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A NOTE ON WANSING'S EXPANSION OF NELSON'S LOGIC

A CORRECTION TO AN AXIOMATIZATION OF WANSING'S EXPANSION OF NELSON'S LOGIC

A b s t r a c t. The present note corrects an error made by the author in answering an open problem of axiomatizing an expansion of Nelson's logic introduced by Heinrich Wansing. It also gives a correct axiomatization that answers the problem by importing some results on subintuitionistic logics presented by Greg Restall.

1. Introduction

In [2], the author presented an axiomatization of an expansion of Nelson's logic, introduced by Heinrich Wansing, motivated by an open problem for-

mulated in [4, p.52]. The axiomatization, however, turned out to be incorrect.[1] The purpose of this note is to correct my error and present another axiomatization which is complete with respect to the semantics formulated by Wansing.

2. Semantics and proof theory

After setting up the language, we first present the semantics, and then turn to the proof theory.

Definition 2.1. The language $\mathcal{L}$ consists of a finite set $\{\bot, \sim, \mathsf{M}, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2.1. Semantics

Let us now state the semantics. Although Wansing's focus was on one of the Nelson's logics known as **N3** in the literature[2], we will take a little more general system $\mathbf{N4}^{\bot}$, introduced by Sergei Odintsov in [1], as the base system and add the consistency connective.

Definition 2.2. A model for the language $\mathcal{L}$ is a quadruple $\langle g, W, \leq, V \rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); $\leq$ is a reflexive and transitive relation on W with g being the least element; and $V : W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0,1\}$. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$,

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[1] I would like to thank Mitio Takano for pointing out the error.

[2] In [4], Wansing refers to the system as **N**, but here we will use the updated notation from later publications.

- $1 \notin I(w, \bot)$,

- $0 \in I(w, \bot)$,

- $1 \in I(w, \sim A)$ iff $0 \in I(w, A)$,

- $0 \in I(w, \sim A)$ iff $1 \in I(w, A)$,

- $1 \in I(w, A \wedge B)$ iff $1 \in I(w, A)$ and $1 \in I(w, B)$,

- $0 \in I(w, A \wedge B)$ iff $0 \in I(w, A)$ or $0 \in I(w, B)$,

- $1 \in I(w, A \vee B)$ iff $1 \in I(w, A)$ or $1 \in I(w, B)$,

- $0 \in I(w, A \vee B)$ iff $0 \in I(w, A)$ and $0 \in I(w, B)$,

- $1 \in I(w, A \to B)$ iff for all $x \in W$: if $w \leq x$ and $1 \in I(x, A)$ then $1 \in I(x, B)$,

- $0 \in I(w, A \to B)$ iff $1 \in I(w, A)$ and $0 \in I(w, B)$,

- $1 \in I(w, \mathsf{M}A)$ iff for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$,

- $0 \in I(w, \mathsf{M}A)$ iff $0 \in I(w, A)$.

Finally, semantic consequence is now defined as follows:

$\Sigma \models A$ iff for all models $\langle g, W, \leq, I \rangle$, $1 \in I(g, A)$ if $1 \in I(g, B)$ for all $B \in \Sigma$.

Remark 2.3. Note that persistence is *not* preserved in the presence of M. This was not handled carefully in [2]. More specifically, the soundness result ([2, Theorem 3.1]) does not hold for formulas of the form $\mathsf{M}A \to (B \to \mathsf{M}A)$. However, if we assume further that the reflexive and transitive relation is also directed, then the Hilbert-style system introduced in [2] is sound and complete with respect to the modified semantics. This additional constraint on the binary relation also explains why the remarks in [2, §4] related to Jankov's logic hold in the axiomatic system.

Remark 2.4. First, note that we included the base state. This constraint does not affect the consequence relation of Nelson's logics but is needed for the completeness proof below. Second, if we eliminate the clause for M, then we obtain the semantics for the system $\mathbf{N4}^{\perp}$. Note also that two falsity conditions are considered for M in [4]. Based on the observation given by Wansing in [4, p. 51], we take the simpler version.

2.2. Proof Theory

We now turn to the proof theory. Since Nelson's logic is presented in terms of a Hilbert-style calculus in [4], we will follow that path, and present some axioms for the new connective.

Definition 2.5. The system $\mathbf{N4}^{\perp}(\mathbf{Md})$ consists of the following axiom schemata:

$$A \to A \tag{Ax1}$$

$$A \to (B \to B) \tag{Ax2}$$

$$((A \to B) \land (B \to C)) \to (A \to C) \tag{Ax3}$$

$$(A \land B) \to A \tag{Ax4}$$

$$(A \land B) \to B \tag{Ax5}$$

$$((C \to A) \land (C \to B)) \to (C \to (A \land B)) \tag{Ax6}$$

$$A \to (A \lor B) \tag{Ax7}$$

$$B \to (A \lor B) \tag{Ax8}$$

$$((A \to C) \land (B \to C)) \to ((A \lor B) \to C) \tag{Ax9}$$

$$(A \land (B \lor C)) \to ((A \land B) \lor (A \land C)) \tag{Ax10}$$

$$(A \to B) \to ((B \to C) \to (A \to C)) \tag{Ax11}$$

$$(A \land (A \to B)) \to B \tag{Ax12}$$

$$A \to (B \to A) \text{ where } A \text{ is M-free} \tag{Ax13}$$

$$\sim\sim A \leftrightarrow A \tag{Ax14}$$

$$\sim(A \land B) \leftrightarrow (\sim A \lor \sim B) \tag{Ax15}$$

$$\sim(A \lor B) \leftrightarrow (\sim A \land \sim B) \tag{Ax16}$$

$$\sim(A \to B) \leftrightarrow (A \land \sim B) \tag{Ax17}$$

$$\perp \to A \tag{Ax18}$$

$$A \to \sim\perp \tag{Ax19}$$

$$(\mathsf{M}A \land (A \to \perp)) \to B \tag{Ax20}$$

$$B \to (\mathsf{M}A \lor (A \to \perp)) \tag{Ax21}$$

$$\sim\mathsf{M}A \leftrightarrow \sim A \tag{Ax22}$$

In addition to these axioms, we have the following rules of inference.

$$\frac{A \quad A \to B}{B} \quad \text{(R1)} \qquad\qquad \frac{A \vee C \quad (A \to B) \vee C}{B \vee C} \quad \text{(R2)}$$

$$\frac{A \quad B}{A \wedge B} \quad \text{(R3)} \qquad \frac{(A \to B) \vee E \quad (C \to D) \vee E}{((B \to C) \to (A \to D)) \vee E} \quad \text{(R4)}$$

Following the usual convention, we define $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$. Finally, we write $\Gamma \vdash A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{N4}^\perp(\mathbf{Md})$; (iii) is obtained by one of the rules (R1)–(R4) from formulas preceding it in sequence.

Remark 2.6. The subsystem of $\mathbf{N4}^\perp(\mathbf{Md})$ consisting of axioms (Ax1) through (Ax10) together with the rules of inference (R1) through (R4) and the following rule is the system $\mathbf{SJ}$, introduced and studied by Greg Restall in [3].

$$\frac{A \to B \quad C \to D}{(B \to C) \to (A \to D)} \tag{1}$$

Note that the above rule, included in the original formulation of $\mathbf{SJ}$, is derivable in view of (R4), (R1), (Ax1) and (Ax9). Indeed, assume $A \to B$ and $C \to D$. Then by (Ax7) and (R1) we obtain $(A \to B) \vee E$ and $(C \to D) \vee E$ where E is $(B \to C) \to (A \to D)$. Then by applying (R4), we have $E \vee E$, and by (R1), (Ax1) and (Ax9), we obtain E, as desired.

Remark 2.7. It deserves noting that the following rules, known as Prefixing, Suffixing and Transitivity respectively, are derivable in $\mathbf{SJ}$ in view of (1), (R1) and (Ax1):

$$\frac{C \to D}{(A \to C) \to (A \to D)} \quad \frac{A \to B}{(B \to C) \to (A \to C)} \quad \frac{A \to B \quad B \to C}{A \to C}.$$

Indeed, for Prefixing, substitute A for B in (1). For Suffixing, substitute C for D in (1). And finally, for Transitivity, substitute B for C in (1).

Remark 2.8. The disjunctive form of (R3), namely the following rule is derivable:
$$\frac{A \vee C \quad B \vee C}{(A \wedge B) \vee C}$$

Note first that the following formulas and rules are provable:

$$(A \wedge B) \to (B \wedge A) \qquad (2)$$
$$(A \vee B) \to (B \vee A) \qquad (3)$$
$$((A \vee B) \vee C) \to (A \vee (B \vee C)) \quad (4)$$

$$\frac{A \vee B \quad B \to C}{A \vee C} \qquad (5)$$

$$\frac{B \vee A \quad B \to C}{C \vee A} \qquad (6)$$

For proofs, use (Ax4), (Ax5), (R3), (Ax6), (R1) for (2) and (Ax7), (Ax8), (R3), (Ax9), (R1) for (3) and (4).

For (5), assume $A \vee B$ and $B \to C$. Then by the latter, (Ax8) and (R1), we obtain $A \vee (B \to C)$, and then we only need to apply (R2) to obtain the desired result. Finally, for (6), use (3), (5) and (R1).

We can then derive the concerned rule as follows.

1	$A \vee C$	[sup.]
2	$B \vee C$	[sup.]
3	$((A \vee C) \wedge B) \vee ((A \vee C) \wedge C)$	[1, 2, (R3), (Ax10), (R1)]
4	$((A \vee C) \wedge C) \to C$	[(Ax4)]
5	$((A \vee C) \wedge B) \vee C$	[3, 4, (5)]
6	$((A \vee C) \wedge B) \to ((B \wedge A) \vee (B \wedge C))$	[(2), (Ax10), Transitivity]
7	$((B \wedge A) \vee (B \wedge C)) \vee C$	[5, 6, (6)]
8	$((B \wedge C) \vee C) \to C$	[(Ax5), (Ax1), (R3), (Ax9), (R1)]
9	$(B \wedge A) \vee C$	[7, (4), (R1), 8, (5)]
10	$(A \wedge B) \vee C$	[9, (2), (6)]

Proposition 2.9. *If $A \vdash B$ then $C \vee A \vdash C \vee B$.*

Proof. By induction on the length n of the proof of $A \vdash B$. $\square$

3. Soundness and completeness

3.1. Soundness

As usual, the soundness part is rather straightforward.

Theorem 3.1 (Soundness). *For $\Gamma \cup \{A\} \subseteq$ Form, if $\Gamma \vdash A$ then $\Gamma \models A$.*

Proof. By induction on the length of the proof. We only note that axioms (Ax1) through (Ax10) and (Ax14) through (Ax19) are valid in view of truth and falsity conditions. Moreover, (Ax11) and (Ax12) are valid in

view of transitivity and reflexivity of $\leq$ respectively. Furthermore, (Ax13) is valid in view of the fact, as noted by Restall in [3, Lemma 3.1], that persistence is preserved for M-free formulas since we assume the transitivity of $\leq$. For the three axioms for M, we can check their validity as follows.

For (Ax20): For any $w \in W$, the following holds.

$1 \in I(w, \mathsf{M}A \wedge (A \to \bot))$

 iff $1 \in I(w, \mathsf{M}A)$ and $1 \in I(w, A \to \bot)$

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  and (for all $x \in W$: if $w \leq x$ and $1 \in I(x, A)$ then $1 \in I(x, \bot)$)

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  and (for all $x \in W$: if $w \leq x$ then $1 \notin I(x, A)$)

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  and not (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

Therefore, we obtain that $1 \notin I(w, \mathsf{M}A \wedge (A \to \bot))$, and thus $\models (\mathsf{M}A \wedge (A \to \bot)) \to B$.

For (Ax21): For any $w \in W$, the following holds.

$1 \in I(w, \mathsf{M}A \vee (A \to \bot))$

 iff $1 \in I(w, \mathsf{M}A)$ or $1 \in I(A \to \bot)$

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  or (for all $x \in W$: if $w \leq x$ and $1 \in I(x, A)$ then $1 \in I(x, \bot)$)

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  or (for all $x \in W$: if $w \leq x$ then $1 \notin I(x, A)$)

 iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

  or not (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

Therefore, we obtain that $1 \in I(w, \mathsf{M}A \vee (A \to \bot))$, and thus $\models B \to (\mathsf{M}A \vee (A \to \bot))$.

For (Ax22): For any $w \in W$, the following holds.

$$1 \in I(w, {\sim}\mathsf{M}A) \text{ iff } 0 \in I(w, \mathsf{M}A)$$
$$\text{iff } 0 \in I(w, A)$$

Therefore, $\models {\sim}\mathsf{M}A \leftrightarrow {\sim}A$, and this completes the proof. □

3.2. Completeness

First, we introduce some notions following [3].

Definition 3.2. We introduce the following notions.

1. If Π is a set of sentences, let $\Pi_\to$ be the set of all members of Π of the form $A \to B$.
2. $\Sigma \vdash_\Pi A$ iff $\Sigma \cup \Pi_\to \vdash A$.
3. Σ is a Π-*theory* iff:

   (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$

   (b) if $\vdash_\Pi A \to B$ then (if $A \in \Sigma$ then $B \in \Sigma$).

4. Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
5. If X is any set of sets of formulas the binary relation R on X is defined thus:

   $$\Sigma R \Delta \text{ iff (if } A \to B \in \Sigma \text{ then (if } A \in \Delta \text{ then } B \in \Delta)).$$

6. $\Sigma \vdash_\Pi \Delta$ iff for some $D_1, \ldots, D_n \in \Delta$ $(n \geq 1), \Sigma \vdash_\Pi D_1 \vee \ldots \vee D_n$.
7. $\vdash_\Pi \Sigma \to \Delta$ iff for some $C_1, \ldots, C_n \in \Sigma$ $(n \geq 1)$ and $D_1, \ldots, D_m \in \Delta$ $(m \geq 1)$:
   $$\vdash_\Pi C_1 \wedge \ldots \wedge C_n \to D_1 \vee \ldots \vee D_m.$$

8. Σ is Π-*deductively closed* iff (if $\Sigma \vdash_\Pi A$ then $A \in \Sigma$).
9. $\langle \Sigma, \Delta \rangle$ is a Π-*partition* iff (i) $\Sigma \cup \Delta = \mathsf{Form}$ and (ii) $\nvdash_\Pi \Sigma \to \Delta$.

In all the above, if Π is $\emptyset$, then the prefix 'Π-' will simply be omitted.

With these notions in mind, some lemmas are listed without their proofs. For the details, see [3].

Lemma 3.3. *If $\langle \Sigma, \Delta \rangle$ is a Π-partition then Σ is a prime Π-theory.*

Lemma 3.4. *If $\Sigma \nvdash \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a partition, and Σ' is deductively closed.*

Corollary 3.5. *If $\Sigma \nvdash A$ then there is a $\Pi \supseteq \Sigma$ such that $A \notin \Pi$, Π is a prime Π-theory and Π is Π-deductively closed.*

Lemma 3.6. *If $\nvdash_\Pi \Sigma \to \Delta$ then there are $\Sigma' \supseteq \Sigma$ and $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a Π-partition.*

Lemma 3.7. *Let Σ be a prime Π-theory and $A \to B \notin \Sigma$. Then there is a prime Π-theory, Δ such that $\Sigma R \Delta$, $A \in \Delta$, $B \notin \Delta$.*

Now, we are ready to prove the completeness.

Theorem 3.8 (Completeness). *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash A$. Then, by Corollary 3.5, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle \Pi, X, R, I \rangle$, where

$$X = \{\Delta : \Delta \text{ is a non-empty, non-trivial prime } \Pi\text{-theory}\},$$

R as in Definition 3.2 and I is defined thus. For every state, Σ and propositional variable, p:

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } \sim p \in \Sigma \qquad (\dagger)$$

Note here that persistence holds for $p \in \mathsf{Prop}$. We only consider one of them since the other holds in a similar manner. Suppose that $1 \in I(\Sigma, p)$ and $\Sigma R \Delta$. Then by the definition of I, we obtain $p \in \Sigma$. Since Δ is nonempty, let D be an element of Δ. In view of (Ax13), we have $\vdash p \to (D \to p)$, and this together with $p \in \Sigma$ implies $D \to p \in \Sigma$ since Σ is a Π-theory. Moreover, by the definition of R and that $D \in \Delta$, we obtain $p \in \Delta$, i.e. $1 \in I(\Delta, p)$, as desired. Note further that reflexivity and transitivity of R is guaranteed by (Ax12) and (Ax11) respectively.

In the remainder of the proof, we show that the above condition ($\dagger$) holds for any arbitrary formula, B:

$$1 \in I(\Sigma, B) \text{ iff } B \in \Sigma \text{ and } 0 \in I(\Sigma, B) \text{ iff } \sim B \in \Sigma \qquad (*)$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models A$. The proof of ($*$) is by a simultaneous induction on the complexity of B with respect to the positive and the negative clause.

For bottom: For the positive clause, note that the semantic clause is $1 \notin I(\Sigma, \bot)$ and that (Ax18) together with the non-triviality of Σ gives us $\bot \notin \Sigma$. Therefore, we obviously have $1 \notin I(\Sigma, \bot)$ iff $\bot \notin \Sigma$, and so, by contraposition, the desired result is proved. For the negative clause, we have the semantic clause $0 \in I(\Sigma, \bot)$. Moreover, since Σ is nonempty, let

D be an element of Σ. In view of (Ax19), we have $\vdash D \rightarrow \sim\perp$, and this together with $D \in \Sigma$ implies $\sim\perp \in \Sigma$ since Σ is a Π-theory. Therefore, we obtain $0 \in I(\Sigma, \perp)$ iff $\sim\perp \in \Sigma$.

For negation: We begin with the positive clause.

$$1 \in I(\Sigma, \sim C) \text{ iff } 0 \in I(\Sigma, C)$$
$$\text{iff } \sim C \in \Sigma \qquad\qquad \text{IH}$$

The negative clause is also straightforward.

$$0 \in I(\Sigma, \sim C) \text{ iff } 1 \in I(\Sigma, C)$$
$$\text{iff } C \in \Sigma \qquad\qquad \text{IH}$$
$$\text{iff } \sim\sim C \in \Sigma \qquad\qquad (\text{Ax14})$$

For disjunction: We begin with the positive clause.

$$1 \in I(\Sigma, C \vee D) \text{ iff } 1 \in I(\Sigma, C) \text{ or } 1 \in I(\Sigma, D)$$
$$\text{iff } C \in \Sigma \text{ or } D \in \Sigma \qquad\qquad \text{IH}$$
$$\text{iff } C \vee D \in \Sigma \qquad\qquad \Sigma \text{ is a prime theory}$$

The negative clause is also straightforward.

$$0 \in I(\Sigma, C \vee D) \text{ iff } 0 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D)$$
$$\text{iff } \sim C \in \Sigma \text{ and } \sim D \in \Sigma \qquad\qquad \text{IH}$$
$$\text{iff } \sim C \wedge \sim D \in \Sigma \qquad\qquad \Sigma \text{ is a theory}$$
$$\text{iff } \sim(C \vee D) \in \Sigma \qquad\qquad (\text{Ax16})$$

For conjunction: Similar to the case for disjunction, and thus we leave the details to the reader.

For implication: We begin with the positive clause.

$$1 \in I(\Sigma, C \rightarrow D) \text{ iff for all } \Delta \text{ s.t. } \Sigma R \Delta, \text{ if } 1 \in I(\Delta, C) \text{ then } 1 \in I(\Delta, D)$$
$$\text{iff for all } \Delta \text{ s.t. } \Sigma R \Delta, \text{ if } C \in \Delta \text{ then } D \in \Delta \qquad \text{IH}$$
$$\text{iff } C \rightarrow D \in \Sigma$$

For the last equivalence, assume $C \rightarrow D \in \Sigma$ and $C \in \Delta$ for any Δ such that $\Sigma R \Delta$. Then by the definition of $\Sigma R \Delta$, we obtain $\Delta \vdash D$, i.e. $D \in \Delta$, as desired. On the other hand, suppose $C \rightarrow D \notin \Sigma$. Then by Lemma 3.7,

there is a Δ such that $\Sigma R\Delta$, $C \in \Delta$, $D \notin \Delta$ and Δ is a prime Π-theory. Furthermore, non-triviality of Δ is obvious by $D \notin \Delta$. Thus, we obtain the desired result.

As for the negative clause, it is straightforward.

$$0 \in I(\Sigma, C \to D) \text{ iff } 1 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D)$$
$$\text{iff } C \in \Sigma \text{ and } {\sim}D \in \Sigma \qquad\qquad \text{IH}$$
$$\text{iff } C \wedge {\sim}D \in \Sigma \qquad\qquad \Sigma \text{ is a theory}$$
$$\text{iff } {\sim}(C \to D) \in \Sigma \qquad\qquad \text{(Ax17)}$$

For consistency: We begin with the positive clause.

$$1 \in I(\Sigma, \mathsf{M}C) \text{ iff for some } \Delta, \Sigma R\Delta \text{ and } 1 \in I(\Delta, C)$$
$$\text{iff for some } \Delta, \Sigma R\Delta \text{ and } C \in \Delta \qquad\qquad \text{IH}$$
$$\text{iff } \mathsf{M}C \in \Sigma$$

For the last equivalence, assume $\mathsf{M}C \in \Sigma$. Then, we have $C \to \bot \notin \Sigma$. Indeed, if $C \to \bot \in \Sigma$, then since Σ is a Π-theory, we obtain $\mathsf{M}C \wedge (C \to \bot) \in \Sigma$ which implies $\bot \in \Sigma$ by (Ax20). But since Σ is non-trivial, i.e. $\bot \notin \Sigma$, we obtain $C \to \bot \notin \Sigma$. Once this is established, then by Lemma 3.7, there is a Σ' such that $\Sigma R\Sigma'$, $C \in \Sigma'$, $\bot \notin \Sigma'$ and Σ' is a prime Π-theory, as desired. For the other half, assume $\mathsf{M}C \notin \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma R\Delta$. Then by the former, (Ax21) and the primeness of Σ, we obtain $C \to \bot \in \Sigma$. This together with $C \in \Delta$ and the definition of R implies that $\bot \in \Delta$ which contradicts to the assumption that Δ is non-trivial.

As for the negative clause, it runs as follows.

$$0 \in I(\Sigma, \mathsf{M}C) \text{ iff } 0 \in I(\Sigma, C)$$
$$\text{iff } {\sim}C \in \Sigma \qquad\qquad \text{IH}$$
$$\text{iff } {\sim}\mathsf{M}C \in \Sigma \qquad\qquad \text{(Ax22)}$$

Thus, we obtain the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.9. Note that if we drop transitivity, reflexivity, and persistence conditions for the binary relation $\leq$, then the corresponding axiomatization is obtained by eliminating (Ax11), (Ax12) and (Ax13).

Acknowledgements

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