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ON HOMOMORPHIC IMAGES AND THE FREE DISTRIBUTIVE LATTICE EXTENSION OF A DISTRIBUTIVE NEARLATTICE

A b s t r a c t. In this paper we will introduce N-Vietoris families and prove that homomorphic images of distributive nearlattices are dually characterized by N-Vietoris families. We also show a topological approach of the existence of the free distributive lattice extension of a distributive nearlattice.

1. Introduction and preliminaries

A correspondence between Tarski algebras, called also implication algebras, and join-semilattices with greatest element in which every principal filter is a Boolean lattice was developed by Abbott in [1]. The variety of Tarski algebras is the algebraic semantics of the $\{\to\}$-fragment of classical propositional logic and are a special case of more general algebraic structures

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called *nearlattices*, i.e., join-semilattices with greatest element in which every principal filter is a bounded lattice. In [14] and [11] it is proved that the class of nearlattices forms a variety and in [2] proves that the variety of nearlattices is 2-based. An important class of nearlattices is the class of *distributive nearlattices*. These algebras have been studied in [12] and [14], and recently by several authors in [10], [9], [13], [8] and [5].

In [8], a full duality between distributive nearlattices with greatest element and certain topological spaces with a distinguished basis, called *N-spaces*, was developed. The N-spaces are a generalization of Stone space, also called spectral space [16]. This paper has two objectives. First, motivated by similar results given in [4] and [7], and the duality developed in [8], we will show that the homomorphic images of a distributive nearlattice can be characterized in terms of families of basic saturated irreducible subsets of the N-space $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle$ endowed with a lower Vietoris topology. The second one is to give a topological approach, different from that given in [12], of the existence of the free distributive lattice extension of a distributive nearlattice.

In the remainder of this section we will recall some results and definitions on the representation and topological duality for distributive nearlattices. In Section 2 we will give the mentioned characterization of the homomorphic images of a distributive nearlattice. In Section 3 we shall give the topological proof of the existence of the free distributive lattice extension of a distributive nearlattice.

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element. In this paper and in order to shorten the terminology we will call them semilattices [9]. Recall that the binary relation $\leq$ defined by $x \leq y$ if and only if $x \vee y = y$ is a partial order. A *filter* of $\mathbf{A}$ is a subset $F \subseteq A$ such that $1 \in F$, if $x \leq y$ and $x \in F$ then $y \in F$ and if $x, y \in F$ then $x \wedge y \in F$, whenever $x \wedge y$ exists. The *filter generated by a subset* X of $\mathbf{A}$, in symbols $F(X)$, is the least filter containing X. A filter G is said to be *finitely generated* if $G = F(X)$ for some finite subset X of A. Note that if $X = \{a\}$ then $F(\{a\}) = [a)$, called the principal filter of a. We will denote by $\mathrm{Fi}(\mathbf{A})$ and $\mathrm{Fi}_f(\mathbf{A})$ the set of all filters and finitely generated filters of $\mathbf{A}$, respectively. A subset I of A is called an *ideal* if for every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$ and if $x, y \in I$, then $x \vee y \in I$. The set of all ideals of $\mathbf{A}$ is denoted by $\mathrm{Id}(\mathbf{A})$. A non-empty proper ideal P is *prime* if for all $x, y \in A$, if $x \wedge y \in P$, whenever

$x \wedge y$ exists, then $x \in P$ or $y \in P$. We will denoted by $X(\mathbf{A})$ the set of all prime ideals of $\mathbf{A}$.

Definition 1.1. Let $\mathbf{A}$ be a semilattice. Then $\mathbf{A}$ is a *nearlattice* if for each $a \in A$ the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded lattice.

The Tarski algebras are examples of nearlattices where each principal filter is a Boolean lattice [1]. Nearlattices can be considered as algebras with one ternary operation: if $x, y, z \in A$, the element $m(x, y, z) = (x \vee z) \wedge_z (y \vee z)$ is correctly defined since both $x \vee z, y \vee z \in [z)$ and $[z)$ is a lattice, where $\wedge_z$ denotes the meet in $[z)$. This fact was proved by Hickman in [14] and by Chajda and Kolařík in [11]. In [2] Araújo and Kinyon found a smaller equational base.

Theorem 1.2. *[2] Let $\mathbf{A}$ be a nearlattice. The following identities are satisfied:*

1. $m(x, y, x) = x$,
2. $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,
3. $m(x, x, 1) = 1$.

Conversely, let $\mathbf{A} = \langle A, m, 1\rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1)–(3). If we define $x \vee y = m(x, x, y)$, then $\mathbf{A}$ is a semilattice and for each $z \in A$, $[z)$ is a bounded lattice, where for $x, y \in [z)$ their infimum is $x \wedge_z y = m(x, y, z)$. Hence $\mathbf{A}$ is a nearlattice.

As in lattice theory, the class of distributive nearlattices is very important.

Definition 1.3. Let $\mathbf{A}$ be a nearlattice. Then $\mathbf{A}$ is *distributive* if for each $a \in A$ the principal filter $[a) = \{x \in A : a \leq x\}$ is a bounded distributive lattice.

Theorem 1.4. *[11] Let $\mathbf{A}$ be a nearlattice. Then $\mathbf{A}$ is distributive if and only if it satisfies either of the following identities:*

1. $m(x, m(y, y, z), w) = m(m(x, y, w), m(x, y, w), m(x, z, w))$,
2. $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

We denote by $\mathcal{DN}$ the variety of distributive nearlattices. If $\mathbf{A} \in \mathcal{DN}$, we note that from the results given in [12] we have the following characterization of the filter generated by a subset X of A:

$$F(X) = \{a \in A \,:\, \exists x_1, ..., x_n \in [X) \;\; (x_1 \wedge ... \wedge x_n = a)\}.$$

We note that in the characterization of $F(X)$ we suppose that there exists the meet of the set $\{x_1, ..., x_n\}$. The following result, analogue of the Prime Ideal theorem, was proved in [13].

Theorem 1.5. *Let $\mathbf{A} \in \mathcal{DN}$. Let $I \in \mathrm{Id}(\mathbf{A})$ and let $F \in \mathrm{Fi}(\mathbf{A})$ such that $I \cap F = \emptyset$. Then there exists $P \in X(\mathbf{A})$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

We recall some topological notions. A topological space with a base $\mathcal{K}$ will be denoted by $\langle X, \mathcal{K}\rangle$. We consider the set $D_{\mathcal{K}}(X) = \{U \,:\, U^c \in \mathcal{K}\}$. A subset $Y \subseteq X$ is *basic saturated* if it is an intersection of basic open sets, i.e., $Y = \bigcap\{U_i \in \mathcal{K} \,:\, Y \subseteq U_i\}$. The basic saturation $\mathrm{Sb}(Y)$ of a subset Y is the smallest basic saturated set containing Y. If $Y = \{y\}$, we write $\mathrm{Sb}(\{y\}) = \mathrm{Sb}(y)$. We denote by $\mathcal{S}(X)$ the family of all basic saturated subsets of $\langle X, \mathcal{K}\rangle$. On X is defined a binary relation $\leq$ as $x \leq y$ if and only if $y \in \mathrm{Sb}(x)$. The relation $\leq$ is reflexive and transitive, but not necessarily antisymmetric. It is easy to see that the relation $\leq$ is a partial order if and only if $\langle X, \mathcal{K}\rangle$ is T_0. We note that $\mathrm{Sb}(x) = [x)$. Let Y be a non-empty subset of X. We say that Y is *irreducible* if for every $U, V \in D_{\mathcal{K}}(X)$ such that $U \cap V \in D_{\mathcal{K}}(X)$ and $Y \cap (U \cap V) = \emptyset$ implies $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$. We say that Y is *dually compact* if for every family $\mathcal{F} = \{U_i \,:\, i \in I\} \subseteq \mathcal{K}$ such that $\bigcap\{U_i \,:\, i \in I\} \subseteq Y$ implies that there exists a finite family $\{U_1, ..., U_n\} \subseteq \mathcal{F}$ such that $U_1 \cap ... \cap U_n \subseteq Y$. We denote by $\mathcal{S}_{\mathrm{Irr}}(X)$ the family of all basic saturated irreducible subsets of $\langle X, \mathcal{K}\rangle$. The following definition is introduced in [8].

Definition 1.6. Let $\langle X, \mathcal{K}\rangle$ be a topological space. Then $\langle X, \mathcal{K}\rangle$ is an *N-space* if:

1. $\mathcal{K}$ is a basis of open, compact and dually compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X.
2. For every $U, V, W \in \mathcal{K}$, $(U \cap W) \cup (V \cap W) \in \mathcal{K}$.
3. For every irreducible basic saturated subset Y of X there exists a unique $x \in X$ such that $Y = \mathrm{Sb}(x)$.

If $\langle X, \mathcal{K}\rangle$ is an N-space, then the relation $\leq$ is a partial order and $\langle X, \mathcal{K}\rangle$ is T_0.

Proposition 1.7. *[8] Let $\langle X, \mathcal{K}\rangle$ be a topological space where $\mathcal{K}$ is a basis of open and compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X. Suppose that $(U \cap W) \cup (V \cap W) \in \mathcal{K}$ for every $U, V, W \in \mathcal{K}$. The following conditions are equivalent:*

1. *$\langle X, \mathcal{K}\rangle$ is T_0, and if $A = \{U_i : i \in I\}$ and $B = \{V_j : j \in J\}$ are non-empty families of $D_{\mathcal{K}}(X)$ such that*
$$\bigcap \{U_i : i \in I\} \subseteq \bigcup \{V_j : j \in J\},$$
then there exist $U_1, ..., U_n \in [A)$ and $V_1, ..., V_k \in B$ such that $U_1 \cap ... \cap U_n \in D_{\mathcal{K}}(X)$ and $U_1 \cap ... \cap U_n \subseteq V_1 \cup ... \cup V_k$.
2. *$\langle X, \mathcal{K}\rangle$ is T_0, every $U \in \mathcal{K}$ is dually compact and the assignment $H : X \to X(D_{\mathcal{K}}(X))$ defined by*
$$H(x) = \{U \in D_{\mathcal{K}}(X) : x \notin U\},$$
for each $x \in X$, is onto.
3. *Every $U \in \mathcal{K}$ is dually compact and for every irreducible basic saturated subset Y of X, there exists a unique $x \in X$ such that $Y = \mathrm{Sb}(x)$.*

If $\langle X, \mathcal{K}\rangle$ is an N-space, then $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ is a distributive nearlattice. We note that if $\langle X, \mathcal{K}\rangle$ is an N-space then $X \in \mathcal{K}$ if and only if $D_{\mathcal{K}}(X)$ is a bounded distributive lattice. So, $\mathcal{K}$ is the set of all compact and open subsets of X and we obtain the topological representation for bounded distributive lattices given by Stone in [16]. If $\langle X, \mathcal{K}\rangle$ is an N-space, then the map $H : X \to X(D_{\mathcal{K}}(X))$ defined in the Proposition 1.7 is a homeomorphism such that $x \leq y$ if and only if $H(x) \subseteq H(y)$.

Let $\mathbf{A} \in \mathcal{DN}$. Let us consider the poset $\langle X(\mathbf{A}), \subseteq\rangle$ and the mapping $\varphi_{\mathbf{A}} : A \to \mathcal{P}_d(X(\mathbf{A}))$ defined by $\varphi_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) : a \notin P\}$. Let $\varphi_{\mathbf{A}}[\mathbf{A}] = \{\varphi_{\mathbf{A}}(a) : a \in A\}$. Then $\mathbf{A}$ is isomorphic to the subalgebra $\varphi_{\mathbf{A}}[\mathbf{A}]$ of $\mathcal{P}_d(X(\mathbf{A}))$ and the pair $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle$ is an N-space, called the *dual space* of $\mathbf{A}$, where the topology $\mathcal{T}_{\mathbf{A}}$ is generated by taking as base of opens the family $\mathcal{K}_{\mathbf{A}} = \{\varphi_{\mathbf{A}}(a)^c : a \in A\}$. Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$. A mapping $h : A \to B$ is a *semi-homomorphism* if $h(1) = 1$ and $h(a \vee b) = h(a) \vee h(b)$

for all $a, b \in A$. A mapping $h : A \to B$ is a *homomorphism* if it is a semi-homomorphism such that if $a \wedge b$ exists then $h(a \wedge b) = h(a) \wedge h(b)$. Note that if $a \wedge b$ exists, then $h(a) \wedge h(b)$ exists. If $h : A \to B$ is a onto homomorphism, then we shall say that $\mathbf{B}$ is a *homomorphic image* of $\mathbf{A}$.

There exists a duality between homomorphisms of distributive nearlattices and certain binary relations. Let X_1 and X_2 be two sets, $\mathcal{P}(X_1)$ and $\mathcal{P}(X_2)$ the set of all subsets of X_1 and X_2, respectively, and $R \subseteq X_1 \times X_2$ be a binary relation. For each $x \in X_1$, let $R(x) = \{y \in X_2 \ : \ (x, y) \in R\}$. We define the mapping $h_R : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ by

$$h_R(U) = \{x \in X_1 \ : \ R(x) \cap U \neq \emptyset\}.$$

It is easy to verify that h_R is a homomorphism between $\mathcal{P}(X_2)$ and $\mathcal{P}(X_1)$.

Definition 1.8. Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation. Then R is an *N-relation* if:

1. $h_R(U) \in D_{\mathcal{K}_1}(X_1)$ for every $U \in D_{\mathcal{K}_2}(X_2)$.
2. $R(x)$ is a basic saturated subset of X_2 for each $x \in X_1$.
3. $R(x) \neq \emptyset$ for each $x \in X$, i.e., R is serial.

We say that R is an *N-functional relation* if R is an N-relation satisfying that for each $x \in X_1$, there exists $y \in X_2$ such that $R(x) = \mathrm{Sb}(y)$.

Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$ and $h : A \to B$ be a mapping. In [8] it was proved that h is a homomorphism if and only if the relation $R_h \subseteq X(\mathbf{B}) \times X(\mathbf{A})$ defined by $(P, Q) \in R_h$ if and only if $h^{-1}(P) \subseteq Q$ is an N-functional relation. We are interested here a particular class of N-relations.

Definition 1.9. Let $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be an N-relation. Then R is *1-1* if for each $x \in X_1$ and $U \in D_{\mathcal{K}_1}(X_1)$ with $x \notin U$, there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$.

If $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$ and $h : A \to B$ a homomorphism, then h is onto if and only if R_h is 1-1. Also, if $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ be two N-spaces and $R \subseteq X_1 \times X_2$ be an N-functional relation, then R is 1-1 if and only if h_R is onto (see [8]).

2. Homomorphic images

Let $\langle X, \mathcal{K}\rangle$ be a topological space and $\mathcal{C}(X)$ the family of all non-empty closed subsets of $\langle X, \mathcal{K}\rangle$. Let $\mathcal{F}$ be a non-empty family of non-empty irreducible basic saturated subsets of $\langle X, \mathcal{K}\rangle$. For each $U \in \mathcal{C}(X)$ we consider the set

$$M_U = \{Y \in \mathcal{F} \ : \ Y \cap U = \emptyset\}.$$

The *lower Vietoris topology* $\mathcal{T}_L$ defined on $\mathcal{F}$ is the topology generated by the collection of sets

$$\mathcal{B}_L = \{M_U \ : \ U \in \mathcal{C}(X)\}$$

as subbasis for $\mathcal{T}_L$ [15].

Let $\mathbf{A} \in \mathcal{DN}$ and $\langle X, \mathcal{K}\rangle$ be an N-space. Let $R \subseteq X \times X(\mathbf{A})$ be an 1-1 N-functional relation and consider

$$\mathcal{F}_R = \{R(x) \ : \ x \in X\}.$$

Since R is an N-functional relation, there exists $P \in X(\mathbf{A})$ such that $R(x) = \mathrm{Sb}(P)$ for each $x \in X$. It is easy to see that $\mathrm{Sb}(P)$ is irreducible and therefore $\mathcal{F}_R \subseteq \mathcal{S}_{\mathrm{Irr}}(X(\mathbf{A}))$. For $a \in A$, we consider the set

$$M_a = \{R(x) \in \mathcal{F}_R \ : \ R(x) \cap \varphi_{\mathbf{A}}(a) = \emptyset\}.$$

Lemma 2.1. *Let $\mathbf{A} \in \mathcal{DN}$ and $\langle X, \mathcal{K}\rangle$ be an N-space. Let $R \subseteq X \times X(\mathbf{A})$ be an 1-1 N-functional relation. Then the family*

$$\mathcal{B}_{\mathbf{A}} = \{M_a \ : \ a \in A\}$$

is a basis for the topology $\mathcal{T}_L$ on $\mathcal{F}_R$.

Proof. First, we prove that $\mathcal{F}_R = \bigcup\{M_a \ : \ a \in A\}$. Let $x \in X$ and $R(x) \in \mathcal{F}_R$. Since $\mathcal{K}$ is a basis of $\langle X, \mathcal{K}\rangle$, there exists $U \in D_{\mathcal{K}}(X)$ such that $x \notin U$. Then, as R is 1-1, there exists $V \in D_{\mathcal{K}_{\mathbf{A}}}(X(\mathbf{A}))$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$. So, as $\mathbf{A}$ is isomorphic to $D_{\mathcal{K}_{\mathbf{A}}}(X(\mathbf{A}))$, there exists $a \in A$ such that $V = \varphi_{\mathbf{A}}(a)$. Then $x \notin h_R(\varphi_{\mathbf{A}}(a))$, i.e., $R(x) \cap \varphi_{\mathbf{A}}(a) = \emptyset$ and $R(x) \in M_a$. Therefore $\mathcal{F}_R = \bigcup\{M_a \ : \ a \in A\}$.

Let $a, b \in A$ such that $M_a \cap M_b \neq \emptyset$. We prove that $M_a \cap M_b = M_{a\vee b}$. If $R(x) \in M_{a\vee b}$, then $R(x) \cap \varphi_{\mathbf{A}}(a \vee b) = R(x) \cap [\varphi_{\mathbf{A}}(a) \cup \varphi_{\mathbf{A}}(b)] = \emptyset$. It follows that $R(x) \cap \varphi_{\mathbf{A}}(a) = \emptyset$ and $R(x) \cap \varphi_{\mathbf{A}}(b) = \emptyset$, i.e., $R(x) \in M_a \cap M_b$. The other inclusion is similar. So, $\mathcal{B}_{\mathbf{A}}$ is a basis for the topology $\mathcal{T}_L$ on $\mathcal{F}_R$. □

Remark 2.2. Let $H_a = \{R(x) \in \mathcal{F}_R : R(x) \cap \varphi_{\mathbf{A}}(a) \neq \emptyset\}$. Then $H_a = \mathcal{F}_R - M_a = M_a^c$ and by Lemma 2.1, $H_a \cup H_b = H_{a\vee b}$. Also, since $R(x)$ is serial, $H_1 = \mathcal{F}_R$. Therefore

$$\langle D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}_R), \cup, \mathcal{F}_R\rangle$$

is a semilattice.

Let $\mathbf{A} \in \mathcal{DN}$ and $I \in \mathrm{Id}(\mathbf{A})$. In [8] it was defined the set

$$\alpha(I) = \{P \in X(\mathbf{A}) : I \nsubseteq P\}.$$

It is easy to prove that $\alpha(I) = \bigcup\{\varphi_{\mathbf{A}}(a) : a \in I\}$. We have the following result.

Lemma 2.3. *Let $\mathbf{A} \in \mathcal{DN}$ and $\langle X, \mathcal{K}\rangle$ be an N-space. Let $R \subseteq X \times X(\mathbf{A})$ be an 1-1 N-functional relation.*

1. *$\langle \mathcal{F}_R, \mathcal{B}_{\mathbf{A}}\rangle$ is T_0.*
2. *For every $a, b, c \in A$, $(M_a \cap M_c) \cup (M_b \cap M_c) \in \mathcal{B}_{\mathbf{A}}$.*
3. *Let $\{H_b : b \in B\}$ and $\{H_c : c \in C\}$ non-empty families of $D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}_R)$. Then*
$$\bigcap\{H_b : b \in B\} \subseteq \bigcup\{H_c : c \in C\}$$
if and only if
$$\bigcap\{h_R(\varphi_{\mathbf{A}}(b)) : b \in B\} \subseteq \bigcup\{h_R(\varphi_{\mathbf{A}}(c)) : c \in C\}.$$
4. *A subset $Y \subseteq \mathcal{F}_R$ is basic saturated of $\langle \mathcal{F}_R, \mathcal{B}_{\mathbf{A}}\rangle$ if and only if there exists $J \in \mathrm{Id}(\mathbf{A})$ such that $Y = \{R(x) : R(x) \subseteq \alpha(J)^c\}$.*

Proof. (1) Let $x, y \in X$ such that $R(x) \neq R(y)$. Suppose that there exists $P \in X(\mathbf{A})$ such that $P \in R(x)$ and $P \notin R(y)$. Since R is an N-relation, $R(y)$ is a basic saturated subset of $X(\mathbf{A})$ and there exists a subset $B \subseteq A$ such that

$$R(y) = \bigcap\{\varphi_{\mathbf{A}}(b)^c : b \in B\}.$$

As $P \notin R(y)$, there exists $b_0 \in B$ such that $P \notin \varphi_{\mathbf{A}}(b_0)^c$, i.e., $P \in \varphi_{\mathbf{A}}(b_0)$. Then $P \in R(x) \cap \varphi_{\mathbf{A}}(b_0)$ and $R(x) \notin M_{b_0}$. On the other hand, if $Q \in R(y)$, then $Q \in \bigcap \{\varphi_{\mathbf{A}}(b)^c \ : \ b \in B\}$ and in particular, $Q \in \varphi_{\mathbf{A}}(b_0)^c$, and this is true for all $Q \in R(y)$. So, $R(y) \cap \varphi_{\mathbf{A}}(b_0) = \emptyset$ and $R(y) \in M_{b_0}$. Therefore, $\langle \mathcal{F}_R, \mathcal{B}_{\mathbf{A}} \rangle$ is T_0.

(2) Let $a, b, c \in A$. So, $M_a, M_b, M_c \in \mathcal{B}_{\mathbf{A}}$. By Lemma 2.1, $M_a \cap M_b = M_{a \vee b}$ and $(M_a \cap M_c) \cup (M_b \cap M_c) = M_{a \vee c} \cup M_{b \vee c}$. Note that $(a \vee c) \wedge_c (b \vee c)$ exists in $[c)$. So, $\varphi_{\mathbf{A}}((a \vee c) \wedge_c (b \vee c)) = \varphi_{\mathbf{A}}(a \vee c) \cap \varphi_{\mathbf{A}}(b \vee c)$. We prove that $M_{a \vee c} \cup M_{b \vee c} = M_{(a \vee c) \wedge_c (b \vee c)}$. If $R(x) \in M_{(a \vee c) \wedge_c (b \vee c)}$, then

$$R(x) \cap \varphi_{\mathbf{A}}((a \vee c) \wedge_c (b \vee c)) = R(x) \cap [\varphi_{\mathbf{A}}(a \vee c) \cap \varphi_{\mathbf{A}}(b \vee c)] = \emptyset.$$

Since $R(x)$ is irreducible, $R(x) \cap \varphi_{\mathbf{A}}(a \vee c) = \emptyset$ or $R(x) \cap \varphi_{\mathbf{A}}(b \vee c) = \emptyset$. So, $R(x) \in M_{a \vee c} \cup M_{b \vee c}$. The converse is similar, and $(M_a \cap M_c) \cup (M_b \cap M_c) \in \mathcal{B}_{\mathbf{A}}$.

(3) Let $\{H_b \ : \ b \in B\}$ and $\{H_c \ : \ c \in C\}$ non-empty families of $D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}_R)$ such that

$$\bigcap \{H_b \ : \ b \in B\} \subseteq \bigcup \{H_c \ : \ c \in C\}.$$

Let $x \in \bigcap \{h_R(\varphi_{\mathbf{A}}(b)) \ : \ b \in B\}$. Then $x \in h_R(\varphi_{\mathbf{A}}(b))$, i.e., $R(x) \cap \varphi_{\mathbf{A}}(b) \neq \emptyset$ for every $b \in B$. So, $R(x) \in \bigcap \{H_b \ : \ b \in B\}$ and by hypothesis, $R(x) \in \bigcup \{H_c \ : \ c \in C\}$. Then there exists $c_0 \in C$ such that $R(x) \in H_{c_0}$. Therefore, $x \in h_R(\varphi_{\mathbf{A}}(c_0))$ and $x \in \bigcup \{h_R(\varphi_{\mathbf{A}}(c)) \ : \ c \in C\}$. The converse is analogous.

(4) Let $Y \subseteq \mathcal{F}_R$ be a basic saturated subset of $\langle \mathcal{F}_R, \mathcal{B}_A \rangle$. Then there exists a subset $B \subseteq A$ such that $Y = \bigcap \{M_b \ : \ b \in B\}$. Let us consider the ideal $J = I(B)$. It is easy to see that $Y = \bigcap \{M_b \ : \ b \in J\}$. If $R(x) \in Y$, then $R(x) \in M_b$, i.e., $R(x) \cap \varphi_{\mathbf{A}}(b) = \emptyset$ and $R(x) \subseteq \varphi_{\mathbf{A}}(b)^c$ for every $b \in J$. It follows that $R(x) \subseteq \bigcap \{\varphi_{\mathbf{A}}(b)^c \ : \ b \in J\} = \alpha(J)^c$. For the other inclusion is similar. Thus, $Y = \{R(x) \ : \ R(x) \subseteq \alpha(J)^c\}$.

Reciprocally, suppose that $Y = \{R(x) \ : \ R(x) \subseteq \alpha(J)^c\}$ for some $J \in \mathrm{Id}(\mathbf{A})$. Then

$$\begin{array}{lll} R(x) \in Y & \text{iff } R(x) \subseteq \alpha(J)^c & \text{iff } R(x) \subseteq \bigcap \{\varphi_{\mathbf{A}}(b)^c \ : \ b \in J\} \\ & \text{iff } \forall b \in J \ (R(x) \subseteq \varphi_{\mathbf{A}}(b)^c) & \text{iff } \forall b \in J \ (R(x) \cap \varphi_{\mathbf{A}}(b) = \emptyset) \\ & \text{iff } \forall b \in J \ (R(x) \in M_b) & \text{iff } R(x) \in \bigcap \{M_b \ : \ b \in J\}. \end{array}$$

Therefore $Y = \bigcap \{M_b \ : \ b \in J\}$ and Y is a basic saturated subset of $\langle \mathcal{F}_R, \mathcal{B}_{\mathbf{A}} \rangle$. □

Remark 2.4. Note that by item (2) of Lemma 2.3 it is easy to check that the structure $\langle D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}_R), \cup, \mathcal{F}_R \rangle$ is a distributive nearlattice.

Theorem 2.5. *Let* $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$. *Let* $h : A \to B$ *be an onto homomorphism. Then* $\langle \mathcal{F}_{R_h}, \mathcal{B}_{\mathbf{A}} \rangle$ *is an* N*-space which is homeomorphic to* $\langle X(\mathbf{B}), \mathcal{K}_{\mathbf{B}} \rangle$.

Proof. By Lemmas 2.1 and 2.3, $\mathcal{B}_{\mathbf{A}}$ is a basis of open and compact subsets for a topology $\mathcal{T}_L$ on $\mathcal{F}_R$ such that $(M_a \cap M_c) \cup (M_b \cap M_c) \in \mathcal{B}_{\mathbf{A}}$ for every $a, b, c \in A$. Also, by Lemma 2.3, $\langle \mathcal{F}_{R_h}, \mathcal{B}_{\mathbf{A}} \rangle$ is T_0 and if $\{H_b : b \in B\}$ and $\{H_c : c \in C\}$ are non-empty families of $D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}_R)$ such that $\bigcap \{H_b : b \in B\} \subseteq \bigcup \{H_c : c \in C\}$, then there exist $b_1, ..., b_n \in [B)$ and $c_1, ..., c_k \in C$ such that $H_{b_1} \cap ... \cap H_{b_n} \in D_{\mathcal{K}}(X)$ and $H_{b_1} \cap ... \cap H_{b_n} \subseteq H_{c_1} \cup ... \cup H_{c_k}$. So, by Proposition 1.7, $\langle \mathcal{F}_{R_h}, \mathcal{B}_{\mathbf{A}} \rangle$ is an N-space.

Now, we prove that $\langle \mathcal{F}_{R_h}, \mathcal{B}_{\mathbf{A}} \rangle$ is homeomorphic to $\langle X(\mathbf{B}), \mathcal{K}_{\mathbf{B}} \rangle$. We define the mapping $f : X(\mathbf{B}) \to \mathcal{F}_{R_h}$ by

$$f(P) = R_h(P).$$

Let $P, Q \in X(\mathbf{B})$ such that $R_h(P) = R_h(Q)$. Suppose that $P \nsubseteq Q$, i.e., $Q \notin \mathrm{Sb}(P)$. Then there exists $b \in B$ such that $P \in \varphi_{\mathbf{B}}(b)^c$ and $Q \notin \varphi_{\mathbf{B}}(b)^c$, i.e., $P \notin \varphi_{\mathbf{B}}(b)$ and $Q \in \varphi_{\mathbf{B}}(b)$. Since h is onto, R_h is 1-1. As $P \notin \varphi_{\mathbf{B}}(b)$, there exists $a \in A$ such that $\varphi_{\mathbf{B}}(b) \subseteq h_{R_h}(\varphi_{\mathbf{A}}(a))$ and $P \notin h_{R_h}(\varphi_{\mathbf{A}}(a))$. So, $R_h(P) \cap \varphi_{\mathbf{A}}(a) = \emptyset$. On the other hand, $Q \in \varphi_{\mathbf{B}}(b) \subseteq h_{R_h}(\varphi_{\mathbf{A}}(a))$ and $R_h(Q) \cap \varphi_{\mathbf{A}}(a) \neq \emptyset$. Since $R_h(P) = R_h(Q)$ we have that $R_h(P) \cap \varphi_{\mathbf{A}}(a) \neq \emptyset$, which is a contradiction. Then $P = Q$ and f is 1-1. It is clear that f is onto. Thus, f is a bijection.

Let $a \in A$ and $P \in X(\mathbf{B})$. Then

$$\begin{aligned} P \in f^{-1}(M_a) \quad &\text{iff} \quad f(P) \in M_a && \text{iff} \quad R_h(P) \in M_a \\ &\text{iff} \quad R_h(P) \cap \varphi_{\mathbf{A}}(a) = \emptyset && \text{iff} \quad P \notin h_{R_h}(\varphi_{\mathbf{A}}(a)) \\ &\text{iff} \quad P \notin \varphi_{\mathbf{B}}(h(a)) && \text{iff} \quad P \in \varphi_{\mathbf{B}}(h(a))^c. \end{aligned}$$

So, $f^{-1}(M_a) = \varphi_{\mathbf{B}}(h(a))^c$ and f is continuous.

We prove that f is an open map. Let $b \in B$. Since h is onto, there exists $a \in A$ such that $h(a) = b$. So,

$$\begin{aligned} R_h(P) \in f(\varphi_{\mathbf{B}}(b)^c) \quad &\text{iff} \quad P \in \varphi_{\mathbf{B}}(b)^c && \text{iff} \quad P \in \varphi_{\mathbf{B}}(h(a))^c \\ &\text{iff} \quad P \notin \varphi_{\mathbf{B}}(h(a)) && \text{iff} \quad P \notin h_{R_h}(\varphi_{\mathbf{A}}(a)) \\ &\text{iff} \quad R_h(P) \cap \varphi_{\mathbf{A}}(a) = \emptyset && \text{iff} \quad R_h(P) \in M_a. \end{aligned}$$

Then $f(\varphi_{\mathbf{B}}(b)^c) = M_a$ and f is open. Therefore, f is a homeomorphism. □

Definition 2.6. Let $\langle X, \mathcal{K} \rangle$ be an N-space. We say that a non-empty family $\mathcal{F}$ of non-empty basic saturated irreducible subsets of $\langle X, \mathcal{K} \rangle$ is an *N-Vietoris family* if $\langle \mathcal{F}, \mathcal{B}_L \rangle$ is an N-space.

Let $\mathbf{A} \in \mathcal{DN}$ and $\mathcal{F} \subseteq \mathcal{S}_{\mathrm{Irr}}(X(\mathbf{A}))$ be an N-Vietoris family. Then $\langle \mathcal{F}, \mathcal{B}_{\mathbf{A}} \rangle$ is an N-space and the structure $\langle D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}), \cup, \mathcal{F} \rangle$ is a distributive nearlattice. We define a binary relation $R_{\mathcal{F}} \subseteq \mathcal{F} \times X(\mathbf{A})$ by

$$(Y, P) \in R_{\mathcal{F}} \text{ iff } P \in Y.$$

Lemma 2.7. *Let $\mathbf{A} \in \mathcal{DN}$. Let $\mathcal{F}$ be an N-Vietoris family of $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$. Then $R_{\mathcal{F}}$ is an 1-1 N-functional relation.*

Proof. First we show that $R_{\mathcal{F}}$ is an N-functional relation. Let $a \in A$. Then

$$h_{R_{\mathcal{F}}}(\varphi_{\mathbf{A}}(a)) = \{R_{\mathcal{F}}(Y) \in \mathcal{F} \ : \ R_{\mathcal{F}}(Y) \cap \varphi_{\mathbf{A}}(a) \neq \emptyset\} = H_a \in D_{\mathcal{B}_{\mathbf{A}}}(\mathcal{F}).$$

Let $Y \in \mathcal{F}$. By definition, $R_{\mathcal{F}}(Y) = Y$. Since $R_{\mathcal{F}}(Y)$ is a basic saturated irreducible subset of $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$, there exists $P \in X(\mathbf{A})$ such that $R_{\mathcal{F}}(Y) = \mathrm{Sb}(P)$. On the other hand, as $\mathcal{F}$ is a family of non-empty subsets, $R_{\mathcal{F}}(Y) \neq \emptyset$ and $R_{\mathcal{F}}(Y)$ is serial. So, $R_{\mathcal{F}}$ is an N-functional relation. Finally, we show that $R_{\mathcal{F}}$ is 1-1. Let $a \in A$ and $Y \in \mathcal{F}$ such that $Y \notin H_a$. Then $Y \cap \varphi_{\mathbf{A}}(a) = \emptyset$. As $R_{\mathcal{F}}(Y) = Y$, we get $Y \notin h_{R_{\mathcal{F}}}(\varphi_{\mathbf{A}}(a))$. It follows that $H_a \subseteq h_{R_{\mathcal{F}}}(\varphi_{\mathbf{A}}(a))$, and therefore $R_{\mathcal{F}}$ is an 1-1 N-functional relation. $\square$

Lemma 2.8. *Let $\mathbf{A} \in \mathcal{DN}$. Let $\langle X, \mathcal{K} \rangle$ be an N-space.*

1. *If $R \subseteq X \times X(\mathbf{A})$ is an 1-1 N-functional relation, then for each $x \in X$ and $P \in X(\mathbf{A})$ we have*

$$(x, P) \in R \text{ iff } (R(x), P) \in R_{\mathcal{F}_R}.$$

2. *If $\mathcal{F} \subseteq \mathcal{S}_{\mathrm{Irr}}(X(\mathbf{A}))$ is an N-Vietoris family, then $\mathcal{F} = \mathcal{F}_{R_{\mathcal{F}}}$.*

Proof. (1) Let $x \in X$ and $P \in X(\mathbf{A})$. Then

$$(R(x), P) \in R_{\mathcal{F}_R} \quad \text{iff} \quad P \in R(x) \quad \text{iff} \quad (x, P) \in R.$$

(2) Let $Y \in \mathcal{F}_{R_{\mathcal{F}}}$. Then there exists $G \in \mathcal{F}$ such that $Y = R_{\mathcal{F}}(G)$, but as $R_{\mathcal{F}}(G) = G$, we have that $Y \in \mathcal{F}$ and $\mathcal{F}_{R_{\mathcal{F}}} = \mathcal{F}$. □

Since homomorphic images of a distributive nearlattice $\mathbf{A}$ are dually characterized by 1-1 N-functional relations of $X(\mathbf{A})$, by Theorem 2.5 and Lemmas 2.7 and 2.8 we obtain the following result.

Theorem 2.9. *Let $\mathbf{A} \in \mathcal{DN}$ and $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$ be the dual space of $\mathbf{A}$. Then the homomorphic images of $\mathbf{A}$ are dually characterized by N-Vietoris families of $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$.*

3. The free distributive lattice extension

In [12] the authors proved that every distributive nearlattice has a free distributive lattice extension. In this section, following the duality developed in [8], we show a topological approach of the existence of the free distributive lattice extension. Also, we study the relation between the filters of a distributive nearlattice and the filters of its free distributive lattice extension.

Definition 3.1. Let $\mathbf{A} \in \mathcal{DN}$. A pair $\mathbf{L} = \langle L, e \rangle$, where L is a bounded distributive lattice and $e : A \to L$ a 1-1 homomorphism, is a *free distributive lattice extension of* A if the following universal property holds: for every bounded distributive lattice $\overline{\mathbf{L}}$ and every homomorphism $h : A \to \overline{L}$, there exists a unique homomorphism $\overline{h} : L \to \overline{L}$ such that $h = \overline{h} \circ e$.

Let $\mathbf{A} \in \mathcal{DN}$ and $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$ be the dual space of $\mathbf{A}$. We will denote by $\mathcal{KO}(X(\mathbf{A}))$ the family of all open and compact subsets of $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$. It follows that if $U \in \mathcal{KO}(X(\mathbf{A}))$, then there exist $a_1, ..., a_n \in A$ such that $U = \varphi_{\mathbf{A}}(a_1)^c \cup ... \cup \varphi_{\mathbf{A}}(a_n)^c$. Moreover, the structure $\mathcal{KO}(X(\mathbf{A}))$ is a distributive lattice. We consider the family

$$D_{\mathcal{KO}}[X(\mathbf{A})] = \{U \ : \ U^c \in \mathcal{KO}(X(\mathbf{A}))\}.$$

So, $\langle D_{\mathcal{KO}}[X(\mathbf{A})], \cup, \cap, \emptyset, X(\mathbf{A}) \rangle$ is a bounded distributive lattice. We take the 1-1 homomorphism $\varphi_{\mathbf{A}} : \mathbf{A} \to D_{\mathcal{KO}}[X(\mathbf{A})]$ defined by $\varphi_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) \ : \ a \notin P\}$ and prove that the pair $\langle D_{\mathcal{KO}}[X(\mathbf{A})], \varphi_{\mathbf{A}} \rangle$ is the free distributive lattice extension of $\mathbf{A}$.

Theorem 3.2. *Let* $\mathbf{A} \in \mathcal{DN}$ *and* $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}} \rangle$ *be the dual space of* $\mathbf{A}$. *Let* $\mathbf{L}$ *be a bounded distributive lattice and* $h : A \to L$ *be a homomorphism. Then there exists a unique homomorphism* $\overline{h} : D_{\mathcal{KO}}[X(\mathbf{A})] \to L$ *such that* $h = \overline{h} \circ \varphi_{\mathbf{A}}$. *Moreover,* h *is 1-1 if and only if* $\overline{h}$ *is 1-1 and if* h *is onto, then* $\overline{h}$ *is onto.*

Proof. Let $\mathbf{L}$ be a bounded distributive lattice and $h : A \to L$ a homomorphism. We define $\overline{h} : D_{\mathcal{KO}}[X(\mathbf{A})] \to L$ by

$$\overline{h}[\varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)] = h(a_1) \wedge \ldots \wedge h(a_n).$$

Let $U, V \in D_{\mathcal{KO}}[X(\mathbf{A})]$. Then there exist $a_1, ..., a_n, b_1, ..., b_m \in A$ such that $U = \varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)$ and $V = \varphi_{\mathbf{A}}(b_1) \cap ... \cap \varphi_{\mathbf{A}}(b_m)$. We show that $\overline{h}$ is well defined. If $U = V$, then $\varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n) = \varphi_{\mathbf{A}}(b_1) \cap ... \cap \varphi_{\mathbf{A}}(b_m)$. We prove that $\overline{h}[U] = \overline{h}[V]$, i.e., $h(a_1) \wedge \ldots \wedge h(a_n) = h(b_1) \wedge \ldots \wedge h(b_m)$. Suppose that $h(a_1) \wedge \ldots \wedge h(a_n) \neq h(b_1) \wedge \ldots \wedge h(b_m)$. Then $h(a_1) \wedge \ldots \wedge h(a_n) \nleq h(b_1) \wedge \ldots \wedge h(b_m)$ or $h(b_1) \wedge \ldots \wedge h(b_m) \nleq h(a_1) \wedge \ldots \wedge h(a_n)$. If $h(a_1) \wedge \ldots \wedge h(a_n) \nleq h(b_1) \wedge \ldots \wedge h(b_m)$, then there exists $P \in X(\mathbf{L})$ such that $h(b_1) \wedge \ldots \wedge h(b_m) \in P$ and $h(a_1) \wedge \ldots \wedge h(a_n) \notin P$. Since P is prime, there exists $j \in \{1, ..., m\}$ such that $h(b_j) \in P$. On the other hand, $h(a_i) \notin P$ for all $i \in \{1, ..., n\}$. Thus, $b_j \in h^{-1}(P)$ and $a_i \notin h^{-1}(P)$ for all $i \in \{1, ..., n\}$. As h is a homomorphism, $h^{-1}(P) \in X(\mathbf{A})$. It follows that $h^{-1}(P) \notin \varphi_{\mathbf{A}}(b_1) \cap ... \cap \varphi_{\mathbf{A}}(b_m)$ and $h^{-1}(P) \in \varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)$, which is a contradiction.

We see that $\overline{h}$ is a homomorphism. By definition, it is easy to see $\overline{h}[U \cap V] = \overline{h}[U] \wedge \overline{h}[V]$. Also, we have

$$\begin{aligned} \overline{h}[U \cup V] &= \overline{h}[(\varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)) \cup (\varphi_{\mathbf{A}}(b_1) \cap ... \cap \varphi_{\mathbf{A}}(b_m))] \\ &= \overline{h}[\varphi_{\mathbf{A}}(a_1 \vee b_1) \cap ... \cap \varphi_{\mathbf{A}}(a_1 \vee b_m) \cap ... \cap \varphi_{\mathbf{A}}(a_n \vee b_m)] \\ &= h(a_1 \vee b_1) \wedge ... \wedge h(a_1 \vee b_m) \wedge ... \wedge h(a_n \vee b_m) \\ &= (h(a_1) \wedge ... \wedge h(a_n)) \vee (h(b_1) \wedge ... \wedge h(b_m)) \\ &= \overline{h}[U] \vee \overline{h}[V]. \end{aligned}$$

So, $\overline{h}$ is a homomorphism.

To see that $\overline{h}$ is unique, suppose there exists a homomorphism $\widetilde{h}$: $D_{\mathcal{KO}}[X(\mathbf{A})] \to L$ such that $h = \widetilde{h} \circ \varphi_{\mathbf{A}}$. Then $\overline{h}[\varphi_{\mathbf{A}}(a)] = \widetilde{h}[\varphi_{\mathbf{A}}(a)]$ for all $a \in A$. If $W \in D_{\mathcal{KO}}[X(\mathbf{A})]$, then there exists $c_1, ..., c_k \in A$ such that

$W = \varphi_{\mathbf{A}}(c_1) \cap ... \cap \varphi_{\mathbf{A}}(c_k)$. So,

$$\begin{aligned}
\overline{h}[W] &= \overline{h}[\varphi_{\mathbf{A}}(c_1) \cap ... \cap \varphi_{\mathbf{A}}(c_k)] \\
&= \overline{h}[\varphi_{\mathbf{A}}(c_1)] \wedge \ldots \wedge \overline{h}[\varphi_{\mathbf{A}}(c_k)] \\
&= \widetilde{h}[\varphi_{\mathbf{A}}(c_1)] \wedge ... \wedge \widetilde{h}[\varphi_{\mathbf{A}}(c_k)] \\
&= \widetilde{h}[\varphi_{\mathbf{A}}(c_1) \cap ... \cap \varphi_{\mathbf{A}}(c_k)] \\
&= \widetilde{h}[W].
\end{aligned}$$

Therefore, $\overline{h}$ is unique.

Now, we prove that h is 1-1 if and only if $\overline{h}$ is 1-1. Suppose that h is 1-1 and suppose that $\overline{h}[U] = \overline{h}[V]$ such that $U \neq V$, i.e., there exists $P \in \varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)$ such that $P \notin \varphi_{\mathbf{A}}(b_1) \cap ... \cap \varphi_{\mathbf{A}}(b_m)$. Then $a_i \notin P$ for all $i \in \{1, ..., n\}$ and there exists $j \in \{1, ..., m\}$ such that $b_j \in P$. Let us consider the set $h(P) = \{h(p) \ : p \in P\}$ and we prove that $h(P) \in X(h(\mathbf{A}))$. It is obvious that $h(P)$ is a non-empty proper subset of $h(A)$. If $a, b \in h(A)$ are such that $a \leq b$ and $b \in h(P)$, then there exists $p_1 \in A$ and there exists $p_2 \in P$ such that $h(p_1) \leq h(p_2)$. Since h is 1-1, $p_1 \leq p_2$ and as P is a ideal, $p_1 \in P$ and $a \in h(P)$. Let $a, b \in h(P)$. Then there exist $p_1, p_2 \in P$ such that $h(p_1) = a$ and $h(p_2) = b$. Let $p = p_1 \vee p_2 \in P$. Since h is a homomorphism, $a \vee b = h(p)$ and $a \vee b \in h(P)$. Thus, $h(P) \in \mathrm{Id}(h(\mathbf{A}))$. Let $a, b \in h(A)$ such that exists $a \wedge b$ and $a \wedge b \in h(P)$. Then there exist $p_1, p_2 \in A$ and there exists $p_3 \in P$ such that $a = h(p_1), b = h(p_2)$ and $a \wedge b = h(p_3)$. So, $h(p_3) = h(p_1) \wedge h(p_2)$. It follows that

$$\begin{aligned}
h(p_3) &= [h(p_1) \wedge h(p_2)] \vee h(p_3) \\
&= [h(p_1) \vee h(p_3)] \wedge [h(p_2) \vee h(p_3)] \\
&= h(p_1 \vee p_3) \wedge h(p_2 \vee p_3) \\
&= h((p_1 \vee p_3) \wedge_{p_3} (p_2 \vee p_3))
\end{aligned}$$

because $p_1 \vee p_3, p_2 \vee p_3 \in [p_3)$ and $[p_3)$ is a bounded distributive lattice. As h is 1-1, $p_3 = (p_1 \vee p_3) \wedge (p_2 \vee p_3) \in P$ and by the primality of P, $p_1 \vee p_3 \in P$ or $p_2 \vee p_3 \in P$. Then $p_1 \in P$ or $p_2 \in P$, i.e., $a \in h(P)$ or $b \in h(P)$. So, $h(P) \in X(h(\mathbf{A}))$. Since $h(b_j) \in h(P)$ and $h(b_1) \wedge ... \wedge h(b_m) \in h(P)$, we have that $h(a_1) \wedge ... \wedge h(a_n) \in h(P)$. Then, as $h(P)$ is a prime ideal, there exists $k \in \{1, ..., n\}$ such that $h(a_k) \in h(P)$, i.e., $a_k \in P$ which is a contradiction. Therefore, $\overline{h}$ is 1-1. Reciprocally, if $\overline{h}$ is 1-1 and $a, b \in A$ such that $h(a) = h(b)$, then $\overline{h}[\varphi_{\mathbf{A}}(a)] = \overline{h}[\varphi_{\mathbf{A}}(b)]$ and $\varphi_{\mathbf{A}}(a) = \varphi_{\mathbf{A}}(b)$. Since $\varphi_{\mathbf{A}}$ is 1-1 it follows that $a = b$ and h is 1-1.

Suppose that h is onto. Let $b \in L$. Then there exists $a \in A$ such that $h(a) = b$. Thus, $\varphi_{\mathbf{A}}(a) \in D_{\mathcal{KO}}[X(\mathbf{A})]$ and $\overline{h}[\varphi_{\mathbf{A}}(a)] = h(a) = b$. Hence, $\overline{h}$ is onto. $\square$

Theorem 3.3. *Let $\mathbf{A} \in \mathcal{DN}$ and $\langle D_{\mathcal{KO}}[X(\mathbf{A})], \varphi_{\mathbf{A}}\rangle$ be the free distributive lattice extension of $\mathbf{A}$. Then the lattices $\mathrm{Fi}(\mathbf{A})$ and $\mathrm{Fi}(D_{\mathcal{KO}}[X(\mathbf{A})])$ are isomorphic.*

Proof. Let us consider the mapping $\Psi : \mathrm{Fi}(D_{\mathcal{KO}}[X(\mathbf{A})]) \to \mathrm{Fi}(\mathbf{A})$ defined by

$$\Psi(G) = \{a \in A \ : \ \varphi_{\mathbf{A}}(a) \in G\}.$$

First, we prove that Ψ is well defined, i.e., if $G \in \mathrm{Fi}(D_{\mathcal{KO}}[X(\mathbf{A})])$ then $\Psi(G) \in \mathrm{Fi}(\mathbf{A})$. If G is a filter of $D_{\mathcal{KO}}[X(\mathbf{A})]$, then $\varphi_{\mathbf{A}}(1) \in G$ and $1 \in \Psi(G)$. Let $a, b \in A$ such that $a \leq b$ and $a \in \Psi(G)$. Then $\varphi_{\mathbf{A}}(a) \subseteq \varphi_{\mathbf{A}}(b)$ and $\varphi_{\mathbf{A}}(a) \in G$. Therefore $\varphi_{\mathbf{A}}(b) \in G$ and $b \in \Psi(G)$. If $a, b \in \Psi(G)$ are such that there $a \wedge b$ exists, then $\varphi_{\mathbf{A}}(a), \varphi_{\mathbf{A}}(b) \in G$. Since G is a filter, $\varphi_{\mathbf{A}}(a) \cap \varphi_{\mathbf{A}}(b) = \varphi_{\mathbf{A}}(a \wedge b) \in G$ and $a \wedge b \in \Psi(G)$. Therefore, $\Psi(G) \in \mathrm{Fi}(\mathbf{A})$.

We see that Ψ is a homomorphism. Let $G_1, G_2 \in \mathrm{Fi}(D_{\mathcal{KO}}[X(\mathbf{A})])$. It follows that $\Psi(G_1 \cap G_2) = \Psi(G_1) \cap \Psi(G_2)$. Let $a \in \Psi(G_1) \vee \Psi(G_2) = F(\Psi(G_1) \cup \Psi(G_2))$. Then there exist $x_1, ..., x_n \in \Psi(G_1) \cup \Psi(G_2)$ such that $x_1 \wedge ... \wedge x_n$ exists and $x_1 \wedge ... \wedge x_n = a$. Thus $\varphi_{\mathbf{A}}(x_1), ..., \varphi_{\mathbf{A}}(x_n) \in G_1 \cup G_2$ and $\varphi_{\mathbf{A}}(x_1) \cap ... \cap \varphi_{\mathbf{A}}(x_n) = \varphi_{\mathbf{A}}(a)$. It follows that $\varphi_{\mathbf{A}}(a) \in F(G_1 \cup G_2) = G_1 \vee G_2$ and $a \in \Psi(G_1 \vee G_2)$. So, $\Psi(G_1) \vee \Psi(G_2) \subseteq \Psi(G_1 \vee G_2)$. Conversely, if $a \notin \Psi(G_1) \vee \Psi(G_2) = F(\Psi(G_1) \cup \Psi(G_2))$, then for every subset $\{x_1, ..., x_n\} \subseteq \Psi(G_1) \cup \Psi(G_2)$ such that $x_1 \wedge ... \wedge x_n$ exists we have $x_1 \wedge ... \wedge x_n \neq a$. It follows that for every subset $\{\varphi_{\mathbf{A}}(x_1), ..., \varphi_{\mathbf{A}}(x_n)\} \subseteq G_1 \cup G_2$ such that $x_1 \wedge ... \wedge x_n$ exists we have $\varphi_{\mathbf{A}}(x_1) \cap ... \cap \varphi_{\mathbf{A}}(x_n) \neq \varphi_{\mathbf{A}}(a)$, i.e., $\varphi_{\mathbf{A}}(a) \notin F(G_1 \cup G_2) = G_1 \vee G_2$. Then $a \notin \Psi(G_1 \vee G_2)$ and $\Psi(G_1 \vee G_2) \subseteq \Psi(G_1) \vee \Psi(G_2)$. So, $\Psi(G_1 \vee G_2) = \Psi(G_1) \vee \Psi(G_2)$.

We prove that Ψ is 1-1. Let $G_1, G_2 \in \mathrm{Fi}(D_{\mathcal{KO}}[X(\mathbf{A})])$ such that $\Psi(G_1) = \Psi(G_2)$. If $U \in G_1$, then there exist $a_1, ..., a_n \in A$ such that $U = \varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n)$. So, $\varphi_{\mathbf{A}}(a_i) \in G_1$, i.e., $a_i \in \Psi(G_1) = \Psi(G_2)$ for all $i \in \{1, ..., n\}$. Then $\varphi_{\mathbf{A}}(a_i) \in G_2$ for all $i \in \{1, ..., n\}$ and $\varphi_{\mathbf{A}}(a_1) \cap ... \cap \varphi_{\mathbf{A}}(a_n) = U \in G_2$. Similarly, if $U \in G_2$ then $U \in G_1$ and $G_1 = G_2$. Thus, Ψ is 1-1.

Finally, we prove that Ψ is onto. Let $G \in \mathrm{Fi}(\mathbf{A})$ and we consider $\varphi_{\mathbf{A}}(G) = \{\varphi_{\mathbf{A}}(a) : a \in G\}$. Then the filter generated $F(\varphi_{\mathbf{A}}(G)) \in \mathrm{Fi}((D_{\mathcal{KO}}[X(\mathbf{A})]))$. We prove that $\Psi(F(\varphi_{\mathbf{A}}(G))) = G$. If $a \in G$, then $\varphi_{\mathbf{A}}(a) \in \varphi_{\mathbf{A}}(G)$ and $\varphi_{\mathbf{A}}(a) \in F(\varphi_{\mathbf{A}}(G))$. So, $a \in \Psi(F(\varphi_{\mathbf{A}}(G)))$. Reciprocally, suppose that $a \notin G$. Then $\varphi_{\mathbf{A}}(a) \notin \varphi_{\mathbf{A}}(G)$. We see that $\varphi_{\mathbf{A}}(a) \notin F(\varphi_{\mathbf{A}}(G))$. If $\varphi_{\mathbf{A}}(a) \in F(\varphi_{\mathbf{A}}(G))$, then there exist $x_1, ..., x_n \in G$ such that $\varphi_{\mathbf{A}}(x_1) \cap ... \cap \varphi_{\mathbf{A}}(x_n) = \varphi_{\mathbf{A}}(a)$. On the other hand, since $a \notin G$, there exists $P \in X(\mathbf{A})$ such that $a \in P$ and $P \cap G = \emptyset$, i.e., $P \notin \varphi_{\mathbf{A}}(a)$ and $P \in \varphi_{\mathbf{A}}(x_i)$ for all $i \in \{1, ..., n\}$, which is a contradiction. Then $\varphi_{\mathbf{A}}(a) \notin F(\varphi_{\mathbf{A}}(G))$ and $a \notin \Psi(F(\varphi_{\mathbf{A}}(G)))$. Therefore $\Psi(F(\varphi_{\mathbf{A}}(G))) = G$ and Ψ is onto. □

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References

[1] J. C. Abbott, Semi-boolean algebra, *Mat. Vestnik* **4** (1967), 177–198.

[2] J. Araújo and M. Kinyon, Independent axiom systems for nearlattices, *Czechoslovak Mathematical Journal* **61** (2011), 975–992.

[3] G. Bezhanishvili and R. Jansana, Priestley style duality for distributive meet-semilattices, *Studia Logica* **98** (2011), 83–122.

[4] G. Bezhanishvili and R. Jansana, Generalized Priestley quasi-orders, *Order* **28** (2011), 201–220.

[5] I. Calomino and S. Celani, A note on annihilators in distributive nearlattices, *Miskolc Mathematical Notes* **16** (2015), 65–78.

[6] S. Celani, Topological representation of distributive semilattices, *Scientiae Math. Japonicae* **8** (2003), 41–51.

[7] S. Celani and I. Calomino, Some remarks on distributive semilattices, *Comment. Math. Univ. Carolin.* **54** (2013), 407–428.

[8] S. Celani and I. Calomino, Stone style duality for distributive nearlattices, *Algebra Universalis* **71** (2014), 127–153.

[9] I. Chajda, R. Halaš, and J. Kühr, Semilattice Structures, Research and Exposition in Mathematics, Heldermann Verlag 2007.

[10] I. Chajda and M. Kolařík, Ideals, congruences and annihilators on nearlattices, *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math* **46** (2007), 25–33.

[11] I. Chajda and M. Kolařík, Nearlattices, *Discrete Math.* **308** (2008), 4906–4913.

[12] W. Cornish and R. Hickman, Weakly distributive semilattices, *Acta Math. Acad. Sci. Hungar.* **32** (1978), 5–16.

[13] R. Halaš, Subdirectly irreducible distributive nearlattices, *Miskolc Mathematical Notes* **7** (2006), 141–146.

[14] R. Hickman, Join algebras, *Communications in Algebra* **8** (1980), 1653–1685.

[15] P. Johnstone, Stone spaces, Cambridge University Press, 1982.

[16] M. Stone, Topological representation of distributive lattices and Brouwerian logics, *Casopis pest. mat. fys.* **67** (1937), 1–25.

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