REPORTS ON MATHEMATICAL LOGIC
51 (2016), 105–131
doi:10.4467/20842589RM.16.008.5285

Gemma ROBLES

THE QUASI-RELEVANT 3-VALUED LOGIC RM3 AND SOME OF ITS SUBLOGICS LACKING THE VARIABLE-SHARING PROPERTY

A b s t r a c t. The logic RM3 is the 3-valued extension of the logic R-Mingle (RM). RM (and so, RM3) does not have the variable-sharing property (vsp), but RM3 (and so, RM) lacks the more "offending" "paradoxes of relevance", such as $A \to (B \to A)$ or $\neg A \to (A \to B)$. Thus, RM and RM3 can be useful when "some relevance", but not the full vsp, is needed. Sublogics of RM3 with the vsp are well known, but this is not the case with those lacking this property. The first aim of this paper is to define an ample family of sublogics of RM3 without the vsp. The second one is to provide these sublogics and RM3 itself with a general Routley-Meyer semantics, that is, the semantics devised for relevant logics in the early seventies of the past century.

Received 9 December 2015
Keywords and phrases: RM3; R-Mingle; relevant logics; quasi-relevant logics; Routley-Meyer semantics; relational semantics; substructural logics.

1. Introduction

The first aim of this paper is to define an ample family of sublogics of RM3 without the variable-sharing property. The second one is to endow these sublogics and RM3 itself with a general Routley-Meyer semantics.

As it is well known (cf. [1]), a necessary property of any relevant logic S is the following:

Definition 1.1 (Variable-sharing property – vsp)**.** If $A \to B$ is a theorem of S, then A and B share at least a propositional variable.

In [1] (§22.1.3), it is shown that the logic of relevance R, and so any logic included in it (as, for example, the logic of entailment E or the logic of ticket entailment T), has the vsp. There is, however, an interesting logic related to R, the logic R-Mingle (RM), that does not have the vsp. The logic RM is axiomatized by adding the axiom mingle $A \to (A \to A)$ to the logic of relevance R (the axiom mingle is labelled A9 below; cf. Definition 2.6). In RM "paradoxes of relevance" (i.e., conditionals in which antecedent and consequent do not share propositional variables) are almost immediate (cf. [1], §29.5). But, despite this fact, RM has been given considerable attention in volume 1 of *Entailment* (cf. [1], §29.3) because it lacks the most "offensive" paradoxes of relevance ($A \to (B \to A)$ or $\neg A \to (A \to B)$, for example) and, in fact, it has the following property, akin to the vsp (cf. [1], p. 417):

Definition 1.2 (Quasi-relevance property)**.** If $A \to B$ is a theorem, then either (1) A and B share at least a propositional variable or (2) both $\neg A$ and B are theorems.

(Cf. Definition 2.1. below on the logical languages used in the paper.)

Thus, as Meyer puts it (cf. [1], p.393): "Sometimes one doesn't need the whole relevance principle and, in those occasions, RM is good enough, when some relevance is desirable." (By the 'whole relevance principle', Meyer refers to the vsp.)

The logic RM3 is the 3-valued extension of RM and its has some of the more important properties of the latter logic such as the quasi-relevance property (cf. Section 4 of the appendix). The logic RM3 can be axiomatized by adding any of the axioms $A \vee (A \to B)$ or $\neg A \to [A \vee (A \to B)]$ to RM. (These axioms are labelled t18 and A11, respectively, below. Cf.

Proposition 4.1.) So, RM3 can be axiomatized by adding A9 $A \to (A \to A)$ and either t18 $A \vee (A \to B)$ or A11 $\neg A \to [A \vee (A \to B)]$ to R (cf. Section 3 of the appendix). Concerning these axioms, it should be noted that t18 can be added to R, the vsp being preserved, as shown in the 5-valued relevant logic CL, one axiom of which is t18. The logic CL is an extension of R axiomatizing Meyer's Crystal lattice CL, and it has the vsp (cf. [5], p. 114, ff.). The axiom mingle A9, however, causes the breaking of the vsp when added to weak systems with this property (cf. [12]). RM3 has been given an algebraic semantics in [7] (cf. also [1], §29.4). Also, it has been endowed in [2] with a Belnap-Dunn "bivalent" type semantics as well as a 2 set-up model structure of the kind defined in Routley-Meyer semantics (cf. [14] and references therein).

Next, we describe the aims of the paper. Sublogics of RM and RM3 with the vsp are well known: they include the logic of relevance R and their sublogics such as the logic of entailment E, the logic of ticket entailment T, not to mention weak relevant logics among which Brady's logic DR and its sublogics are undoubtedly the more important (cf. [3], [4], [6]). Nevertheless, sublogics of RM3 without the vsp have not been studied yet, as far as we know. But these logics are interesting and useful, since, as remarked above, there are situations where we need "some relevance", but not necessarily the full vsp. Consequently, the first aim of this paper is to define an ample family of sublogics of RM3 without the vsp. The minimal logic among these is the result of adding to Routley and Meyer's basic logic B (cf. [14]; cf. Definition 2.5 below) the characteristic axioms of RM3, A9 and A11, referred to above, together with the auxiliary axiom A10 $(A \wedge \neg A) \to (B \vee \neg B)$ labeled "safety" in its rule form in [8], p. 14. The second aim of this paper is to endow these sublogics of RM3 and RM3 itself with a general Routley-Meyer semantics (RM-semantics), the semantics devised for relevant logics in the early seventies of the past century (cf. [14]). In this sense, we remark that although there is more than one semantical approach to RM3, as it was pointed above, we do not have a general Routley-Meyer semantics for RM3 or any of its sublogics without the vsp yet, as far as we know.

The paper is organized as follows. In Section 2, the logic BRM3 is defined, a Routley-Meyer semantics for BRM3 is provided and soundness is proved w.r.t. this semantics. The label BRM3 stands for 'Basic non-relevant logic included in RM3'. In Section 3, it is proved that BRM3 is complete w.r.t. the semantics defined in the previous section, by using a canonical model construction. In Section 4, we define an ample family of sublogics of BMR3 without the vsp and endow each one of them with a general RM-semantics. The section is ended by providing two alternative ways of defining models for RM3 in the RM-semantics. In Section 5, we draw some conclusions from the results obtained and suggest some directions for further work on the same topic. Finally, we have included an appendix recording some matrices upon which some proofs in the preceding sections are based and some other additional material.

2. The logic BRM3 and its semantics

Definition 2.1 (Languages, logics)**.** The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ...$ and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, C, etc., are metalinguistic variables. From the proof-theoretical point of view, we shall consider propositional logics formulated in the Hilbert-style way, that is, logics axiomatized by means of a set of axioms (actually, axiom schemes) and a set of rules of derivation. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems. By $\vdash_{\mathrm{S}} A$, it is indicated that A is a theorem of S.

Definition 2.2 (Logical matrix)**.** A (logical) matrix is a structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper set of $\mathcal{V}$ (the set of designated values); and (3) $\mathtt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \mathtt{F} : \mathcal{V}^n \to \mathcal{V}$.

Definition 2.3 (M-interpretations, M-validity)**.** Let M be a matrix for (a propositional language) L. An M-interpretation I is a function from the set of all wffs to $\mathcal{V}$ according to the functions in $\mathtt{F}$. Then, $\vDash_{\mathrm{M}} A$ (A is M-valid; A is valid in the matrix M) iff $I(A) \in D$ for all M-intepretations I.

Definition 2.4 (The matrix MRM3)**.** The propositional language consists of the connectives $\to$, $\wedge$, $\vee$ and $\neg$. The matrix MRM3 is the structure

$(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is $\{0, 1, 2\}$ and it is ordered as shown in the following diagram:

$$0 \_\_\_ 1 \_\_\_ 2$$

(2) $D = \{1, 2\}$; (3) $\mathtt{F} = \{f_\to, f_\wedge, f_\vee, f_\neg\}$ and these functions are defined according to the following tables.

$\to$	0	1	2	$\neg$
0	2	2	2	2
1	0	1	2	1
2	0	0	2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

The notions of an RM3-interpretation and RM3-validity are defined according to the general Definition 2.3.

The logic RM3 axiomatizing the matrix MRM3 is recorded in section 3 of the appendix. (A logic S axiomatizes a matrix MS iff for every wff A, $\vdash_{\mathrm{S}} A$ iff $\vDash_{\mathrm{MS}} A$.) The logic BRM3 (the label intends to abbreviate 'Basic non-relevant logic included in RM3') is defined from Routley and Meyer's basic logic B as follows.

Definition 2.5 (The logics B_+ and B)**.** Routley and Meyer's basic positive logic B_+ is formulated with the following axioms and rules of inference (cf. [13] or [14]).

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A \;/\; (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \;/\; B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules of derivation

Modus Ponens (MP): $A \;\&\; A \to B \Rightarrow B$

Adjunction (Adj): $A \;\&\; B \Rightarrow A \wedge B$

Suffixing (Suf): $A \to B \Rightarrow (B \to C) \to (A \to C)$

Prefixing (Pref): $B \to C \Rightarrow (A \to B) \to (A \to C)$

Next, Routley and Meyer's basic logic B is axiomatized when adding the following axioms and rule to B_+ (cf. [14]):

A7. $A \to \neg\neg A$

A8. $\neg\neg A \to A$

Contraposition (Con) $A \to B \Rightarrow \neg B \to \neg A$

Definition 2.6 (The logic BRM3)**.** The logic BRM3 is formulated by adding the following axioms to B:

A9. $A \to (A \to A)$

A10. $(A \wedge \neg A) \to (B \vee \neg B)$

A11. $\neg A \to [A \vee (A \to B)]$

We record some theorems of BRM3 and a couple of facts about this logic.

Proposition 2.7 (Some theorems of BRM3)**.** *The following are theorems and rule of BRM3 (a proof is sketched to the right of each one of them)*

T1. $\neg A \to B \Rightarrow \neg B \to A$ Con, A8

T2. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ A3, A4, Con; A3, A5, Con, A7

T3. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ A3, A4, T1; A3, A4, A8, Con

T4. $A \to [B \to (A \vee B)]$ A4, A9

T5. $A \to [\neg A \vee (\neg A \to B)]$ A7, A11

Proposition 2.8 (On the axiomatization of BRM3)**.** *The logic BRM3 is well axiomatized w.r.t. B_+. That is, given the logic B_+, A7, A8, A9, A10, A11 and Con are independent of each other.*

Proof. See the appendix. □

Remark 1 (On BRM3 and vsp)**.** Axiom A10 $(A \wedge \neg A) \to (B \vee \neg B)$ clearly breaks the variable-sharing property (vsp). So, any extension of BRM3 lacks the vsp whence all of them (BRM3 included) are not relevant logics. (A logic is relevant in the minimal sense of the term if it has the vsp.)

Next, models and the notion of validity are defined. Then, the soundness theorem is proved.

Definition 2.9 (BRM3-models)**.** A BRM3-model is a structure $(K, O, R, *, \vDash)$ where K is a set, $O \subseteq K$, R is a ternary relation on K and $*$ is a unary operation on K subject to the following definitions and postulates for all $a, b, c \in K$:

d1. $a \leq b =_{\text{df}} (\exists x \in O)Rxab$

P1. $a \leq a$

P2. $(a \leq b \;\;\&\;\; Rbcd) \Rightarrow Racd$

P3. $a \leq a^{**}$

P4. $a^{**} \leq a$

P5. $a \leq b \Rightarrow b^{*} \leq a^{*}$

P6. $Rabc \Rightarrow (a \leq c \text{ or } b \leq c)$

P7. $a \leq a^{*}$ or $a^{*} \leq a$

P8. $Rabc \Rightarrow (b \leq a \text{ or } b \leq a*)$

Finally, $\vDash$ is a relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i). $(a \leq b \;\;\&\;\; a \vDash p) \Rightarrow b \vDash p$

(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$

(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$

(iv). $a \vDash A \rightarrow B$ iff for all $b, c \in K$, $(Rabc$ and $b \vDash A) \Rightarrow c \vDash B$

(v). $a \vDash \neg A$ iff $a^{*} \nvDash A$

Definition 2.10 (Truth in a BRM3-model)**.** A wff A is true in a BRM3-model iff $a \vDash A$ for all $a \in O$ in this model.

Definition 2.11 (BRM3-validity)**.** A formula A is BRM3-valid (in symbols, $\vDash_{\text{BRM3}} A$) iff $a \vDash A$ for all $a \in O$ in all BRM3-models.

Remark 2 (B-models)**.** A B-model, that is, a model for Routley and Meyer's basic logic B (Definition 2.5) is defined similarly as a BRM3-model, except that postulates P6, P7 and P8 are dropped (cf. [14], Chapter 4).

Models for logics extending BRM3 are simply defined by adding to P1-P8 the appropriate semantical postulates while defining 'truth in a model' and 'validity' similarly as in Definitions 2.10 and 2.11. In this way, an RM-semantics for a wealth of extensions of BRM3 (RM3 included) is defined in Section 4.

In order to prove soundness, the following lemmas are useful. (An adequate version of these lemmas is immediate for each extension of BRM3 considered in this paper.)

Lemma 2.12 (Hereditary condition)**.** *For any BRM3-model, $a, b \in K$ and wff A, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.*

Proof. Induction on the length of A. The conditional case is proved with P2 and the negation case with P5. □

Lemma 2.13 (Entailment lemma)**.** *For any wffs A, B, $\vDash_{BRM3} A \to B$ iff $(a \vDash A \Rightarrow a \vDash B$, for all $a \in K)$ in all BRM3-models.*

Proof. From left to right: by P1; from right to left: by Lemma 2.12.□

We can now prove soundness.

Theorem 2.14 (Soundness of BRM3)**.** *For each wff A, if $\vdash_{BRM3} A$, then $\vDash_{BRM3} A$.*

Proof. Axioms A1-A8 and the rules MP, Adj, Suf, Pref and Con are proved as in B-models (cf. Remark 2; cf. [14]). Then, it remains to prove that A9, A10 and A11 are BRM3-valid. We proceed by 'reductio ad absurdum' (by clauses ii-v, we refer to those in Definition 2.9; the proofs are simplified by leaning on Lemma 2.13).

(a) A9 $A \to (A \to A)$ is BRM3-valid: Suppose that there is $a \in K$ in some BRM3-model and wff A such that (1) $a \vDash A$ but $a \nvDash A \to A$. Then, (2) $b \vDash A$, $c \nvDash A$ for $b, c \in K$ such that $Rabc$ (clause iv). By P6, (3) $a \leq c$ or $b \leq c$. By applying Lemma 2.12 to 1, 2 and 3, we get (4) $c \vDash A$, contradicting 2.

(b) A10 $(A \wedge \neg A) \to (B \vee \neg B)$ is BRM3-valid: Suppose that there is $a \in K$ in some BRM3-model and wffs A, B such that (1) $a \vDash A \wedge \neg A$ but (2) $a \nvDash B \vee \neg B$. Then, (3) $a \vDash A$, $a^* \nvDash A$ (i.e., $a \vDash \neg A$), $a \nvDash B$, $a^* \vDash B$ (i.e., $a \nvDash \neg B$) by clauses ii, iii and v. By P7, (4) $a \leq a^*$ or $a^* \leq a$. So, (5) either $a^* \vDash A$ or $a \vDash B$ (3, 4, Lemma 2.12), a contradiction.

(c) A11 $\neg A \to [A \vee (A \to B)]$ is BRM3-valid: Suppose that there is $a \in K$ in some BRM3-model and wffs A, B such that (1) $a \vDash \neg A$ but $a \nvDash A \vee (A \to B)$, i.e., $a \nvDash A$ and $a \nvDash A \to B$ (clause iii). Then, (2) $a^* \nvDash A$, $a \nvDash A$, $b \vDash A$, $c \nvDash B$ for $b, c \in K$ such that $Rabc$ (clauses iv, v). By P8, (3) $b \leq a$ or (4) $b \leq a^*$. But if 3 is the case, $b \nvDash A$ follows (2 and Lemma 2.12), contradicting 2; and if 4 is the case, then we have $a^* \vDash A$ (2 and Lemma 2.12), also contradicting 2. □

In the next section, we prove the completeness of BRM3 w.r.t. the RM-semantics for this logic developed in the present section.

3. Completeness of BRM3

We begin by defining some preliminary concepts necessary in order to define the canonical model (cf. [14], Chapter 4).

Definition 3.1 (BRM3-theories)**.** A BMR3-theory (theory, for short) is a set of formulas closed under Adjunction (Adj) and BRM3-implication (BRM3-imp). That is, a is a theory if whenever $A, B \in a$, $A \wedge B \in a$; and if whenever $A \to B$ is a theorem of BRM3 and $A \in a$, $B \in a$.

Definition 3.2 (Classes of theories)**.** Let a be a theory. We set (1) a is prime iff whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; (2) a is empty iff it contains no wffs; (3) a is regular iff a contains all theorems of BRM3; (4) a is trivial iff every wff belongs to it.

Next, the canonical model is defined.

Definition 3.3 (The canonical BRM3-model)**.** Let K^T be the set of all theories and R^T be defined on K^T as follows: for all $a, b, c \in K^T$ and wffs A, B, $R^T abc$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Now, let K^C be the set of all non-trivial, non-empty prime theories and O^C be the subset of K^C formed by the regular theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for each $a \in K^C$, $a^* = \{A \mid \neg A \notin a\}$. Finally, $\vDash^C$is defined as follows: for any $a \in K^C$ and wff A, $a \vDash^C A$ iff $A \in a$. Then, the canonical model is the structure $(K^C, O^C, R^C, *^C, \vDash^C)$.

The canonical model for any extension S of BRM3 is defined in a similar way but referring its items to S-theories.

Before proceeding into the completeness proof, let us note an important fact.

Remark 3 (On canonical relevant models)**.** There is an important feature distinguishing the canonical BRM3-model (and those for the extensions of BRM3 considered in this paper) from the canonical models for standard relevant logics such as E (the logic of entailment) or R (the logic of relevant conditional). Namely, in the former, theories need to be non-empty and non-trivial, unlike in the latter. This fact permeates the entire completeness proof sharply distinguishing it from standard proofs for E, R or their subsystems: each time a theory is built, one has to show that it contains (and lacks) at least one wff. In this sense, A9 and A11 are essential. On the other hand, the canonical model for BRM3 (and those for its extensions) is differentiated from those (in the RM-semantics) for Łukasiewicz's 3-valued logic Ł3 (cf [11]) or 3-valued logic $G3_{Ł}$ (cf. [9]), say, in the following respect: unlike in BRM3 and its extensions included in RM3, non-triviality is equivalent to weak consistency (absence of the negation of any theorem; cf. [10] on this notion)in Ł3 and $G3_{Ł}$. Actually, as shown in Proposition A6, there are regular, non-trivial, prime theories containing the negation of a theorem.

We proceed into the completeness proof. Firstly, a series of lemmas is proved leaning on which it will be shown that the structure defined in Definition 3.3 is indeed a BRM3-model.

Lemma 3.4 (Defining x for a, b in R^T)**.** *Let a, b be non-empty theories. The set $x = \{B \mid \exists A[A \to B \in a \ \ \& \ \ A \in b]\}$ is a non-empty theory such that $R^T abx$.*

Proof. Assume the hypothesis of Lemma 3.4 and define x as indicated. It is easy to show that x is a theory. Then, $R^T abx$ is immediate by definition of R^T (Definition 3.3). Moreover, x is non-empty. Let $A \in a$, $B \in b$. By T4, $A \to [B \to (A \vee B)]$. So, $B \to (A \vee B) \in a$ and thus, $A \vee B \in x$. □

Lemma 3.5 (Extending a in $R^T abc$ to a member in K^C)**.** *Let a, b be non-empty theories and c be a non-trivial prime theory such that $R^T abc$. Then, there is a non trivial (and non-empty) prime theory x such that $a \subseteq x$ and $R^T xbc$.*

Proof. Given the hypothesis of Lemma 3.5, we can build a non-empty prime theory x such that $a \subseteq x$ and $R^T xbc$, following [14], Chapter 4. (The proof works for almost any logic including B_+ —cf. Definition 2.5.) Suppose now that x is trivial and let $A \in b$ and B be an arbitrary wff. As x is trivial, $A \to B \in x$. Then, $B \in c$ ($R^T xbc$, $A \to B \in x$, $A \in b$ and definition of R^T —cf. Definition 3.3), contradicting the non-triviality of c. □

Lemma 3.6 (Extending b in $R^T abc$ to a member in K^C)**.** *Let a and b be non-empty theories and c be a non-trivial prime theory such that $R^T abc$. Then, there is a non trivial (and non-empty) prime theory x such that $b \subseteq x$ and $R^T axc$.*

Proof. Similarly as in the preceding lemma, we build a non-empty prime theory x such that $R^T axc$. Suppose that x is trivial and let $A \in a$ and B be an arbitrary wff. By T5, $A \to [\neg A \vee (\neg A \to B)]$. So, $\neg A \vee (\neg A \to B) \in a$, whence, by primeness of a, either (1) $\neg A \in a$ or (2) $\neg A \to B \in a$. Let us first consider case 2. As x is trivial, $\neg A \in x$. But then $B \in c$ ($R^T axc$, $\neg A \to B \in a$, $\neg A \in x$ and definition of R^T), contradicting the a-consistency of c. Let us now examine case 1. Firstly, notice that $A \wedge \neg A \in a$. Then, $B \vee \neg B \in a$, by applying A10 $(A \wedge \neg A) \to (B \vee \neg B)$. By hypothesis, $B \notin a$, so $\neg B \in a$. By A11,$\neg B \to [B \vee (B \to B)]$ is a theorem. Thus, $B \vee (B \to B)] \in a$, that is, $B \to B \in a$ because $B \notin a$. But $B \in x$ for x is trivial. Then, $B \in c$ ($R^T axc$, $B \to B \in a$, $B \in x$ and definition of R^T), contradicting the a-consistency of c. □

Lemma 3.7 below shows that the canonical relation $\leq^C$ is just set inclusion between non-trivial and non-empty prime theories.

Lemma 3.7 ($\leq^C$ and $\subseteq$ are coextensive)**.** *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$.*

Proof. From left to right, it is immediate. So, suppose $a \subseteq b$ for $a, b \in K^C$. Clearly, $R^T \mathrm{BRM3} aa$ (cf. Definition 3.1 and Definition 3.3). Then, by using Lemma 3.5, there is some regular non-trivial theory x such that BRM3 $\subseteq x$ and $R^C xaa$. By the hypothesis $R^C xab$, i.e., $a \leq^C b$, since $x \in O^C$. □

Lemma 3.8 (Primeness of $*$-images)**.** *Let a be prime theory. Then, (1) a^* is a prime theory as well; (2) for any wff A, $\neg A \in a^*$ iff $A \notin a$.*

Proof. (Cf. [14], Chapter 4.) (1) a^* is closed under BRM3-imp by Con; a^* is closed under Adj by T3; a^* is prime by T2. (2) By A7 and A8. □

Lemma 3.9 ($*^C$ is an operation on K^C)**.** *Let a be a non-trivial and non-empty prime theory. Then, a^* is a non-trivial and non-empty prime theory as well.*

Proof. By Lemma 3.8(1), a^* is a prime theory. Next, it is shown that if a is non-trivial and non-empty, then a^* is also non-trivial and non-empty. (1) a^* is non-empty: as a is non-trivial, there is some wff A such that $A \notin a$. Then, $\neg A \in a^*$ by Lemma 3.8(2). (2) a^* is non-trivial. As a is non-empty, there is some wff A such that $A \in a$. Then, $\neg A \notin a^*$ by Lemma 3.8(2). □

Concerning Lemma 3.9, we note the following remark.

Remark 4 ($*^C$ is not an operation on O^C)**.** The canonical operation $*^C$ is not an operation on O^C: a would have to be weak consistent in order to prove that a^* is regular (cf. Remark 3).

In what follows, we prove the following two facts: (1) postulates P1-P8 hold in the canonical model; and (2) $\vDash^C$ is a (valuation) relation satisfying clauses i-v in Definition 3.3.

Lemma 3.10 (P1-P8 hold canonically)**.** *Postulates P1-P8 hold in the canonical BRM3-model.*

Proof. The use of Lemma 3.7 will greatly simplify the proof. By leaning on this lemma, P1-P5 are proved similarly as in [14], Chapter 4. So, let us prove P6, P7 and P8. We proceed by 'reductio ad absurdum'.

P6 ($R^C abc \Rightarrow (a \leq^C c$ or $b \leq^C c)$) holds in the canonical BRM3-model: suppose that there are $a, b, c \in K^C$ and wffs A, B such that (1) $R^C abc$ but (2) $A \in a$, $A \notin c$, $B \in b$ and $B \notin c$. By T4, (3) $A \to [B \to (A \vee B)]$. So, we have (4) $B \to (A \vee B) \in a$ (by 2, 3) and (5) $A \vee B \in c$ (by 1, 2, 4). But 5 contradicts 2.

P7 ($a \leq a^*$ or $a^* \leq a$) holds in the canonical RM3-model: suppose there is $a \in K^C$ and wffs A, B such that (1) $A \in a$, $A \notin a^*$ (i.e., $\neg A \in a$), $B \in a^*$ (i.e., $\neg B \notin a$) and $B \notin a$. By A10, (2) $(A \wedge \neg A) \to (B \vee \neg B)$. So, we have (3) $B \vee \neg B \in a$ (by 1). But 3 contradicts 1.

P8 ($R^{C}abc \Rightarrow (b \leq^{C} a$ or $b \leq^{C} a^{*}))$ holds in the canonical BRM3–model: suppose that there are $a, b, c \in K^{C}$ and wffs A, B such that (1) $R^{C}abc$ but (2) $A \in b$, $A \notin a$, $B \in b$ and $B \notin a^{*}$ (i.e., $\neg B \in a$). By A11, (3) $\neg B \to [B \vee (B \to C)]$ for arbitrary wff C. Then, (4) $B \vee (B \to C) \in a$ (by 2, 3) whence (5) $B \in a$ or (6) $B \to C \in a$. Let us consider the second alternative, 6. By applying the definition of R^{C} to 1, 2 ($B \in b$) and 6, we have (7) $C \in c$, contradicting the non-triviality of c. So, let us consider the first alternative, 5. We have (8) $B \wedge \neg B \in a$ (by 2, 5), whence (9) $A \vee \neg A \in a$ follows by A10 $(B \wedge \neg B) \to (A \vee \neg A)$. Now, (10) $\neg A \in a$ for $A \notin a$ by 2. Next, again by A11, we have (11) $\neg A \to [A \vee (A \to C)]$ for arbitrary wff C. So, (12) $A \vee (A \to C) \in a$ and, since $A \notin a$, then (13) $A \to C \in a$. Finally, by 1, 2 and (13), $C \in c$, contradicting the non-triviality of c. □

Before showing that the clauses hold canonically, we prove the primeness lemma.

Lemma 3.11 (Extension to prime theories)**.** *Let a be a theory and A a wff such that $A \notin a$. Then, there is a prime theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Cf. [14], Chapter 4, where it is shown how to proceed in an ample class of logics including the logic B_{+} (cf. Definition 2.5). □

Lemma 3.12 (Clauses i-v hold canonically)**.** *Clauses i-v in Definition 2.9 are satisfied by the canonical BRM3-model.*

Proof. Clause i is immediate by Lemma 3.7 and clauses ii, iii and iv from left to right are very easy. So, let us prove iv from right to left. For wffs A, B and $a \in K^{C}$, suppose $A \to B \notin a$ (i.e., $a \nvDash^{C} A \to B$). We prove that there are $x, y \in K^{C}$ such that $R^{C}axy$, $A \in x$ (i.e., $x \vDash^{C} A$) and $B \notin y$ (i.e., $y \nvDash^{C} B$). Consider the sets $z = \{C \mid\vdash_{\text{BRM3}} A \to C\}$ and $u = \{C \mid \exists D[D \to C \in a \ \& \ D \in z]\}$. They are theories such that $R^{T}azu$. Now, $A \in z$ (by A1) and $B \notin u$ (if $B \in u$, then $A \to B \in a$, contradicting the hypothesis). So, z is non-empty and u is non-trivial. Moreover, u is non-empty by Lemma 3.4. Now, by applying Lemma 3.11, u is extended to a non-trivial, non-empty prime theory y such that $u \subseteq y$, $B \notin y$ and $R^{T}azy$. Next, by using Lemma 3.6, z is extended to a non-trivial, non-empty prime theory x such that $z \subseteq x$ and $R^{C}axy$. Clearly, $A \in x$. Therefore, we have non-trivial and non-empty prime theories x, y such that $A \in x$, $B \notin y$ and $R^C axy$, as was to be proved. □

After showing that the canonical model is indeed a model, we finally prove completeness.

Lemma 3.13 (The canonical model is in fact a model)**.** *The canonical BRM3-model is in fact a BRM3-model.*

Proof. Since R^C is clearly a ternary relation on K^C, $*^C$ is an operation on K^C (Lemma 3.9) and K^C is non-empty (Lemma 3.11: BRM3 is non-empty and non-trivial), Lemma 3.13 follows by Lemma 3.10 and 3.12. □

Theorem 3.14 (Completeness of BRM3)**.** *For each wff A, if* $\vDash_{BRM3} A$*, then* $\vdash_{BRM3} A$*.*

Proof. Suppose $\nvdash_{\mathrm{BRM3}} A$. By Lemma 3.11, there is a non-trivial, non-empty prime theory x such that BRM3 $\subseteq x$ and $A \notin x$. By Definition 3.3 and Lemma 3.13, $x \nvDash^C A$. Therefore, $\nvDash_{\mathrm{BRM3}} A$ by Definition 2.11. □

4. Extensions of BRM3 included in RM3

In this section, we provide an RM-semantics for a wealth of extensions of BRM3 included in the logic RM3 among which RM3 itself is to be found. Consider the theses recorded in the following proposition.

Proposition 4.1 (Theses provable in RM3)**.** *The theses and the rule that follow are derivable in RM3. (In some cases some equivalent (w.r.t.*

BRM3) variants are remarked):

t1. $(A \to B) \to [(B \to C) \to (A \to C)]$

t2. $(B \to C) \to [(A \to B) \to (A \to C)]$

t3. $[(A \to B) \wedge (B \to C)] \to (A \to C)$

t4. $[A \wedge (A \to B)] \to B$ *(* $= \neg B \to [\neg A \vee \neg(A \to B)]$ *)*

t5. $[A \to (A \to B)] \to (A \to B)$ *(* $= [A \to (B \to C)] \to [(A \wedge B) \to C]$ *)*

t6. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$

t7. $(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$

t8. $A \Rightarrow (A \to B) \to B$

t9. $(A \to B) \to [[A \wedge (B \to C)] \to C]$

t10. $(A \wedge B) \to [[A \to (B \to C)] \to C]$ *(* $= A \to [[A \to (A \to B)] \to B]$ *)*

t11. $[A \wedge (B \to C)] \to [(A \to B) \to C)]$

t12. $A \to [(A \to B) \to B]$

t13. $[A \to (B \to C)] \to [B \to (A \to C)]$

t14. $B \to [[A \to (B \to C)] \to (A \to C)]$

t15. $(A \to B) \vee (B \to A)$

t16. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$

t17. $[A \to (B \vee C)] \to [(A \to B) \vee (A \to C)]$

t18. $A \vee (A \to B)$

t19. $(A \to B) \to (\neg B \to \neg A)$ *(* $= (A \to \neg B) \to (B \to \neg A) =$
$(\neg A \to B) \to (\neg B \to A) = (\neg A \to \neg B) \to (B \to A)$ *)*

t20. $\neg(A \wedge \neg A)$ *(* $= A \vee \neg A$ *)*

t21. $(A \wedge \neg B) \to \neg(A \to B)$ *(* $= (A \to B) \to (\neg A \vee B)$ *)*

t22. $[(A \to B) \wedge \neg B] \to \neg A$ *(* $= A \to [B \vee \neg(A \to B)]$ *)*

t23. $(\neg A \wedge B) \to (A \to B)$ *(* $= \neg(A \to B) \to (A \vee \neg B)$ *)*

t24. $\neg(A \to B) \to (B \to A)$

t25. $\neg A \to [\neg B \vee (A \to B)]$

t26. $(A \vee \neg B) \vee (A \to B)$

t27. $\neg A \vee (B \to A)$

t28. $A \vee [A \to (B \vee \neg B)]$

t29. $[\neg(A \to B) \wedge B] \to \neg B$ *(* $= B \to [\neg B \vee (A \to B)]$ *)*

t30. $B \to [(A \wedge \neg A) \to B]$

Proof. By the matrix MRM3 (cf. Definition 2.4): t8 preserves RM3–validity and t1-t7, t9-t30 are RM3-valid. □

On the other hand, we have:

Proposition 4.2 (t1-t30 are not provable in BRM3)**.** *Theses t1-t30 are not provable in BRM3.*

Proof. See the appendix. □

Let now S be an extension of BRM3 axiomatized by any selection of t1-t30. The aim of this section is to define an RM-semantics for S. The key concept is "corresponding postulate (cp) to a thesis or rule", which can be rendered as follows (cf. [14], p. 301).

Definition 4.3 (Corresponding postulate —cp)**.** Let ti be any of t1-t30, and let pj be a semantical postulate. Then, given the logic BRM3 and BRM3-models, pj is the cp to ti iff (1) ti is true in any BRM3-model in which pj holds; and (2) pj holds in the canonical BRM3-model if ti is added as an axiom (or rule) to BRM3.

It must be clear that if, given the logic BRM3 and BRM3-semantics, pj is the cp to ti, then the logic BRM3 + ti (i.e., BRM3 plus ti) is sound and complete w.r.t. BRM3-models (i.e., BRM3-models where pj holds).

Given a BRM3-model M, consider the following definition and semantical postulates for all $a, b, c, d \in K$ with quantifiers ranging over K (in some cases, some equivalent (w.r.t. BRM3-models) variants are remarked):

d2. $R^2abcd =_{\text{df}} \exists x(Rabx \ \ \& \ \ Rxcd)$

Pt1. $R^2abcd \Rightarrow \exists x(Racx \ \ \& \ \ Rbxd)$

Pt2. $R^2abcd \Rightarrow \exists x(Rbcx \ \ \& \ \ Raxd)$

Pt3. $Rabc \Rightarrow \exists x(Rabx \ \ \& \ \ Raxc)$

Pt4. $Raaa$ ($= Ra^*a^*a^*$)

Pt5. $Rabc \Rightarrow R^2abbc$

Pt6. $R^2abcd \Rightarrow \exists x, y(Racx \ \ \& \ \ Rbcy \ \ \& \ \ Rxyd)$

Pt7. $R^2abcd \Rightarrow \exists x, y(Racx \ \ \& \ \ Rbcy \ \ \& \ \ Ryxd)$

Pt8. $(\exists x \in O)\ Raxa$

Pt9. $Rabc \Rightarrow \exists x(Rabx\ \ \&\ \ Rbxc)$

Pt10. $Rabc \Rightarrow R^2baac$

Pt11. $Rabc \Rightarrow \exists x(Rbax\ \ \&\ \ Raxc)$

Pt12. $Rabc \Rightarrow Rbac$

Pt13. $R^2abcd \Rightarrow R^2acbd$

Pt14. $R^2abcd \Rightarrow R^2bcad$

Pt15. $(a \in O\ \ \&\ \ Rabc\ \ \&\ \ Rade) \Rightarrow (b \leq e \text{ or } d \leq c)$

Pt16. $(Rabc\ \ \&\ \ Rade) \Rightarrow \exists x[b \leq x\ \ \&\ \ d \leq x\ \ \&\ \ (Raxc \text{ or } Raxe)]$

Pt17. $(Rabc\ \ \&\ \ Rade) \Rightarrow \exists x[x \leq c\ \ \&\ \ x \leq e\ \ \&\ \ (Rabx \text{ or } Radx)]$

Pt18. $(a \in O\ \ \&\ \ Rabc) \Rightarrow b \leq a$

Pt19. $Rabc \Rightarrow Rac^*b^*$

Pt20. $a \in O \Rightarrow a \leq a^*\ (= a \in O \Rightarrow a^* \leq a^{**})$

Pt21. $Raa^*a\ (= Ra^*aa^*)$

Pt22. $Ra^*aa\ (= Raa^*a^*)$

Pt23. $Rabc \Rightarrow (a \leq c \text{ or } b \leq a^*)$

Pt24. $(Rabc\ \ \&\ \ Ra^*de) \Rightarrow (b \leq e \text{ or } d \leq c)]$

Pt25. $Rabc \Rightarrow (b \leq a^* \text{ or } a^* \leq c)$

Pt26. $(a \in O\ \ \&\ \ Rabc) \Rightarrow (a^* \leq c \text{ or } b \leq a)$

Pt27. $(a \in O\ \ \&\ \ Rabc) \Rightarrow a^* \leq c$

Pt28. $Rabc \Rightarrow (b \leq a \text{ or } c^* \leq c)$

Pt29. $Ra^*bc \Rightarrow (a \leq c \text{ or } a^* \leq c)$

Pt30. $Rabc \Rightarrow (a \leq c \text{ or } b \leq b^*)$

We have the following proposition:

Proposition 4.4 (Corresponding postulates to t1-t30)**.** *Given the logic BRM3 and BRM3-models, ptk is the corresponding postulate (cp) to tk* *(*$1 \leq k \leq 30$*).*

Proof. The proof is similar to that provided in [14], Chapter 4, for extensions of Routley and Meyer's basic logic B (cf. Definition 2.5). Actually, t1-t8, t12, t13, t15, t16, t18, t19-t21 and t24 are among the theses and rules considered in [14]. However, notice that, as it was remarked above (cf. Remark 3), any new theory introduced has to be shown non-empty and non-trivial. Consequently, most of the proofs in [14] have to modified in the context of the present paper. Nevertheless, the required modifications are easy to get by using Lemmas 3.4-3.7, as actually shown above in the case of postulates P6, P7 and P8. Anyway, let us prove a couple of examples (we follow the pattern set on in Theorem 2.14 and Lemma 3.10).

(a) *pt11 is the cp to t11.* (ai) *t11 is true in any BRM3 + pt11-model*: suppose that there is $a \in K$ in some BRM3 + pt11-model and wffs A, B such that (1) $a \vDash A \wedge (B \to C)$ but (2) $a \nvDash (A \to B) \to C$. Then, (3) $a \vDash A$, $a \vDash B \to C$ (clause ii, 1) and (4) $b \vDash A \to B$, $c \nvDash C$ for $b, c \in K$ such that $Rabc$ in this model (clause iv, 2). By pt11, we have (5) $Rbax$ and (6) $Raxc$ for some $x \in K$. Then, we get (7) $x \vDash B$ (3, 4, 5, clause iv) and, finally, (8) $c \vDash C$ (3, 6, 7, clause iv), contradicting 4. (aii) *pt11 holds in the canonical BRM3 + t11-model*: for $a, b, c \in K^C$, suppose (1) $R^C abc$. Consider now the following set $y = \{B \mid \exists A[A \to B \in b \ \ \& \ \ A \in a]\}$. By Lemma 3.4, y is a non-empty theory such that (2) $R^T bay$. Next, we show that $R^T ayc$ holds. Suppose (3) $A \to B \in a$ and (4) $A \in y$ for wffs A, B. We have to show that $B \in c$. By definition of y, we have (5) $C \to A \in b$ for some wff C such that (6) $C \in a$. By t11, $[C \wedge (A \to B)] \to [(C \to A) \to B]$ is a theorem. So, (7) $(C \to A) \to B \in a$ since $C \wedge (A \to B) \in a$ (by 3, 6). Then, (8) $B \in c$ (by 1, 5, 7). Thus, we have a non-empty theory y such that (9) $R^T bay$ and $R^T ayc$. By Lemma 3.6, y is extended to a non-trivial, non-empty prime theory x such that $y \subseteq x$ and $R^C axc$. Obviously, $R^C bax$. Therefore, we have $x \in K^C$ such that $R^C bax$ and $R^C axc$, as it was to be proved.

(b) *pt26 is the cp to t26.* (bi) *t26 is true in any BRM3 + pt26-model*: suppose that there is $a \in O$ in some BRM3 + pt26-model and wffs A, B such that (1) $a \nvDash (A \vee \neg B) \vee (A \to B)$, i.e., (2) $a \nvDash A$, (3) $a^* \vDash B$ (i.e., $a \nvDash \neg B$ by clause v) and (4) $a \nvDash A \to B$ by applying clause iii to 1. Then, (5) $b \vDash A$ and $c \nvDash B$ for $b, c \in K$ such that $Rabc$ in this model (by clause iv in 4). By pt26, (6) $a^* \leq c$ or (7) $b \leq a$. Suppose 6. Then, (8) $c \vDash B$ (by 3), contradicting 5. On the other hand, suppose 7. Then, (9) $a \vDash A$ (by 5), contradicting 2. (bii) *pt26 holds in the canonical BRM3 + t26-model*: suppose that there are $a \in O^C$ and $b, c \in K^C$ and wffs A, B, C such that (1) $R^C abc$ but (2) $A \in a^*$ (i.e., $\neg A \notin a$ by clause v), $A \notin c$, $B \in b$ and $B \notin a$. By t26, (3) $(B \vee \neg A) \vee (B \to A) \in a$. So, (4) $B \to A \in a$ follows by

2 and 3. Finally, (5) $A \in c$ (by 1, 2 and 4), contradicting 2. □

The section is ended by defining a general Routley-Meyer semantics for RM3.

As pointed out in the appendix, following Anderson and Belnap, RM3 can be axiomatized by adding A9 ($A \to (A \to A)$) and t18 ($A \vee (A \to B)$) to the logic of relevance R (cf. [1], pp. 469, ff.). Now, a model for R can be defined as follows (cf. [14], Chapter 4).

Definition 4.5 (R-models)**.** An R-model is structure $(K, O, R, *, \vDash)$ where $K, O, R, *$ and $\vDash$ are defined similarly as in B-models (cf. Remark 2), save for the addition of the following semantical postulates: pt1 ($R^2abcd \Rightarrow \exists x(Racx\ \&\ Rbxd)$), pt5 ($Rabc \Rightarrow R^2abbc$), pt12 ($Rabc \Rightarrow Rbac$) and pt19 ($Rabc \Rightarrow Rac^*b^*$).

Then, an RM3-model is defined as follows:

Definition 4.6 (RM3-models 1)**.** An RM3-model is defined similarly as an R-model, save for the addition of postulates: pt6 ($Rabc \Rightarrow (a \leq c$ or $b \leq c)$) and pt18 ($(a \in O\ \&\ Rabc) \Rightarrow b \leq a$).

But, on the other hand, following Brady, RM3 can be axiomatized as an extension of the logic DW with the following axioms: A11, t4, t21 and t23 (cf. [2]; cf. the appendix). So, RM3-models can alternatively be defined as follows (firstly, DW-models are defined).

Definition 4.7 (DW-models)**.** A DW-model is defined similarly as a B-model (cf. Remark 2), except that P5 is changed for pt19 ($Rabc \Rightarrow Rac^*b^*$).

Definition 4.8 (RM3-models 2)**.** An RM3-model is defined similarly as a DW-model, save for the addition of the following semantical postulates: P8 ($Rabc \Rightarrow (b \leq a$ or $b \leq a^*)$), pt4 ($Raaa$), pt21 (Raa^*a) and pt23 ($Rabc \Rightarrow (a \leq c$ or $b \leq a^*)$).

Notice that in Definition 4.5 and 4.6, postulate P5 is derivable immediately by pt19. Also, remark that P7 is not necessary in Definitions 4.6 and 4.8, since A10 is, of course, derivable in RM3 and P7 is the cp to A10 w.r.t. B-models (cf. Remark 2 and Lemma 3.10).

5. Conclusions

In this paper, we have provided an RM-semantics for an ample family of sublogics of RM3 including the basic logic BRM3. Actually, in Section 4, it has been shown that each one of t1-t30 can be added to BMR3 independently, the resulting logic being characterized by an RM-semantics. And, as pointed out in the introduction to this paper, these logics can be useful when "some relevance", but not the full vsp, is needed. Nonetheless, we have not paused to discuss neither BRM3 nor any of its extensions. So, future work on the topic could focus on this question or on proposing alternative extensions of BRM3 not defined in this paper. On our part, we will close these brief considerations by remarking that the matrices in section 2 in the appendix provide a rough first selection of sublogics of RM3. Thus, for example, M3 shows (when 1 is designated in addition to 2) that the characteristic axioms of RM3 (A9 and A11 together with A10) can be added to ticket entailment logic, T (minus the reductio axiom t21, but with the 'Principium of tertium non datur', t20), without the result collapsing in RM3 (T is axiomatized when dropping t12 and adding t2 to the formulation of R in Definition A3 in the appendix).

A. Appendix

A.1. Independence in BRM3

The following matrices show that A7, A8, A9, A10, A11 and Con are independent of each other, given the logic B_+ (cf. Proposition 2.5 on the logic B_+; cf. Definitions 2.2, 2.3 on the notion of a logical matrix. Designated values are starred):

Matrix I. Independence of A7:

$\to$	0	1	$\neg$
0	1	1	0
*1	0	1	0

$\wedge$	0	1
0	0	0
*1	0	1

$\vee$	0	1
0	0	1
*1	1	1

Falsifies A7 ($A = 1$).

Matrix II. Independence of A8:

Same set as in matrix I (the positive classical truth tables), save for the table for negation which is now as follows:

	$\neg$
0	1
*1	1

Falsifies A8 ($A = 0$).

Matrix III. Independence of Con:

$\to$	0	1	2	3	4	$\neg$
0	4	4	4	4	4	4
1	2	3	2	3	4	2
2	1	1	3	3	4	1
*3	0	1	2	3	4	3
*4	0	0	0	0	4	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	0	1	1
2	0	0	2	2	2
*3	0	1	2	3	3
*4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
*3	3	3	3	3	4
*4	4	4	4	4	4

Falsifies Con ($A = 1$, $B = 3$).

Matrix IV. Independence of A9:

$\to$	0	1	2	$\neg$
0	2	2	2	2
*1	0	1	1	1
*2	0	0	1	0

$\wedge$	0	1	2
0	0	0	0
*1	0	1	1
*2	0	1	2

$\vee$	0	1	2
0	0	1	2
*1	1	1	2
*2	2	2	2

Falsifies A9 ($A = 2$).

Matrix V. Independence of A10:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	3	0	3	1
*2	0	0	2	3	2
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
*2	0	0	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
*2	2	3	2	3
*3	3	3	3	3

Falsifies A10 ($A = 2$, $B = 1$).

Matrix VI. Independence of A11:

The tables for $\wedge$, $\vee$, $\neg$ are as in Matrix IV, but the conditional table is as follows:

→	0	1	2
0	1	1	2
*1	0	1	2
*2	0	0	2

Falsifies A11 ($A = B = 0$).

A.2. t1-t30 are not provable in BRM3

We prove that t1-t30 are not derivable in BRM3. We shall use eight matrices. Each one of them verifies BRM3 and falsifies some subset of t1-t30 (designated values are starred).

Matrix M1:

The tables for $\wedge$, $\vee$, $\neg$ are as in Matrix IV in the preceding section, but the conditional table is as follows:

→	0	1	2
0	2	2	2
1	1	2	2
*2	0	0	2

Falsifies t1 ($A = 1$, $B = C = 0$); t2 ($A = B = 1$, $C = 0$); t4 ($A = 1$, $B = 0$); t5 ($A = 1$, $B = 0$); t6 ($A = B = 1$, $C = 0$); t7 ($A = B = 1$, $C = 0$); t9 ($A = B = 1$, $C = 0$); t19 ($A = 1$, $B = 0$); t20 ($A = 1$); t27 ($A = 1$, $B = 2$).

Matrix M2:

The tables for $\wedge$, $\vee$, $\neg$ are as in M1, but the conditional table is the following:

→	0	1	2
0	2	2	2
1	1	2	2
*2	0	1	2

Falsifies t3 ($A = 2$, $B = 1$, $C = 0$).

Matrix M3:

The tables for $\wedge$, $\vee$, $\neg$ are as in M1, but the conditional table is as follows:

$\to$	0	1	2
0	2	2	2
*1	0	2	2
*2	0	0	2

Falsifies t8 ($A = B = 1$); t10 ($A = B = C = 1$); t11 ($A = B = C = 1$); t12 ($A = B = 1$); t13 ($A = 2,\ B = C = 1$); t14 ($A = 2,\ B = C = 1$); t18 ($A = 1,\ B = 0$); t21 ($A = B = 1$).

Matrix M4:

The tables for $\wedge$, $\vee$, $\neg$ are as in M1, but the conditional table is the following:

$\to$	0	1	2
0	2	2	2
1	0	2	2
*2	0	1	2

Falsifies t22 ($A = 2,\ B = 1$).

Matrix M5:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	2	3	3	3	2
2	0	0	3	3	1
*3	0	0	0	3	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
*3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
*3	3	3	3	3

Falsifies t23 ($A = 2,\ B = 1$); t26 ($A = 2,\ B = 1$); t29 ($A = 3,\ B = 2$).

Matrix M6:

$\to$	0	1	2	3	4	$\neg$
0	4	4	4	4	4	4
1	3	4	4	4	4	3
2	0	0	4	4	4	2
3	0	0	0	4	4	1
*4	0	0	0	0	4	0

$\wedge$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	2	2	2
3	0	1	2	3	3
*4	0	1	2	3	4

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	3	4
3	3	3	3	3	4
*4	4	4	4	4	4

Falsifies t28 ($A = 3,\ B = 2$); t30($A = 2,\ B = 1$).

Matrix M7:

→	0	1	2	3	4	5	¬
0	5	5	5	5	5	5	5
1	4	5	5	5	5	5	4
2	3	3	5	3	5	5	3
3	2	2	2	5	5	5	2
4	1	1	2	3	5	5	1
*5	0	0	0	0	0	5	0

∧	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	2	2
3	0	1	1	3	3	3
4	0	1	2	3	4	4
*5	0	1	2	3	4	5

∨	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	3	4	5
2	2	2	2	4	4	5
3	3	3	4	3	4	5
4	4	4	4	4	4	5
*5	5	5	5	5	5	5

Falsifies t15 ($A = 3,\ B = 2$); t16 ($A = 3,\ B = 2,\ C = 1$); t17 ($A = 4,\ B = 3,\ C = 2$).

Matrix M8:

→	0	1	2	3	4	5	6	¬
0	6	6	6	6	6	6	6	6
1	5	6	5	6	6	6	6	5
2	4	4	6	6	6	6	6	4
3	3	4	5	6	6	6	6	3
4	0	0	0	0	6	0	6	2
5	0	0	0	0	0	6	6	1
*6	0	0	0	0	0	0	6	0

∧	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1
2	0	0	2	2	2	2	2
3	0	1	2	3	3	3	3
4	0	1	2	3	4	3	4
5	0	1	2	3	3	5	5
*6	0	1	2	3	4	5	6

∨	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	1	3	3	4	5	6
2	2	3	2	3	4	5	6
3	3	3	3	3	4	5	6
4	4	4	4	4	4	6	6
5	5	5	5	5	6	5	6
*6	6	6	6	6	6	6	6

Falsifies t24 ($A = 5,\ B = 4$); t25 ($A = 5,\ B = 4$).

A.3. Axiomatization of RM3

We record the formulations of RM3 provided in [1] and in [2], respectively.

Definition A.1 (The logic DW)**.** The logic DW is axiomatized when changing the rule Con for t19 in the formulation of Routley and Meyer's logic B (cf. Definition 2.5; Proposition 4.1; cf. [6] about DW and other weak relevant logics).

Definition A.2 (Brady's axiomatization of RM3)**.** RM3 can be axiomatized by adding to DW the following axioms: A11, t4, t21 and t23 (cf. [2]; cf. Proposition 4.1).

Definition A.3 (The logic R)**.** Anderson and Belnap's logic of relevance R can be axiomatized with the following axioms and rules: A1, A2, A3, A4, A5, A6, A7, A8, t1, t5, t12, t19, MP and Adj (cf. [1]; Definition 2.5; Proposition 4.1).

Definition A.4 (Anderson and Belnap's axiomatization of RM3)**.** According to Anderson and Belnap (cf. [1], p. 469, ff.), RM3 can be axiomatized by adding to R the following axioms: t18 ($A \vee (A \to B)$) and $A \to (\neg A \to A)$.

Now, $A \to (\neg A \to A)$ and A9 ($A \to (A \to A)$) are easily shown equivalent, given R. So, RM3 can be axiomatized by adding A9 and t18 to R. Moreover, since t18 is immediate given R and A11 ($\neg A \to [A \vee (A \to B)]$), RM3 can be axiomatized by adding A9 and A11 to R.

A.4. RM3 has the quasi-relevance property

We prove:

Proposition A.5 (RM3 is quasi-relevant)**.** *If $A \to B$ is a theorem of RM3, then either (1) A and B share at least a propositional variable or (2) both $\neg A$ and B are theorems.*

Proof. Suppose that $A \to B$ is a theorem of RM3, but A and B do not share propositional variables and either $\nvdash_{\text{RM3}} \neg A$ or $\nvdash_{\text{RM3}} B$. By completeness w.r.t. MRM3-validity (cf. [2]), $\nvDash_{\text{MRM3}} \neg A$ or $\nvDash_{\text{MRM3}} B$.

Then, there are MRM3-interpretations I and I' such that $I(\neg A) = 0$ or $I'(B) = 0$, that is, $I(A) = 2$ or $I'(B) = 0$. Suppose (1) $I(A) = 2$. Let I'' be exactly as I except that for each propositional variable p_i in B, $I''(p_i) = I'(p_i)$. Clearly, $I''(B) = 0$ and $I''(A) = 2$ since A and B do no share propositional variables. Thus, $I''(A \to B) = 0$, whence $\nvdash_{\mathrm{RM3}} A \to B$ by soundness w.r.t. MRM3-validity (cf. [2]), contradicting the hypothesis. Case (2) ($I'(B) = 0$) is treated similarly. Consequently, we can conclude that RM3 has the quasi-relevance property. □

A.5. There are weak-consistent theories that are non-trivial

Let S be any extension of BRM3 included in or equivalent to RM3. We prove:

Proposition A.6 (w-inconsistent S-theories that are non-trivial)**.** *There are regular, prime, w-inconsistent S-theories that are, nevertheless, non-trivial.*

Proof. Let p_i, p_m be propositional variables and consider the set $y = \{B \mid \exists A[\vdash_{\mathrm{S}} A \;\&\; \vdash_{\mathrm{S}} [A \wedge \neg(p_i \to p_i)] \to B]\}$. It is easy to prove that y is an S-theory. Moreover, it is regular; but y is w-inconsistent: $\neg(p_i \to p_i) \in y$. So, y is inconsistent in the standard sense. Anyway, y is not trivial: for any theorem A of RM3, $[A \wedge \neg(p_i \to p_i)] \to p_m$ is falsified by any MRM3-interpretation I such that $I(p_i) = 1$ and $I(p_m) = 0$. (Recall that if A is a theorem of RM3, $I(A) = 1$ or $I(A) = 2$ for any MRM3-interpretation I. Cf. Definitions 2.3, 2.4.) Consequently, for any theorem A of RM3, $[A \wedge \neg(p_i \to p_i)] \to p_m$ is not provable in RM3 (by soundness of RM3 w.r.t. MRM3, cf. [2]). Therefore, for any theorem A in S, $[A \wedge \neg(p_i \to p_i)] \to p_m$ is not provable in S. Thus, $p_m \notin$ S, and then S is not trivial. Finally, y is extended to the required regular, non-trivial and prime theory, by using Lemma 3.11. □

Acknowledgments

Work supported by research project FFI2014-53919-P financed by the Spanish Ministry of Economy and Competitiveness. - I sincerely thank an anonymous referee of the RML for his (her) comments and suggestions on a previous version of this paper.

References

[1] A. R. Anderson and N. D. Belnap Jr., Entailment. The Logic of Relevance and Necessity, vol I. Princeton University Press, 1975.

[2] R. T. Brady, Completeness Proofs for the Systems RM3 and BN4, *Logique et Analyse* **25** (1982), 9-32.

[3] R. T. Brady, Depth relevance of some paraconsistent logics, *Studia Logica* **43** (1984), 63–73.

[4] R. T. Brady, Hierarchical semantics for relevant logics, *Journal of Philosophical Logic* **25** (1992), 357–374.

[5] R. T. Brady (ed.), Relevant Logics and Their Rivals, Vol. II, Ashgate, Aldershot, 2003

[6] R. T. Brady, Universal Logic, CSLI, Stanford, CA, 2006.

[7] J. M. Dunn, Algebraic completeness results for R-Mingle and its extensions , *Journal of Symbolic Logic* **35** (1970), 1–13.

[8] J. M. Dunn, Partiality and its Dual , *Studia Logica* bf 65 (2000), 5–40.

[9] G. Robles, A Routley-Meyer semantics for Gödel 3-valued logic and its paraconsistent counterpart, *Logica Universalis* **7** (2013), 507–532.

[10] G. Robles and J. M. Méndez, The basic constructive logic for a weak sense of consistency, *Journal of Logic Language and Information* **17** (2008), 89–107.

[11] G. Robles and J. M. Méndez, A Routley-Meyer semantics for truth-preserving and well-determined Łukasiewicz 3-valued logics, *Logic Journal of the IGPL* **22** (2014), 1–23.

[12] G. Robles, F. Salto, and J. M. Méndez, A weak logic with the axiom mingle lacking the variable-sharing property, *Bulletin of the Section of Logic* **40**:3-4 (2011), 195–202.

[13] R. Routley, R. K. Meyer, "The semantics of Entailment III", *Journal of Philosophical Logic* **1** (1972), 192–208.

[14] R. Routley, R. K. Meyer, V. Plumwood, and R. T. Brady,Relevant Logics and their Rivals, vol. 1, Ridgeview Publishing Co., Atascadero, CA 1982.

Dpto. de Psicología, Sociología y Filosofía
Universidad de León
Campus de Vegazana, s/n
24071, León
Spain

gemma.robles@unileon.es
http://grobv.unileon.es