REPORTS ON MATHEMATICAL LOGIC
51 (2016), 35–41
doi:10.4467/20842589RM.16.003.5280

Gert-Jan C. LOKHORST

AN ALTERNATIVE INTUITIONISTIC VERSION OF MALLY'S DEONTIC LOGIC

A b s t r a c t. Some years ago, Lokhorst proposed an intuitionistic reformulation of Mally's deontic logic (1926). This reformulation was unsatisfactory, because it provided a striking theorem that Mally himself did not mention. In this paper, we present an alternative reformulation of Mally's deontic logic that does not provide this theorem.

1. Introduction

Some years ago, Lokhorst proposed an intuitionistic reformulation of Mally's deontic logic (1926) [3]. This reformulation was unsatisfactory, because it provided a striking theorem that Mally himself did not mention, namely $\bigcirc(A \vee \neg A)$. In this paper, we present an alternative reformulation of Mally's deontic logic that does not provide this theorem.

Received 4 October 2015
Keywords and phrases: deontic logic, intuitionistic logic.

2. Definitions

Heyting's system of intuitionistic propositional logic h is defined as follows [1, Ch. 2].

Axioms: (a) $A \to (B \to A)$.
(b) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.
(c) $(A \wedge B) \to A$; $(A \wedge B) \to B$.
(d) $A \to (B \to (A \wedge B))$.
(e) $A \to (A \vee B)$; $B \to (A \vee B)$.
(f) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$.
(g) $\bot \to A$.

Rule: $A, A \to B/B$ (modus ponens, MP).

Definitions: $\neg A = A \to \bot$, $\top = \neg\bot$, $A \leftrightarrow B = (A \to B) \wedge (B \to A)$.

The second-order intuitionistic propositional calculus with comprehension C2h is **h** plus [1, Ch. 9]:

Axioms: Q1 $(\forall x)A(x) \to A(y)$.
Q2 $A(y) \to (\exists x)A(x)$.
Q5 $(\forall x)(B \vee A(x)) \to (B \vee (\forall x)A(x))$, x not free in B.
Q6 $(\exists x)(x \leftrightarrow A)$, x not free in A.

Rules: Q3 $A(x) \to B/(\exists x)A(x) \to B$, x not free in B.
Q4 $B \to A(x)/B \to (\forall x)A(x)$, x not free in B.

Definition: $\bot \stackrel{\text{df}}{=} (\forall x)x$ [1, Ch. 9, Exercise 10].

An intuitionistic version of Mally's deontic logic ○C2h is **C2h** plus [4, Ch. I]:

A1 $((A \to \bigcirc B) \wedge (B \to C)) \to (A \to \bigcirc C)$.

A2 $((A \to \bigcirc B) \wedge (A \to \bigcirc C)) \to (A \to \bigcirc(B \wedge C))$.

A3 $(A \to \bigcirc B) \leftrightarrow \bigcirc(A \to B)$.

A4 $\bigcirc\top$.

A5 $\neg(\top \to \bigcirc\bot)$.

Some comments on **○C2h**:

1. Mally wrote $!A$ instead of $\bigcirc A$. He read $!A$ as "it ought to be case that A" or "it is required that A is the case." He read $A \to !B$ as "A requires B."

2. Definition: $\mathbf{U} \stackrel{\text{df}}{=} \top$. Mally read $\mathbf{U}$ as "the unconditionally required" or "what conforms with what ought to be the case."

3. Definition: $\mathbf{\Omega} \stackrel{\text{df}}{=} \bot$. Mally read $\mathbf{\Omega}$ as "what conflicts with what ought to be the case."

4. Mally wrote $\exists \mathbf{U} \bigcirc \mathbf{U}$ instead of A4. We regard $\exists \mathbf{U} \bigcirc \mathbf{U}$ as ill-formed, because we view $\mathbf{U}$ as a constant. We therefore replace $\exists \mathbf{U} \bigcirc \mathbf{U}$ by $(\exists x)((x \leftrightarrow \mathbf{U}) \wedge \bigcirc x)$ (this is formula T15$''$ in the Appendix below). This agrees with Mally's informal interpretation of $\exists \mathbf{U} \bigcirc \mathbf{U}$.

3. Theorems

Definition 1. Let A be a formula in the language of $\bigcirc$**C2h**. By induction on the number of connectives in A we define two translations, $[A]^+$ and $[A]^-$, of A into the formulas of **C2h** as follows:

1. If A is atomic, then $[A]^\pm \stackrel{\text{df}}{=} A$.
2. $[\bot]^\pm \stackrel{\text{df}}{=} \bot$.
3. $[A_1 \otimes A_2]^\pm \stackrel{\text{df}}{=} [A_1]^\pm \otimes [A_2]^\pm$, where $\otimes$ is $\wedge$, $\vee$ or $\rightarrow$.
4. $[(Qx)A(x)]^\pm \stackrel{\text{df}}{=} (Qx)[A(x)]^\pm$, where (Qx) is $(\forall x)$ or $(\exists x)$.
5. $[\bigcirc A]^+ \stackrel{\text{df}}{=} [A]^+$ and $[\bigcirc A]^- \stackrel{\text{df}}{=} \neg\neg[A]^-$.

Theorem 1. *(After [2, Theorem 1, p. 312].) If A is a theorem of* $\bigcirc$***C2h**, then $[A]^\pm$ is a theorem of* ***C2h**.*

Proof. By induction on the construction of the proof of A. *Base case*: for each axiom A of $\bigcirc$**C2h**, $[A]^\pm$ is a theorem of **C2h**, as can easily be checked. *Inductive step*: MP, Q3 and Q4 preserve this property. Suppose that the theorem holds for A, B and that $\bigcirc$**C2h** provides A/B by rule R (induction hypothesis). We show that **C2h** provides $[A]^\pm/[B]^\pm$ by R.

Case R of:

- MP: let $A \stackrel{\text{df}}{=} C$, $B \stackrel{\text{df}}{=} C \rightarrow D$. **C2h** provides $[A]^\pm/[B]^\pm$ by R, because $[A]^\pm = [C]^\pm$ and $[B]^\pm \stackrel{\text{df}}{=} [C \rightarrow D]^\pm \stackrel{\text{df}}{=} [C]^\pm \rightarrow [D]^\pm$.
- Q3: let $A \stackrel{\text{df}}{=} C(x) \rightarrow D$, $B = (\exists x)C(x) \rightarrow D$, x not free in D. **C2h** provides $[A]^\pm/[B]^\pm$ by R, because $[A]^\pm \stackrel{\text{df}}{=} [C(x) \rightarrow D]^\pm \stackrel{\text{df}}{=} [C(x)]^\pm \rightarrow [D]^\pm$ and $[B]^\pm \stackrel{\text{df}}{=} [(\exists x)C(x) \rightarrow D]^\pm \stackrel{\text{df}}{=} (\exists x)[C(x)]^\pm \rightarrow [D]^\pm$.

- Q4: let $A \stackrel{\text{df}}{=} C \rightarrow D(x)$, $B = [C \rightarrow (\forall x)D(x)]^{\pm}$, x not free in C. **C2h** provides $[A]^{\pm}/[B]^{\pm}$ by R, because $[A]^{\pm} \stackrel{\text{df}}{=} [C \rightarrow D(x)]^{\pm} \stackrel{\text{df}}{=} [C]^{\pm} \rightarrow [D(x)]^{\pm}$ and $[B]^{\pm} \stackrel{\text{df}}{=} [C \rightarrow (\forall x)D(x)]^{\pm} \stackrel{\text{df}}{=} [C]^{\pm} \rightarrow (\forall x)[D(x)]^{\pm}$.

□

Theorem 2. *(After [2, Theorem 1, p. 312].) Let p be an atomic formula. There is no formula A in the language of **C2h** such that* $\bigcirc$***C2h*** $\vdash \bigcirc p \leftrightarrow A$.

Proof. From Theorem 1. If for some formula A of **C2h**, $\bigcirc$**C2h** $\vdash \bigcirc p \leftrightarrow A$, then **C2h** $\vdash \neg\neg p \leftrightarrow A$ and **C2h** $\vdash p \leftrightarrow A$, since $[A]^{\pm}$ is A. Hence **C2h** $\vdash p \leftrightarrow \neg\neg p$, but this is false. □

Definition 2. For theories T based on intuitionistic logic, if A is an arbitrary formula of the language of T, then A is stable in T if and only if T provides $\neg\neg A \rightarrow A$.

Theorem 3. $\bigcirc A$ *is not stable in* $\bigcirc$***C2h****.*

Proof. From Theorem 1. $[\neg\neg\bigcirc p \rightarrow \bigcirc p]^{+}$ ($\stackrel{\text{df}}{=} \neg\neg p \rightarrow p$) is not a theorem of **C2h**. □

Theorem 4. $\bigcirc$***C2h*** *provides A1–A5 and all theorems of [4, Chs. I–II] (see Appendix), except:*

T12c $\bigcirc(A \rightarrow B) \leftrightarrow \bigcirc\neg(A \wedge \neg B)$.

T12d $\bigcirc\neg(A \wedge \neg B) \leftrightarrow \bigcirc(\neg A \vee B)$.

T13a $(A \rightarrow \bigcirc B) \leftrightarrow \neg(A \wedge \neg \bigcirc B)$.

T13b $\neg(A \wedge \neg \bigcirc B) \leftrightarrow (\neg A \vee \bigcirc B)$.

T14 $(A \rightarrow \bigcirc B) \leftrightarrow (\neg B \rightarrow \bigcirc\neg A)$.

Proof. From Theorem 1. For each formula A on the above list, $[A]^{+}$ is not a theorem of **C2h**. Additionally, $[\text{T13b}]^{-}$ is not a theorem of **C2h**. □

Theorem 5. $\bigcirc$***C2h*** *does not provide* $\bigcirc(A \vee \neg A)$.

Proof. From Theorem 1. $[\bigcirc(p \vee \neg p)]^{+}$ ($\stackrel{\text{df}}{=} p \vee \neg p$) is not a theorem of **C2h**. □

4. Conclusion

The intuitionistic reformulation of Mally's deontic logic proposed in [3] provided $\bigcirc(A \vee \neg A)$. This formula is not a theorem of $\bigcirc$**C2h**. Moreover, Mally did not mention this formula. $\bigcirc$**C2h** is, in a sense, therefore more adequate than the intuitionistic reformulation proposed in [3], even though the latter reformulation lacked only T13b (from the formulas mentioned in Theorem 4).

Appendix

All theorems from [4, Ch. II], as listed in [5, pp. 121–123], plus one theorem that seems to have been overlooked in [5, pp. 121–123], namely T15″ (cf. [4, Ch. I, axiom IV]). All theorems are derivable in $\bigcirc$**C2h**, except those marked with a † (Theorem 4).

T01	$(C \to \bigcirc(A \wedge B)) \to ((C \to \bigcirc A) \wedge (C \to \bigcirc B))$
T02	$((C \to \bigcirc A) \wedge (C \to \bigcirc B)) \leftrightarrow (C \to \bigcirc(A \wedge B))$
T1	$(A \to \bigcirc B) \to (A \to \bigcirc\top)$
T2′	$(A \to \bigcirc\bot) \to (\forall x)(A \to \bigcirc x)$
T2″	$(\forall x)(A \to \bigcirc x) \to (A \to \bigcirc\bot)$
T3	$((C \to \bigcirc A) \vee (C \to \bigcirc B)) \to (C \to \bigcirc(A \vee B))$
T4	$((C \to \bigcirc A) \wedge (D \to \bigcirc B)) \to ((C \wedge D) \to \bigcirc(A \wedge B))$
T5a	$\bigcirc A \leftrightarrow (\forall x)(x \to \bigcirc A)$
T5b	$(\forall x)(x \to \bigcirc A) \leftrightarrow (\forall x)(x \to \bigcirc A)$
T6	$(\bigcirc A \wedge (A \to B)) \to \bigcirc B$
T7	$\bigcirc A \to \bigcirc\top$
T8	$((A \to \bigcirc B) \wedge (B \to \bigcirc C)) \to (A \to \bigcirc C)$
T9	$(\bigcirc A \wedge (A \to \bigcirc B)) \to \bigcirc B$
T10	$(\bigcirc A \wedge \bigcirc B) \leftrightarrow \bigcirc(A \wedge B)$
T11	$((A \to \bigcirc B) \wedge (B \to \bigcirc A)) \leftrightarrow \bigcirc(A \leftrightarrow B)$
T12a	$(A \to \bigcirc B) \leftrightarrow (A \to \bigcirc B)$
T12b	$(A \to \bigcirc B) \leftrightarrow \bigcirc(A \to B)$
†T12c	$\bigcirc(A \to B) \leftrightarrow \bigcirc\neg(A \wedge \neg B)$
†T12d	$\bigcirc\neg(A \wedge \neg B) \leftrightarrow \bigcirc(\neg A \vee B)$
†T13a	$(A \to \bigcirc B) \leftrightarrow \neg(A \wedge \neg \bigcirc B)$
†T13b	$\neg(A \wedge \neg \bigcirc B) \leftrightarrow (\neg A \vee \bigcirc B)$
†T14	$(A \to \bigcirc B) \leftrightarrow (\neg B \to \bigcirc\neg A)$
T15	$(\forall x)(x \to \bigcirc\mathbf{U})$

T15$''$	$(\exists x)((x \leftrightarrow \mathbf{U}) \wedge \bigcirc x)$
T16	$(\mathbf{U} \rightarrow A) \rightarrow \bigcirc A$
T17	$(\mathbf{U} \rightarrow \bigcirc A) \rightarrow \bigcirc A$
T18	$\bigcirc\bigcirc A \rightarrow \bigcirc A$
T19	$\bigcirc\bigcirc A \leftrightarrow \bigcirc A$
T20	$(\mathbf{U} \rightarrow \bigcirc A) \leftrightarrow ((A \rightarrow \bigcirc \mathbf{U}) \wedge (\mathbf{U} \rightarrow \bigcirc A))$
T21	$\bigcirc A \leftrightarrow ((A \rightarrow \bigcirc \mathbf{U}) \wedge (\mathbf{U} \rightarrow \bigcirc A))$
T22	$\bigcirc \top$
T23$'$	$\top \rightarrow \bigcirc \mathbf{U}$
T23$''$	$\mathbf{U} \rightarrow \bigcirc \top$
T23$'''$	$\bigcirc(\mathbf{U} \leftrightarrow \top)$
T24	$A \rightarrow \bigcirc A$
T25	$(A \rightarrow B) \rightarrow (A \rightarrow \bigcirc B)$
T26	$(A \leftrightarrow B) \rightarrow ((A \rightarrow \bigcirc B) \wedge (B \rightarrow \bigcirc A))$
T27	$(\forall x)(\mathbf{\Omega} \rightarrow \bigcirc \neg x)$
T27$'$	$(\forall x)(\mathbf{\Omega} \rightarrow \bigcirc x)$
T28	$\mathbf{\Omega} \rightarrow \bigcirc \mathbf{\Omega}$
T29	$\mathbf{\Omega} \rightarrow \bigcirc \mathbf{U}$
T30	$\mathbf{\Omega} \rightarrow \bigcirc \bot$
T31	$(\mathbf{\Omega} \rightarrow \bigcirc \bot) \wedge (\bot \rightarrow \bigcirc \mathbf{\Omega})$
T31$'$	$\bigcirc(\mathbf{\Omega} \leftrightarrow \bot)$
T32	$\neg(\mathbf{U} \rightarrow \bigcirc \bot)$
T33	$\neg(\mathbf{U} \rightarrow \bot)$
T34	$\mathbf{U} \leftrightarrow \top$
T35	$\mathbf{\Omega} \leftrightarrow \bot$

References

[1] D. M. Gabbay, Semantical Investigations in Heyting's Intuitionistic Logic. Reidel, Dordrecht, 1981.

[2] M. Kaminski, Nonstandard connectives of intuitionistic propositional logic. *Notre Dame Journal of Formal Logic* **29** (1988), 309–331.

[3] G. J. C. Lokhorst, An intuitionistic reformulation of Mally's deontic logic. *Journal of Philosophical Logic* **42** (2013), 635–641.

[4] E. Mally, Grundgesetze des Sollens: Elemente der Logik des Willens, Leuschner und Lubensky, Graz 1926.

[5] E. Morscher, Mallys Axiomsystem für die deontische Logik: Rekonstruktion und kritische Würdigung. In A. Hieke (Ed.), Ernst Mally: Versuch einer Neubewertung, pp. 81–165, Academia Verlag, Sankt Augustin 1998.

Section of Philosophy
Faculty of Technology, Policy and Management
Delft University of Technology
P.O. Box 5015
2600 GA Delft
The Netherlands
g.j.c.lokhorst@tudelft.nl