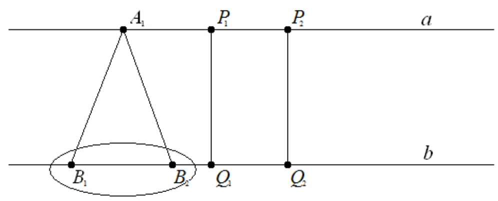

Figure 6: Illustration of the second part of the proof of Lemma 3.6.

□

Lemma 3.7. *If a 2-stripe* $(\mathcal{X}, \mathcal{L})$ *is a homomorphism-homogeneous mixed 2-stripe, then it has just one singularity either of the first type or of the second type.*

Proof. Lemma 3.6 shows that $(\mathcal{X}, \mathcal{L})$ can have at most two singularities of the first type on different regular lines.

Suppose that the mixed 2-stripe has two singularities of the first type $P_1 \in a$ and $P_2 \in b$, where a and b are the regular lines of the 2-stripe. We may assume that there is a point $A_1 \in a$ such that $|\mathcal{N}_b(A_1)| \geq 2$ and let $B_1, B_2 \in \mathcal{N}_b(A_1)$. Then the map $f : \begin{pmatrix} P_1 & B_2 \\ P_1 & P_2 \end{pmatrix}$ is a homomorphism since there is no line containing P_1 and B_2. Then f cannot be extended to the point A_1 since $f(A_1)$ should be collinear with the points $f(P_1) = P_1$ and $f(B_2) = P_2$ lying on the regular line a or b which is impossible. □

Lemma 3.8. *Let* $(\mathcal{X}, \mathcal{L})$ *be a homomorphism-homogeneous* k*-stripe with* $a, b \in \mathcal{L}$ *two regular lines spanning a mixed 2-stripe. Then any two nonsingular points of* a *and* b *are connected by a line.*

Proof. Let P be a singular point of a. Assuming on the contrary that there are nonsingular points A_2 and B_1 which are not connected by any line. Then we may assume $A_2 \in a$ and $B_1 \in b$ such that $B_1 \nsim A_2$. On the regular line a we have at least three points and the point B_1 is a nonsingular point. Thus there exists a point $A_1 \in a$ such that $A_1 \sim B_1$. We distinguish two cases.

If P is a singularity of the first type, then the map $f : \begin{pmatrix} A_2 & P & B_1 \\ B_1 & B_2 & P \end{pmatrix}$ is a homomorphism, but cannot be extended to the point A_1, where $B_2 \in b$ is an arbitrary point different from B_1.

Otherwise there is a singularity of the second type (P, Q), where $Q \in b$. Now, if $|\mathcal{N}_b(A_1)| \geq 2$, let $B_2 \in b$ such that $A_1 \sim B_2$. Then the map $f : \begin{pmatrix} A_2 & P & B_1 & B_2 \\ B_2 & B_1 & P & A_2 \end{pmatrix}$ is a homomorphism, but cannot be extended to the point A_1, because the point $f^*(A_1)$ has to be on the line b collinear with the points P and A_2 which is impossible. If $|\mathcal{N}_b(A_1)| = 1$, then $|\mathcal{N}_a(B_1)| \geq 2$. Let $A_3 \sim B_1$, $A_3 \neq A_1$ and $B_2 \in b$, $B_2 \neq Q$. Then the map $f : \begin{pmatrix} B_2 & Q & A_1 & A_3 \\ P & Q & B_1 & A_3 \end{pmatrix}$ is a homomorphism, but cannot be extended to the point B_1, because the point $f^*(B_1)$ has to be on the line PQ collinear with the points B_1 and A_3 which is impossible. $\square$

Proposition 3.9. *Let $(\mathcal{X}, \mathcal{L})$ be a mixed 2-stripe which has a singularity $P \in a$ of the first type and for all $A, B \in (a \cup b) \setminus \{P\}$ we have $A \sim B$ (Fig. 7). Then $(\mathcal{X}, \mathcal{L})$ is homomorphism-homogeneous.*

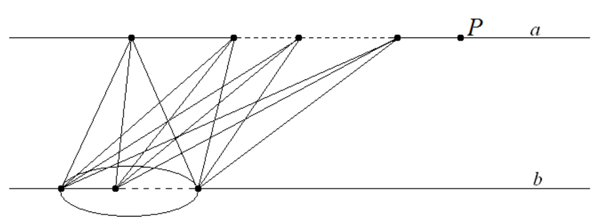

Figure 7: Illustration of Proposition 3.9.

Proof. Let $f : \mathcal{S} \to \mathcal{X}$ be a local homomorphism for some $\mathcal{S} \subsetneq \mathcal{X}$ and let $X \in \mathcal{X} \setminus \mathcal{S}$.

If $a \cap \mathcal{S} = \emptyset$ or $b \cap \mathcal{S} = \emptyset$, then pick any $A \in \mathcal{S}$ and let $f^*(X) = f(A)$. This f^* clearly maps lines to lines.

Now, suppose that $a \cap \mathcal{S} \neq \emptyset$ and $b \cap \mathcal{S} \neq \emptyset$. If $X \in b$ pick some $B \in b \cap \mathcal{S}$ and let $f^*(X) = f(B)$. If $X \in a$ and there is an $A \in (a \cap \mathcal{S}) \setminus \{P\}$, then put $f^*(X) = f(A)$. If $X \in a$ and $(a \cap \mathcal{S}) \setminus \{P\} = \emptyset$, then $f(P)$ is defined. If $f(A)$ is not defined for any other $A \in a$ then let $f^*(X) = X$. This extension f^* is clearly a homomorphism because the point X is collinear with all points including $f(P)$. $\square$

Proposition 3.10. *Let $(\mathcal{X}, \mathcal{L})$ be a mixed 2-stripe which has a singularity (P, Q) of the second type and $A \sim B$ for all $A, B \in (a \cup b) \setminus \{P, Q\}$ (Fig. 8). Then $(\mathcal{X}, \mathcal{L})$ is homomorphism-homogeneous.*

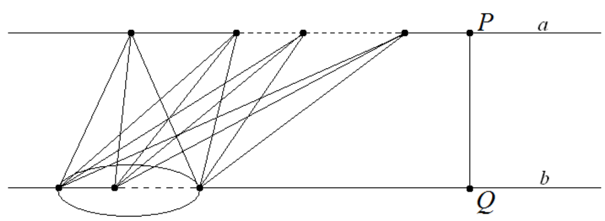

Figure 8: Illustration of Proposition 3.10.

Proof. Let $f : \mathcal{S} \to \mathcal{X}$ be a local homomorphism for some $\mathcal{S} \subsetneq \mathcal{X}$ and let $X \in \mathcal{X} \setminus \mathcal{S}$. Without loss of generality suppose that $X \in a$.

If $a \cap \mathcal{S} = \emptyset$ or $b \cap \mathcal{S} = \emptyset$, then pick any $A \in \mathcal{S}$ and let $f^*(X) = f(A)$. This f^* clearly is a local homomorphism extending f to X.

Now, suppose that $a \cap \mathcal{S} \neq \emptyset$ and $b \cap \mathcal{S} \neq \emptyset$. If $Q \notin \mathcal{S}$, then a similar argument applies as in Proposition 3.9. If $Q \in \mathcal{S}$, then if $X = P$, we put $f^*(P) = P'$, where $P' \in \mathcal{N}'_{a'}(f(Q))$ and $f(a \cap \mathcal{S}) \subseteq a'$ for some line a'. Else, if $P \notin \mathcal{S}$ or $|a \cap (\mathcal{S} \setminus \{P\})| \geq 1$, pick some $A \in (a \cap \mathcal{S}) \setminus \{P\}$ and let $f^*(X) = f(A)$, otherwise $P \in \mathcal{S}$ and $|a \cap (\mathcal{S} \setminus \{P\})| < 1$, i.e. the point P is the only point from the set $a \cap \mathcal{S}$ which is mapped by f. Then we put $f^*(X) = f(Q)$. Obviously, f^* is a homomorphism because $f^*(X)$ is collinear with all points of the line b including $f(P)$. $\square$

3.4. Characterisation of 2-stripes

In this subsection we will show the result in final form.

Theorem 3.11. *Let $(\mathcal{X},\mathcal{L})$ be a 2-stripe induced by the pair (a,b), where $a,b \in \mathcal{L}$ are regular lines. It is homomorphism-homogeneous if and only if it is contained in one of the following classes (Fig. 9):*

1. $|\mathcal{N}(X)| = 0$ *for all* $X \in \mathcal{X}$,
2. $|\mathcal{N}(X)| = 1$ *for all* $X \in \mathcal{X}$,
3. $(\mathcal{X},\mathcal{L})$ *is a full 2-stripe and* $(\{\mathcal{N}_b(A) \mid A \in a\}, \subseteq)$ *is a totally ordered set.*
4. $(\mathcal{X},\mathcal{L})$ *is a mixed 2-stripe with only one singularity of the first or of the second type and any other two points are connected by a line.*

Proof. Propositions 3.2 and 3.5 handle thin and full 2-stripes which correspond to the first three cases. The possible structures of mixed 2-stripes are given in Lemmas 3.7 and 3.8 and the description coincide with the case 4. Propositions 3.9 and 3.10 handle the other direction for mixed 2-stripes. □

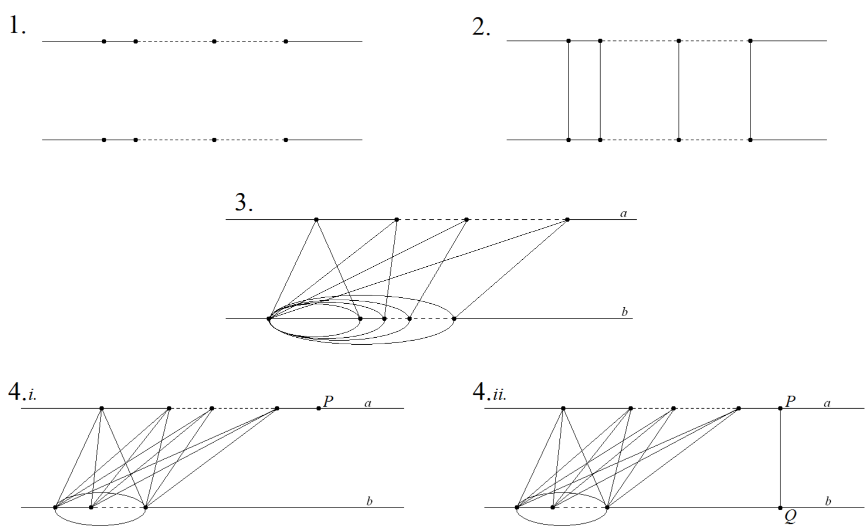

Figure 9: Illustration of Theorem 3.11.

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Mathematical Institute
Eötvös Loránd University
Pázmány Péter sétány 1/C, 1117 Budapest, Hungary
evajungabel@caesar.elte.hu