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SETS WITH NO SUBSETS OF HIGHER WEAK TRUTH-TABLE DEGREE

A b s t r a c t. We consider the weak truth-table reducibility $\leq_{wtt}$ and we prove the existence of wtt-introimmune sets in Δ^0_2. This closes the gap on the existence of arithmetical r-introimmune sets for all the known reducibilities $\leq_r$ strictly contained in the Turing reducibility.

1. Introduction

The existence of sets without subsets of higher Turing degree was proved by Soare [11]. In terms of their complexity, we know by Jockusch [7] that they cannot be arithmetical, and later Simpson [10] even proved that they cannot be hyperarithmetical. A natural question is to consider reducibilities $\leq_r$ that are strictly contained in the Turing reducibility $\leq_T$ and to

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see if there are arithmetical sets without subsets of higher r-degree. The reader unfamiliar with these reducibilities can see e.g. [6, 8, 9, 12]. The approach of to consider such reducibilities $\leq_r$ and to study the existence of arithmetical sets without subsets of higher r-degree was initiated in [5], in which r-*introimmune* sets have been introduced. An infinite set A of natural numbers is r-introimmune if for every subset B of A with $|A \backslash B| = \infty$ we have $A \not\leq_r B$. Some common reducibilities strictly contained in $\leq_T$ studied in Computability Theory are the following, from the smallest to the largest: the *one-one* $\leq_1$, the *many-one* $\leq_m$, the *truth-table* $\leq_{tt}$ and the *weak truth-table* reducibility $\leq_{wtt}$. r-introimmune sets have no subsets of higher r-degree for all the reducibilities $\leq_r$ of the list. In [5] it was proved the existence of arithmetical c-introimmne sets, where $\leq_c$ is the conjunctive reducibility, a particular truth-table reducibility. More specifically, it was proved the existence of c-introimmune Δ^0_4 sets. This was improved by Ambos-Spies [1] by showing the existence of tt-introimmune Δ^0_2 sets. So, from Ambos-Spies' result we know that there are arithmetical r-introimmune sets for all the reducibilities $\leq_r$ of the above list up to $\leq_{tt}$. In this paper we close the gap by considering the *weak truth-table* reducibility $\leq_{wtt}$, and we prove the existence of arithmetical wtt-introimmune sets, in particular wtt-introimmune Δ^0_2 sets. Since we currently do not know intermediate reducibilities between $\leq_{wtt}$ and $\leq_T$, we deduce that for all the known reducibilities $\leq_r$ strictly contained in $\leq_T$ there are arithmetical r-introimmune sets.

2. Notation

Our notation is standard and we mainly refer to [9, 12]. Letter $\mathbb{N}$ denotes the set of natural numbers. We identify each subset of $\mathbb{N}$ with its characteristic function. Given any two sets $A, B \subseteq \mathbb{N}$, $A \backslash B$ denotes the set difference of A and B. We fix a computable permutation $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. A string is any function $\alpha : \{0, 1, \ldots, n\} \to \{0, 1\}$, where $n \in \mathbb{N}$. $\emptyset$ denotes the empty string. The length of a string α, in short $|\alpha|$, is the cardinality of its domain. Given two strings α and β, we write:

- $\alpha \subseteq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha(m) \leq \beta(m)$ for every $m < |\alpha|$,

- $\alpha \sqsubseteq \beta$ if $|\alpha| \leq |\beta|$ and $\alpha(m) = \beta(m)$ for every $m < |\alpha|$,
- $\alpha \sqsubset \beta$ if $\alpha \sqsubseteq \beta$ and $\alpha \neq \beta$.

For every string β and every $m \leq |\beta|$, $\beta \restriction m$ is the string $\alpha \sqsubseteq \beta$ with $|\alpha| = m$. If α is a string and $b \in \{0,1\}$ then αb denotes the string of length $|\alpha| + 1$ such that $\alpha \sqsubset \alpha b$ and $\alpha b(|\alpha|) = b$. We fix an effective acceptable enumeration $\Phi_0, \Phi_1, \dots$ of the Turing functionals. We fix also an effective acceptable enumeration $\varphi_0, \varphi_1, \dots$ of the Turing-computable unary functions. Finally, given two sets $A, B \subseteq \mathbb{N}$, A is *weak truth table* reducible to B, in short $A \leq_{wtt} B$, if there exists a number $e \in \mathbb{N}$ and a total computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that:

i) $\Phi_e^B = A$,

ii) for every $x \in \mathbb{N}$, the computation of the e-th oracle Turing machine with oracle B on input x asks the oracle only numbers less than $\varphi(x)$.

In this case we say that (Φ_e, φ) *wtt*-reduces A to B. The weak truth-table reducibility is also known in literature as the *bounded Turing reducibility* $\leq_{bT}$.

3. Main result

Given any reducibility $\leq_r$ and given any set $A \subseteq \mathbb{N}$, the r-degree of A is the class $\{B \subseteq \mathbb{N} : A \equiv_r B\}$, where $A \equiv_r B$ if and only if $A \leq_r B$ and $B \leq_r A$. A set A does not have subsets of higher r-degree if $A \not<_r B$ for every $B \subseteq A$. So a *wtt*-introimmune set does not have subsets of higher *wtt*-degree. In this section we prove the existence of a *wtt*-introimmune set in the class Δ^0_2. Thus, for each known reducibility $\leq_r$ strictly contained in $\leq_T$ there are arithmetical r-introimmune sets. As for the arithmetical complexity we observe that for each reducibility $\leq_r$ such that $\leq_1 \Rightarrow \leq_r$ there cannot be r-introimmune sets in Σ^0_1, because such sets are immune. This follows from the fact that each 1-introimmune set is immune.

Proposition 3.1. *Each* 1*-introimmune set is immune.*

Proof. Let $A \subseteq \mathbb{N}$ be an infinite set and let us suppose that A is not immune. Then there exists an infinite recursive set $R \subseteq A$. Let $f : \mathbb{N} \to \mathbb{N}$

be a total one-one computable function such that $R = \{f(0), f(1), \ldots\}$. Let us consider the infinite set

$$R_0 = \{f(\langle 0, n\rangle) : n \in \mathbb{N}\} \subseteq R.$$

Then,

$$A \backslash R_0 \subseteq A$$

and

$$|A \backslash (A \backslash R_0)| = |R_0| = \infty.$$

It follows that A is not 1-introimmune, because $A \leq_1 A \backslash R_0$ is witnessed by the total one-one computable function $g : \mathbb{N} \to \mathbb{N}$ defined in the following way:

1. $g(x) = x$ for every $x \notin R$, and
2. $g(f(\langle n, m\rangle)) = f(\langle n+1, m\rangle)$ for every $n, m \in \mathbb{N}$.

It is routine to check that for every $x \in \mathbb{N}$, $x \in A \Leftrightarrow g(x) \in A \backslash R_0$. □

We know of the existence of m-introimmune sets in the class Π^0_1 [3, 4]. We leave as an open question the existence of *wtt*-introimmune sets in Π^0_1.

Theorem 3.2. *There exists a wtt-introimmune set in* Δ^0_2.

Proof. By the finite-extension method we construct a set A satisfying the following requirements for every $a, b, e \in \mathbb{N}$:

$$P_{2e} : |A| \geq e,$$

and

$$N_{2\langle a,b\rangle+1} : (\Phi_a, \varphi_b) \text{ does not } wtt\text{-reduce } A \text{ to any } X \subseteq A \text{ with } |A \backslash X| = \infty.$$

The satisfaction of all the requirements P_{2e} guarantees that A is infinite, while the satisfaction of all the requirements $N_{2\langle a,b\rangle+1}$ guarantees that A is *wtt*-introimmune. □

3.1 Strategy

Set A will be constructed by infinitely many stages $s = 0, 1, \ldots$. At every stage s we define the finite set A_s, and the final set will be

$$A = \lim_{s \to \infty} A_s,$$

with $A_s \subseteq A_{s+1}$ for every $s \geq 0$. Set A will be a subset of $\{h(n) : n \geq 0\}$, where $h : \mathbb{N} \to \mathbb{N}$ is a suitable *dominating* function.

Definition 3.3. (dominating function). A function $g : \mathbb{N} \to \mathbb{N}$ is *dominating* if for every total computable function $\varphi : \mathbb{N} \to \mathbb{N}$, $\varphi(n) < g(n)$ for almost every n.

Let $K = \{x \in \mathbb{N} : \varphi_x(x) \downarrow\}$ be the halting set, and let g be any increasing dominating K-computable function with $g(0) > 0$. Let us define the increasing sequence $(g^n(0) : n \geq 1)$ in the following way: $g^1(0) = g(0)$, and for every $n \geq 1$ $g^{n+1}(0) = g(g^n(0))$. Let us define for every $n \geq 1$

$$h(n) = g^n(0),$$

with $h(0) = 0$. Then, h is a dominating K-computable function which satisfies the following property.

Proposition 3.4. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ be any total computable function. Then for almost every $n \in \mathbb{N}$, for every $m \leq n$*

$$\varphi(h(m)) < h(n+1).$$

Proof. Given any such φ, let us consider the total computable function

$$\tilde{\varphi}(n) = \max\{\varphi(u) : u \leq n\}.$$

Let n_0 be such that for every $n \geq n_0$

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)). \tag{1}$$

Then, for every $n \geq n_0$ and for every $m \leq n$

$$\varphi(h(m)) = \varphi(g^m(0)) \tag{2}$$

by definition of h, and

$$\varphi(g^m(0)) \leq \tilde{\varphi}(g^n(0)) \tag{3}$$

by $(g^n(0) : n \geq 1)$ increasing and by the definition of $\tilde{\varphi}$. Finally

$$\tilde{\varphi}(g^n(0)) < g(g^n(0)) = g^{n+1}(0) = h(n+1) \tag{4}$$

by (1) and by the definition of h. □

3.2 Strategies to satisfy requirements

To satisfy each requirement P_{2e} we add an element to A at e opportune stages. The strategy to satisfy each requirement $N_{2\langle a,b\rangle+1}$ is essentially the method used in [1]. To satisfy $N_{2\langle a,b\rangle+1}$ means in particular to prevent (5):

$$(\exists X)[X \subseteq A \text{ and } |A\backslash X| = \infty \text{ and } \Phi_a^X = A]. \tag{5}$$

But (5) implies that there is an infinite sequence $(n_s : s \geq 0)$ of natural numbers such that

$$\Phi_a^X(h(n_s)) = A(h(n_s)) = 1 \text{ and } X(h(n_s)) = 0. \tag{6}$$

So, we wait for a stage $s+1$ at which

$$\varphi_b(h(s)) < h(s+1) \tag{7}$$

and for some $X \subseteq A_s \subseteq \{h(0), h(1), \ldots, h(s-1)\}$ it is

$$\Phi_a^X(h(0)) = A_s(h(0)), \ldots, \Phi_a^X(h(s-1)) = A_s(h(s-1)) \tag{8}$$

and

$$\Phi_a^X(h(s)) = 1. \tag{9}$$

Then, we force $\Phi_a^X(h(s))$ to be wrong by setting $A_{s+1}(h(s)) = 0$. Observe that by (7) and by $X \subseteq A_s \subseteq \{h(0), \cdots, h(s-1)\}$ the computation of $\Phi_a^X(h(s))$ depends only on number less than or equal to $h(s)$.

3.3 Formalization

We formalize the above strategies and the construction of the set A. First, we define formally the conditions under which a requirement requires attention. Then, we will give an algorithm for the construction of the set A

by defining the actions needed to satisfy all the requirements. In order to better handle some proofs later we introduce first the following notation: given any string α, let X_α be the set

$$\{h(n) : n < |\alpha| \wedge \alpha(n) = 1\}.$$

From now on, Φ^α stands for Φ^{X_α} for each string α. The algorithm with which we will construct our set $A = \bigcup_{s \geq 0} A_s$ will generate by stages infinitely many strings $\alpha_0 \sqsubset \alpha_1 \sqsubset \cdots$. The final set A will be

$$A = \lim_{s \to \infty} \alpha_s,$$

where α_s is the string obtained by the end of stage s with $|\alpha_s| = s$ and denoting $A_s = X_{\alpha_s}$.

3.3.1 Requirements requiring attention

Fix a stage $s+1$, and let α_s be the string constructed by the end of stage s.

- Requirement P_{2e} requires attention at stage $s+1$ if

$$|A_s| < e.$$

- Requirement $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ via the string α with $|\alpha| = |\alpha_s| = s$ if the following conditions hold.

  C1: $\varphi_b(h(s)) < h(s+1)$,

  C2: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$, for every $m < s$,

  C3: $\alpha \subseteq \alpha_s$,

  C4: (for every $m < s$), $[\Phi_a^\alpha(h(m)) = \alpha_s(m)]$,

  C5: $\Phi_a^{\alpha 0}(h(s)) = 1$.

We describe the meaning of each condition. Condition **C1** makes the computation of $\Phi_a^\alpha(h(s))$ depending only on numbers less than or equal to $h(s)$. Condition **C2** says that (Φ_a, φ_b) could be a *wtt*-reduction. Condition **C3** says that the set X_α is a subset of the constructed set A_s. Conditions **C4** and **C5** formalize (8) and (9), that is

$$\Phi_a^{X_\alpha}(h(0)) = A_s(h(0)), \ldots, \Phi_a^{X_\alpha}(h(s-1)) = A_s(h(s-1))$$

and

$$\Phi_a^{X_\alpha}(h(s)) = 1.$$

3.3.2 Construction of the set A

We say that a N-requirement requires attention at stage $s+1$ if it requires attention at stage $s+1$ via some string α of length s. A requirement R_n has higher priority than a requirement R_m if $n < m$. At any stage $s+1$ a requirement R_n is *active* if it is the highest priority requirement requiring attention. The algorithm to construct the set A is the following.

Algorithm

- Stage 0. Set $\alpha_0 = \emptyset$.

- Stage $s+1$. Let α_s be the string constructed by the end of stage s, and let R_n be the active requirement. If n is even, then set $\alpha_{s+1} = \alpha_s 1$, otherwise set $\alpha_{s+1} = \alpha_s 0$.

End of algorithm

Set $A = \lim_{s\to\infty} \alpha_s$. The construction of A is by the finite extension method, thus for every stage $s \geq 0$ and for every $n < |\alpha_s|$, $\alpha_s(n) = A(h(n))$. Now we have to prove that the construction is correct, that is that each requirement is met and that $A \in \Delta_2^0$.

Lemma 3.5. *Every requirement requires attention at most finitely often and is met.*

Proof. By induction on the index n of the requirement R_n. Let $n \geq 0$ be given, and let s_0 be the minimum stage such that no requirement of higher priority than R_n requires attention after s_0. Distinguish two cases on n.

- $R_n = P_{2e}$. Let us suppose that it requires attention at stage $s+1 > s_0$.

By hypothesis P_{2e} is active from stage $s+1$ onwards. At each of these consecutive stages we add one element, so in at most $t \leq e$ stages starting from $s+1$ the cardinality of A_{s+t} will be e, P_{2e} is satisfied and it will no longer require attention.

- $R_n = N_{2\langle a,b\rangle+1}$. By Proposition 3.4 we can make the following further hypothesis on s_0: for every $s \geq s_0$ and for every $m < s$,

$$\varphi_b(h(m)) < h(s). \tag{10}$$

From (10) we get the following

Claim 3.6. *For every string α and α' of length at least s_0, if $\alpha \sqsubseteq \alpha'$, then*

$$(\forall m < |\alpha|)[\Phi_a^{\alpha}(h(m)) = \Phi_a^{\alpha'}(h(m))]. \tag{11}$$

Proof. Let $\alpha \sqsubseteq \alpha'$ with $|\alpha| \geq s_0$. For every $m < |\alpha|$ the computation of $\Phi_a^{X_\alpha}(h(m))$ can ask the oracle only numbers less that $\varphi_b(h(m)) < h(|\alpha|)$, where

$$X_\alpha \subseteq \{h(0), h(1), \ldots, h(|\alpha|-1)\}.$$

On the other hand, $\alpha \sqsubseteq \alpha'$ means that

$$\alpha = \alpha' \upharpoonright |\alpha|,$$

that is X_α is equal to $X_{\alpha'}$ up to $h(|\alpha|-1)$. Therefore the two computations $\Phi_a^{X_\alpha}(h(m))$ and $\Phi_a^{X_{\alpha'}}(h(m))$ are equal for every $m < |\alpha|$. *End of proof of Claim 3.6.*

The proof that $N_{2\langle a,b\rangle+1}$ requires attention at most finitely often is distributed in the following three claims[1].

Claim 3.7. *If $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1 > s_0$ via α, then for every s' with $s_0 \leq s' < s$ it holds that $\alpha(s') = A(h(s'))$.*

Proof. Let α_s be the string constructed by the end of stage s. For the sake of contradiction, let s' be the minimum such that $s_0 \leq s' < s$ and $\alpha(s') \neq A(h(s'))$. By hypothesis $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1$, thus by condition **C3**

[1] Technically, the proofs of these three claims are based on [2].

$$\alpha(s') \leq A(h(s')), \tag{12}$$

that is

$$\alpha(s') = 0 \text{ and } A(h(s')) = 1. \tag{13}$$

Let us consider $\beta = \alpha \upharpoonright s'$, that is

$$\beta \sqsubseteq \alpha \text{ and } |\beta| = s'. \tag{14}$$

We prove that $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β, and this implies $\alpha_{s'+1} = \alpha_{s'}0$, that is $\alpha_{s'+1}(s') = 0$; but $\alpha_{s'+1} \sqsubseteq \alpha_s$, whence $\alpha_s(s') = 0$, that is $A(h(s')) = 0$, contradicting (13). In order to prove that $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β it is enough to check that all the conditions **C1**, **C2**, **C3**, **C4** and **C5** hold for β and $\alpha_{s'}$ at stage $s'+1$.

- **C1**: $\varphi_b(h(s')) < h(s'+1))$ holds by (10) because $s'+1 \geq s_0$.
- **C2**: $\Phi_a^\beta(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s' < |\alpha|$, because **C2** holds at stage $s+1$ w.r.t. α.
- **C3**: $\beta \subseteq \alpha_{s'}$, because $\alpha_{s'} \sqsubseteq \alpha_s$, $\alpha \subseteq \alpha_s$ and $\beta = \alpha \upharpoonright s'$.
- **C4**: $\beta \sqsubseteq \alpha$ with both the lengths of β and α at least s_0, so by Claim 1 for every $m < |\beta|$
$$\Phi_a^\beta(h(m)) = \Phi_a^\alpha(h(m)). \tag{15}$$
Moreover, for every $m < |\beta|$
$$\Phi_a^\alpha(h(m)) = A(h(m)) \tag{16}$$
because **C4** holds at stage $s+1$ w.r.t. α. Thus, by equations (15) and (16)
$$\Phi_a^\beta(h(m)) = A(h(m)) \tag{17}$$
for every $m < |\beta|$.
- **C5**: We observe first that $\beta 0 \sqsubseteq \alpha$, because by (13) it is $\alpha(s') = 0$ and by (14) it is $|\beta| = s'$. Then, by (10) the computation of $\Phi_a^{\beta 0}(h(s'))$ depends only on numbers $\leq h(s')$, which means that
$$\Phi_a^{\beta 0}(h(s')) = \Phi_a^\alpha(h(s')).$$

But by hypothesis $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$, that is at stage $s+1$ condition **C4** holds for every $m < s$, in particular for $m = s' < s$, so by the second equality of (13)

$$\Phi_a^{\alpha}(h(s')) = A(h(s')) = 1.$$

Therefore

$$\Phi_a^{\beta 0}(h(s')) = 1$$

and **C5** is satisfied. Hence, all the conditions **C1**, **C2**, **C3**, **C4** and **C5** are satisfied by β and $\alpha_{s'}$, so $N_{2\langle a,b\rangle+1}$ requires attention at stage $s'+1$ via β with $|\beta| = s'$. But as before observed this causes $A(h(s')) = 0$, contradicting (13). *End of proof of Claim 3.7.*

Claim 3.8. *Let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1 > s_0$, and let α' be such that $\alpha \sqsubset \alpha'$. Then, $N_{2\langle a,b\rangle+1}$ does not require attention via α'.*

Proof. By hypothesis, at the end of stage $s+1$ is

$$A(h(s)) = 0. \tag{18}$$

Let $s' > s$, and for the sake of contradiction let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α' at stage $s'+1$. First, we note that it cannot be $\alpha 1 \sqsubseteq \alpha'$, because otherwise it would be

$$\alpha'(s) = 1$$

and by (18) $A(h(s)) = 0$, that is $\alpha_{s'}(s) = 0$, from which $\alpha' \not\subseteq \alpha_{s'}$, contradicting condition **C3** $\alpha' \subseteq \alpha_{s'}$ at stage $s'+1$. Thus it has to be

$$\alpha 0 \sqsubseteq \alpha'. \tag{19}$$

Since by hypothesis $N_{2\langle a,b\rangle+1}$ requires attention via α at stage $s+1$ it follows that **C5** is satisfied, that is

$$\Phi_a^{\alpha 0}(h(s)) = 1.$$

On the other hand, by (19)

$$\Phi_a^{\alpha'}(h(s)) = \Phi_a^{\alpha 0}(h(s)) = 1.$$

But at stage $s'+1$ $N_{2\langle a,b\rangle+1}$ requires attention via α', so by condition **C4** for $m = s < s'$

$$\Phi_a^{\alpha'}(h(s)) = A(h(s)),$$

that is $A(h(s)) = 1$, which contradicts (18). *End of proof of Claim 3.8.*

Claim 3.9. *For every string α of length s_0, there is at most one string α' properly extending α such that $N_{2\langle a,b\rangle+1}$ requires attention via α'.*

Proof. Let α be a string such that $|\alpha| = s_0$, and let α' and α'' be two strings properly extending α, that is

$$\alpha(m) = \alpha'(m) = \alpha''(m)$$

for every $m < s_0$. Let us suppose that $N_{2\langle a,b\rangle+1}$ requires attention via α' at stage $s'+1 > s_0$ and via α'' at stage $s''+1 > s_0$. Without loss of generality let us suppose that $|\alpha'| \leq |\alpha''|$. By Claim 3.7, for every t with $s_0 \leq t < s'$ it is

$$\alpha'(t) = A(h(t)) = \alpha''(t).$$

If $|\alpha'| = |\alpha''|$, then $\alpha' = \alpha''$. Otherwise $\alpha' \sqsubset \alpha''$, but this contradicts Claim 3.8. *End of proof of Claim 3.9*

Since there are 2^{s_0} string of length s_0, by Claim 4 requirement $N_{2\langle a,b\rangle+1}$ requires attention at most 2^{s_0} times after stage s_0.

We prove now that $N_{2\langle a,b\rangle+1}$ is met. For the sake of contradiction let us suppose that $N_{2\langle a,b\rangle+1}$ is not met. This means that there exists $B \subseteq A$ such that

$$\Phi_a^B = A \tag{20}$$

and

$$|A \backslash B| = \infty. \tag{21}$$

Moreover, for every $x \in \mathbb{N}$ all the queries made in the computation $\Phi_a^B(x)$ are bounded by $\varphi_b(x)$. We proved that $N_{2\langle a,b\rangle+1}$ requires attention at most finitely often. Hence, there is a minimum stage s_0 after which $N_{2\langle a,b\rangle+1}$ does not require attention. By Proposition 3.4 and by (20) and (21) let $s+1 > s_0$ such that the following three conditions are satisfied:

$$\varphi_b(h(s)) < h(s+1), \tag{22}$$

$$\Phi_a^B(h(s) = A(h(s)) = 1 \tag{23}$$

and
$$B(h(s)) = 0. \tag{24}$$

We show that $N_{2\langle a,b\rangle+1}$ requires attention at $s+1$, which is a contradiction. By (22) at stage $s+1$ condition **C1** holds. Let us consider the string α of length s such that
$$\alpha(m) = B(h(m)) \tag{25}$$

for every $m < s$. String α satisfies all the conditions **C2**, **C3**, **C4** and **C5**:

- **C2**: $\Phi_a^\alpha(h(m))$ asks only elements less than $\varphi_b(h(m))$ for every $m < s$, because we are assuming that (Φ_a, φ_b) *wtt*-reduces A to B;
- **C3**: $\alpha \subseteq \alpha_s$ because $B \subseteq A$;
- **C4**: by (20) and (25), for every $m < s$ $\Phi_a^\alpha(h(m)) = A(h(m)) = \alpha_s(m)$;
- **C5**: by (24) and (25), for every $m \leq s$
$$\alpha 0(m) = B(h(m)),$$
therefore by (23)
$$\Phi_a^{\alpha 0}(h(s)) = \Phi_a^B(h(s)) = 1.$$

Thus $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ via α, which is a contradiction. □

It remains to prove that the set A is in Δ_2^0.

Lemma 3.10. *A is in Δ_2^0.*

Proof. We show that A is Turing reducible to the halting set K. It is enough to observe that oracle K suffices to find the active requirement at any stage, hence to generate the sequence $(\alpha_s : s \geq 0)$. We describe first an algorithm that at any stage $s+1$ finds the active requirement and computes the extension α_{s+1} of α_s. Fix a stage $s+1$ and let α_s be the string obtained by the end of stage s. Enumerate and check all the requirements $R_0, R_1, \ldots$, stopping as soon as one of them satisfies the conditions under which it requires attention. For the part concerning the check, let R_n be a requirement of the above list and distinguish two cases:

- $R_n = P_{2n}$. It is decidable whether or no P_{2e} requires attention, and in this case oracle K is unnecessary.

- $R_n = N_{2\langle a,b\rangle+1}$. With oracle K compute first $h(s)$ and $h(s+1)$. Let $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ be the formula obtained by the conjunction of the formulas expressing conditions **C1**, **C2**, **C3**, **C4** and **C5** with X_α in place of α. Then, $N_{2\langle a,b\rangle+1}$ requires attention at stage $s+1$ if the formula

$$(\exists\alpha)[|\alpha| = |\alpha_s| \wedge F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))] \tag{26}$$

  is true. In (26) the existential quantifier on the oracle variable α is bounded, and for each such α oracle K suffices to compute the relative finite set X_α. All the values $h(m)$ for $m < s$ required in the formula are also computable with K. Finally, observe that $F(a, b, X_\alpha, \alpha_s, s, h(s), h(s+1))$ is a Σ^0_1 formula, so oracle K is enough to test its truth. This shows that K suffices to generate $(\alpha_s : s \geq 0)$. To decide A, given any $x \in \mathbb{N}$ generate the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{m+1}$, where m is the minimum such that $h(m) \geq x$. If $h(m) > x$ then reject x. Otherwise, accept x if and only if $\alpha_{m+1}(m) = 1$.

This concludes the proof of Lemma 3.10 and the proof of the theorem. □

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References

[1] K. Ambos-Spies, Problems which Cannot Be Reduced to Any Proper Subproblems, Proc. 28th International Symposium, MFCS 2003, vol. 2747 of Lecture Notes in Computer Science, B. Rovan and P. Vojtáš, editors, Springer, Berlin, 2003, 162–168.

[2] K. Ambos-Spies. Private communication.

[3] P. Cintioli, Sets without Subsets of Higher Many-One Degree, *Notre Dame Journal of Formal Logic* **46**:2 (2005), 207–216.

[4] P. Cintioli, Low sets without subsets of higher many-one degree, *Mathematical Logic Quarterly* **57**:5 (2011), 517–523.

[5] P. Cintioli and R. Silvestri, Polynomial Time Introreducibility, *Theory of Computing Systems* **36**:1 (2003), 1–15.

[6] R.G. Downey and D.R. Hirschfeldt, Algorithmic Randomness and Complexity, Springer-Verlag, New York, 2010.

[7] C.G. Jockusch Jr., Upward closure and cohesive degrees, *Israel Journal of Mathematics* **15**:3 (1973), 332–335.

[8] P. Odifreddi, Strong reducibilities, *Bulletin of the American Mathematical Society* **4**:1 (1981), 37–86.

[9] P. Odifreddi, Classical Recursion Theory, vol. 125 of Studies in Logic and the Foundations of Mathematics, North Holland Publishing Company, Amsterdam, 1989.

[10] S.G. Simpson, Sets Which Do Not Have Subsets of Every Higher Degree, *The Journal of Symbolic Logic* **43**:1 (1978), 135–138.

[11] R.I. Soare, Sets with no Subset of Higher Degrees, *The Journal of Symbolic Logic* **34**:1 (1969), 53–56.

[12] R.I. Soare, Recursively enumerable sets and degrees, Perspectives in Mathematical Logic, Springer-Verlag, Berlin Heidelberg, 1987.

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