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DISCRIMINATOR VARIETIES OF DOUBLE-HEYTING ALGEBRAS

A b s t r a c t. We prove that a variety of double-Heyting algebras is a discriminator variety if and only if it is semisimple if and only if it has equationally definable principal congruences. The result also applies to the class of Heyting algebras with a dual pseudocomplement operation and to the class of regular double p-algebras.

1. Introduction

A well-known theorem for boolean algebras, fundamental to the study of computing and the design of logic circuits, tells us that for all $n \in \omega$, any map $f\colon \{0,1\}^n \to \{0,1\}$ can be defined by a term in the language of boolean algebras. In a general setting, if a finite algebra $\mathbf{A}$ has the property that for

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all $n \in \omega$, any map $f\colon A^n \to A$ can be expressed as a term in the language of $\mathbf{A}$, then $\mathbf{A}$ is called a *primal algebra*. In this terminology, the two-element boolean algebra is a primal boolean algebra; indeed, the only primal boolean algebra. Further examples of primal algebras are found in the context of n-valued logic: the n-element chains are primal as Post algebras. Many properties of primal algebras are encapsulated by the existence of a term that produces the *discriminator function*: the function $t\colon A^3 \to A$ defined by

$$t(x, y, z) = \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

An equational class $\mathcal{V}$, or equivalently, a variety $\mathcal{V}$, is a *discriminator variety* if there exists a term t in the language of $\mathcal{V}$ such that t is the discriminator function on every subdirectly irreducible member of $\mathcal{V}$. Discriminator varieties have been vital in the study of decidability. For instance, Werner [8] proved that for a discriminator variety $\mathcal{V}$, if $\mathcal{V}$ is residually small (that is, it contains up to isomorphism only a set of subdirectly irreducible algebras), then the first-order theory of $\mathcal{V}$ is decidable. Discriminator varieties also played an important role in McKenzie and Valeriote's classification of decidable locally finite varieties [5].

Kowalski [4] classified the discriminator varieties of tense algebras. The formal details can be found in [4], but we note that the proof relies on the existence of a unary term d satisfiying certain properties. For all $n \in \omega$ let $\mathcal{T}_n$ denote the class of tense algebras $\mathbf{A}$ satisfying the equation

$$d^{n+1}x \approx d^n x.$$

Theorem 1.1 (Kowalski [4])**.** *Let $\mathcal{V}$ be a variety of tense algebras. Then the following are equivalent:*

(1) *$\mathcal{V}$ is a discriminator variety,*

(2) *$\mathcal{V}$ is semisimple,*

(3) *$\mathcal{V} \subseteq \mathcal{T}_n$ for some $n \in \omega$,*

(4) *$\mathcal{V}$ has definable principal congruences,*

(5) *$\mathcal{V}$ has equationally definable principal congruences.*

In this paper we utilise techniques from Kowalski's proof to produce a result for double-Heyting algebras that is identical in form. The definitions are laid out in Section 2 and the main result proved in Section 3. Let **A** be a double-Heyting algebra and let $x \in A$. Define $\neg x = x \to 0$ and ${\sim}x = 1 - x$, then let $d^0 x = x$, and for all $n \in \omega$ let $d^{n+1}x = \neg{\sim}d^n x$. For all $n \in \omega$, let $\mathcal{DH}_n$ denote the class of double-Heyting algebras satisfying the equation

$$d^{n+1}x \approx d^n x.$$

Sankappanavar [6] proved that each of the classes $\mathcal{DH}_n$ forms a discriminator variety. Our main result shows that they, and their subvarieties, are the only discriminator varieties of double-Heyting algebras.

Theorem 1.2. *Let $\mathcal{V}$ be a variety of double-Heyting algebras. Then the following are equivalent.*

(1) *$\mathcal{V}$ is a discriminator variety,*

(2) *$\mathcal{V}$ is semisimple,*

(3) *$\mathcal{V} \subseteq \mathcal{DH}_n$ for some $n \in \omega$,*

(4) *$\mathcal{V}$ has definable principal congruences,*

(5) *$\mathcal{V}$ has equationally definable principal congruences.*

2. Preliminaries

2.1. Algebraic preliminaries

We introduce our notation and basic definitions here. For more on universal algebra see Burris and Sankappanavar [2]. For an algebra **A**, the congruence lattice of **A** is written $\mathbf{Con}(\mathbf{A})$. For any $a, b \in A$, the principal congruence generated by identifying a and b is denoted by $\mathrm{Cg}^{\mathbf{A}}(a, b)$. For a complete lattice **L**, an element $\alpha \in L$ is *compact* if, for all $I \subseteq L$, whenever $\alpha \leq \bigvee I$, there exists a finite set $J \subseteq I$ such that $\alpha \leq \bigvee J$. For any lattice **L**, and any $a, b \in L$ we say that *a covers b* if $b < a$ and there is no element $x \in L$ such that $b < x < a$. Note that every principal congruence is compact in $\mathbf{Con}(\mathbf{A})$.

If $\mathbf{Con}(\mathbf{A})$ is a distributive lattice then we say $\mathbf{A}$ is *congruence distributive*. If, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, $\alpha \circ \beta = \beta \circ \alpha$, then $\mathbf{A}$ is *congruence permutable*. It is easy to prove that if $\mathbf{A}$ is congruence permutable then, for all $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$, $\alpha \vee \beta = \alpha \circ \beta$. An algebra is *arithmetical* if it is both congruence distributive and congruence permutable.

If every algebra in a class $\mathcal{K}$ is congruence distributive (congruence permutable, arithmetical) then we say that the class $\mathcal{K}$ is congruence distributive (congruence permutable, arithmetical). If the class $\mathcal{K}$ is closed under taking homomorphic images, subalgebras and direct products then $\mathcal{K}$ is called a *variety*. If there is a set of equations such that $\mathcal{K}$ consists of all algebras satisfying all of those equations, then $\mathcal{K}$ is called an *equational class*. A fundamental result due to Birkhoff tells us that a class is a variety if and only if it is an equational class.

An algebra $\mathbf{A}$ is *subdirectly irreducible* if $\mathbf{Con}(\mathbf{A})$ has a least non-zero element μ. We will call μ the *monolith of* $\mathbf{A}$. An algebra is called *simple* if its congruence lattice has precisely two elements. A variety $\mathcal{V}$ is *semisimple* if every subdirectly irreducible member of $\mathcal{V}$ is simple.

A variety $\mathcal{V}$ has *definable principal congruences* (DPC) if there exists a first-order formula $\varphi(x, y, u, v)$ such that, for all $\mathbf{A} \in \mathcal{V}$ and all $a, b, c, d \in \mathbf{A}$, the following equivalence is satisfied:

$$(a, b) \in \mathrm{Cg}^{\mathbf{A}}(c, d) \iff \mathbf{A} \models \varphi(a, b, c, d).$$

If $\varphi(x, y, u, v)$ can be taken to be a finite conjunction of equations then we say $\mathcal{V}$ has *equationally definable principal congruences* (EDPC).

We will let $\mathbb{N}$ denote the set of natural numbers not including zero.

2.2. Double-Heyting algebras

Definition 2.1. An algebra $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, -, 0, 1\rangle$ is called a *double-Heyting algebra* if $\langle A; \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra and $\langle A; \vee, \wedge, -, 0, 1\rangle$ is a dual Heyting algebra. More precisely, $\mathbf{A}$ is a double-Heyting algebra if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded lattice and the operations $\rightarrow$ and $-$ satisfy the following equivalences:

$$x \wedge y \le z \iff y \le x \rightarrow z,$$
$$x \vee y \ge z \iff y \ge z - x.$$

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, {\sim}, 0, 1\rangle$ is a *Heyting algebra with dual pseudocomplementation* (H^+-algebra for short) if $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\sim$ is a *dual pseudocomplement* operation, that is,

$$x \vee y = 1 \iff y \geq {\sim}x.$$

For a Heyting algebra $\mathbf{A}$ and all $x \in A$ the *pseudocomplement of* x in $\mathbf{A}$ can be given by $\neg x = x \to 0$. Similarly, for a dual-Heyting algebra $\mathbf{A}$ and all $x \in A$, the dual pseudocomplement of x in $\mathbf{A}$ can be defined by ${\sim}x = 1 - x$.

Let $\mathcal{H}^+$ denote the class of H^+-algebras and let $\mathcal{DH}$ denote the class of double-Heyting algebras. It is known (see [6] for example) that the classes $\mathcal{H}^+$ and $\mathcal{DH}$ are both equational classes.

The following result due to Sankappanavar allows us to restrict our attention to H^+-algebras for the remainder of this article.

Theorem 2.2 (Sankappanavar [6])**.** *Let* $\mathbf{A}$ *be a double-Heyting algebra. Then every* H^+ *congruence on* $\mathbf{A}$ *is a double-Heyting congruence on* $\mathbf{A}$*.*

Definition 2.3. Let $\mathbf{A}$ be a H^+-algebra. Define the map $d\colon A \to A$ by $dx = \neg{\sim}x$ and define inductively for each $n \in \omega$ the map $d^n\colon A \to A$ by $d^0x = x$ and $d^{n+1}x = d(d^n x)$. A *normal filter on* $\mathbf{A}$ is a filter $F \subseteq A$ that is closed under d. Let $\mathrm{NF}(\mathbf{A})$ denote the lattice of normal filters of $\mathbf{A}$ and, for all $x \in A$ let $N(x)$ denote the smallest normal filter containing x. For each (normal) filter F, let $\theta(F)$ denote the relation given by

$$(x, y) \in \theta(F) \iff (\exists f \in F)\ x \wedge f = y \wedge f.$$

Lemma 2.4 (Sankappanavar [6])**.** *Let* $\mathbf{A}$ *be an* H^+*-algebra.*

(1) *Let* $x \in A$*. Then* $N(x) = \bigcup_{n\in\omega} {\uparrow} d^n x$*.*

(2) *Let* $F \in \mathrm{NF}(\mathbf{A})$*. Then* $\theta(F)$ *is a congruence on* $\mathbf{A}$*.*

(3) *Let* $\alpha \in \mathrm{Con}(\mathbf{A})$*. Then* $1/\alpha$ *is a normal filter.*

Theorem 2.5 (Sankappanavar [6])**.** *Let* $\mathbf{A} \in \mathcal{H}^+$ *and let* $\theta\colon \mathrm{NF}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A})$ *be the map defined by* $F \mapsto \theta(F)$*. Then* θ *is an isomorphism. Moreover,*

(1) *for all* $F \in \mathrm{NF}(\mathbf{A})$ *and all* $\alpha \in \mathrm{Con}(\mathbf{A})$*, we have* $1/\theta(F) = F$ *and* $\theta(1/\alpha) = \alpha$*,*

(2) *for all* $x, y \in A$*, we have* $\mathrm{Cg}^{\mathbf{A}}(x, y) = \theta(N((x \to y) \wedge (y \to x)))$*. In particular,* $\theta(N(x)) = \mathrm{Cg}^{\mathbf{A}}(1, x)$*.*

The following lemma is a straightforward consequence of Theorem 2.5.

Lemma 2.6. *Let* $\mathbf{A}$ *be an* H^+*-algebra.*

(1) *If* $\mathbf{A}$ *is simple then, for all* $a \in A \backslash \{1\}$*, there exists* $k \in \omega$ *such that* $d^k a = 0$*.*

(2) *If* $\mathbf{A}$ *is subdirectly irreducible with monolith* μ *then, for all* $z \in 1/\mu$ *and all* $a \in A \backslash \{1\}$*, there is some* $k \in \omega$ *such that* $d^k a \leq z$*.*

3. The main result

In this section we state and prove our main reslt. We begin with four preparatory lemmas.

Definition 3.1. Let $\mathcal{V}$ be a variety with unary terms $\neg$ and $\sim$, and let $dx = \neg{\sim}x$. For all $n \in \omega$, let $\mathcal{V}_n$ denote the subvariety of $\mathcal{V}$ satisfying the equation

$$d^{n+1}x \approx d^n x.$$

Lemma 3.2. *Let* $\mathbf{A}$ *be a* H^+*-algebra and let* $x, y \in A$*. Then the following hold for all* $n \in \omega$*.*

(D1) $d^n 1 = 1, \quad d^n 0 = 0.$

(D2) $d^{n+1}x \leq d^n x.$

(D3) *if* $x \leq y$ *then* $d^n x \leq d^n y$*.*

(D4) ${\sim}x \leq d^n{\sim}d^n x.$

Proof. (D1) is obvious, and (D2) and (D3) follow since both $\neg$ and $\sim$ are order-reversing. We prove (D4) by induction. Firstly, we have ${\sim}{\sim}dx \leq dx$ and so $d{\sim}dx = \neg{\sim}{\sim}dx \geq \neg dx = \neg\neg{\sim}x \geq {\sim}x$. Now assume the inequality holds for $k \leq n$. By the inductive hypothesis we have $d^n{\sim}d^n dx \geq {\sim}dx$, and it then follows from (D3) that $d^{n+1}{\sim}d^{n+1}x = dd^n{\sim}d^n dx \geq d{\sim}dx \geq {\sim}x$. □

Lemma 3.3. *Let* $\mathbf{A} \in \mathcal{H}^+$ *and let* $n \in \mathbb{N}$. *If there exists* $b \in A\backslash\{1\}$ *such that* $d^{\lfloor \frac{n-1}{2} \rfloor}a \leq b$, *for all* $a \in A\backslash\{1\}$, *then* $\mathbf{A} \in \mathcal{H}_n^+$.

Proof. We separate the argument into cases where $n = 2k$ and $n = 2k+1$. First, assume $n = 2k$, so that $\lfloor \frac{n-1}{2} \rfloor = k - 1$. Let $b \in A\backslash\{1\}$ and assume, for all $a \in A\backslash\{1\}$, that $d^{k-1}a \leq b$. Suppose that $d^{k+1}b \neq 0$. Then ${\sim}d^k b \neq 1$, and so $d^{k-1}{\sim}d^k b \leq b$ by assumption. By (D2) we have $d^k{\sim}d^k b \leq d^{k-1}{\sim}d^k b$ and so ${\sim}b \leq b$ by (D4), which only happens if $b = 1$ which is not the case. Thus $d^{k+1}b = 0$. Then, for all $x \leq b$ we have by (D3) that $d^{k+1}x = 0$. So, in particular, $d^{k+1}d^{k-1}a = 0$, i.e., $d^{2k}a = 0 = d^{2k+1}a$. We have by (D1) that $d^{2k}1 = d^{2k+1}1 = 1$, and therefore $\mathbf{A} \in \mathcal{H}_n^+$. The argument is essentially identical for the case $n = 2k + 1$. □

For convenience we now introduce the term q, dual to d, given by $qx = {\sim}\neg x$.

Lemma 3.4. *Let* $\mathbf{A} \in \mathcal{H}^+$ *and let* $x \in A$. *For all* $n \in \omega$, *the following hold:*

(1) ${\sim}d^n x = q^n{\sim}x$ *and* $\neg q^n x = d^n\neg x$,

(2) $d^{n+1}x = \neg q^n{\sim}x$ *and* $q^{n+1}x = {\sim}d^n\neg x$,

(3) $q^n d^n x \leq x \leq d^n q^n x$.

Proof. Parts (1) and (2) are obvious. For part (3), we proceed via induction. Firstly, we have $qdx = {\sim}\neg\neg{\sim}x$. Since $\neg\neg{\sim}x \geq {\sim}x$ we then have ${\sim}\neg\neg{\sim}x \leq {\sim}{\sim}x \leq x$ and the inequality holds for $n = 1$. Now let $n > 1$ and assume the inequality holds for all $k \leq n$. By the inductive hypothesis we have $qdd^n x \leq d^n x$. It then follows that $q^{n+1}d^{n+1}x = q^n qdd^n x \leq q^n d^n x$, and once again by the inductive hypothesis we have $q^n d^n x \leq x$, so the inequality holds. A dual argument holds for the remainder of the inequality. □

Lemma 3.5. *Let* $\mathbf{A} \in \mathcal{H}_+$ *be simple and assume that* $\mathbf{A} \notin \mathcal{H}_n^+$, *for some* $n \in \mathbb{N}$. *Then there exists* $p \in A\backslash\{0, 1\}$ *such that* $d^{k+1}{\sim}d^{k-1}p = 0$, *for all* $k \in \{1, \ldots, n\}$.

Proof. Since $\mathbf{A} \notin \mathcal{H}_n^+$, there exists some $x \in A$ falsifying $d^n x = d^{n+1} x$. Since $\mathbf{A}$ is simple, by Lemma 2.6, there exists $m \geq n$ such that $d^m x \neq 0$ and $d^{m+1} x = 0$. Let $a = d^{m-n} x$. Then $d^n a \neq 0$ and $d^{n+1} a = 0$. Let $p = \neg d^n a$. Since $d^n a \neq 0$ we have $p \neq 1$. By Lemma 3.4, we have $p = \neg d^n a = \neg\neg q^{n-1} {\sim} a \geq q^{n-1} {\sim} a = {\sim} d^{n-1} x$. We cannot have ${\sim} d^{n-1} x = 0$ as otherwise $d^{n-1} x = 1$, contradicting $d^n x \neq d^{n+1} x$, and so $p > 0$.

Now let $k \in \{1, \ldots, n\}$. We then have

$$\begin{aligned} d^{k+1} {\sim} d^{k-1} p &= d^{k+1} q^{k-1} {\sim} p && \text{by Lemma 3.4(1)} \\ &= d^{k+1} q^{k-1} {\sim} \neg d^n a && \text{as } p = \neg d^n a \\ &= d^{k+1} q^{k-1} q d^n a && \text{as } qx = {\sim}\neg x \\ &= d^{k+1} q^k d^n a \\ &= d^{k+1} q^k d^k d^{n-k} a. \end{aligned}$$

From Lemma 3.4(3) we have $q^k d^k d^{n-k} a \leq d^{n-k} a$ and hence

$$d^{k+1} q^k d^k d^{n-k} a \leq d^{n+1} a = 0,$$

as required. □

We are now equipped to prove the main result of this paper.

Theorem 3.6. *Let $\mathcal{V}$ be a variety of H^+-algebras. Then the following are equivalent.*

(1) *$\mathcal{V}$ is a discriminator variety,*

(2) *$\mathcal{V}$ is semisimple,*

(3) *$\mathcal{V} \subseteq \mathcal{H}_n^+$, for some $n \in \omega$,*

(4) *$\mathcal{V}$ has DPC,*

(5) *$\mathcal{V}$ has EDPC.*

Proof. (1) $\implies$ (2): This is a known result. See Werner [8].

(2) $\implies$ (3): Suppose $\mathcal{V}$ is semisimple but for all $n \in \omega$ we have $\mathcal{V} \nsubseteq \mathcal{H}_n^+$. Then there exists a sequence $\{\mathbf{A}_i\}_{i \in \omega} \subseteq \mathcal{V}$ such that each $\mathbf{A}_i$ is subdirectly irreducible and $\mathbf{A}_i \notin \mathcal{H}_i^+$. Furthermore, since $\mathcal{V}$ is semisimple, each $\mathbf{A}_i$ is simple and so, by Lemma 3.5, for each $i > 0$ there exists $p_i \in A_i \backslash \{0, 1\}$ such that $d^{k+1} {\sim} d^{k-1} p_i = 0_i$ for each $k \in \{1, \ldots, i\}$.

Take an ultraproduct $\mathbf{A} = \prod_{i \in \mathbb{N}} \mathbf{A}_i/U$ by some non-principal ultrafilter U on $\mathbb{N}$. Let $p = \langle p_i \mid i \in \mathbb{N} \rangle/U$ and let $\alpha = \mathrm{Cg}^{\mathbf{A}}(1, p)$. It is an easy consequence of Zorn's Lemma that since α is compact, there is at least one element $\beta \in \mathrm{Con}(\mathbf{A})$ such that α covers β. Let $\Gamma = \{\gamma \in \mathrm{Con}(\mathbf{A}) \mid \gamma \geq \beta \text{ and } \gamma \ngeq \alpha\}$. It follows from the compactness of α and congruence distributivity that $\bigvee \Gamma \in \Gamma$. Let $\eta = \bigvee \Gamma$. It is easy to see that $\mathbf{A}/\eta$ is subdirectly irreducible, and is consequently simple by the semisimplicity of $\mathcal{V}$.

Since $\mathbf{A}/\eta$ is simple, we have that $\alpha \vee \eta = 1$ in $\mathrm{Con}(\mathbf{A})$. By congruence permutability we then have that $\alpha \vee \eta = \eta \circ \alpha$. Then in particular $(0, 1) \in \eta \circ \alpha$ and so there exists some $c \in A$ such that $(0, c) \in \eta$ and $(c, 1) \in \alpha$. It follows that $(1, {\sim}c) \in \eta$. Since $\alpha = \mathrm{Cg}^{\mathbf{A}}(1, p)$, from Theorem 2.5 we have that $1/\alpha = N(p)$. We then have for some fixed $k > 0$ that $c \geq d^{k-1}p$. Recall that for each $i \geq k$ we have $d^{k+1}{\sim}d^{k-1}p_i = 0_i$, and so $d^{k+1}{\sim}d^{k-1}p = 0$ in the ultraproduct. Then from $c \geq d^{k-1}p$ we have ${\sim}c \leq {\sim}d^{k-1}p$ and so from (D3) we have $d^{k+1}{\sim}c \leq d^{k+1}{\sim}d^{k-1}p = 0$. Then $0 \in N({\sim}c)$ and hence $(0, 1) \in \mathrm{Cg}^{\mathbf{A}}(1, {\sim}c)$. But then since $(1, {\sim}c) \in \eta$ it follows that η is the full congruence on $\mathbf{A}$, contradicting the assumption that $\eta \ngeq \alpha$. Hence we must have $\mathcal{V} \in \mathcal{H}_n^+$, for some $n \in \omega$.

(3) $\implies$ (1): Sankappanavar [6, p. 413] proved that for all $n \in \omega$,

$$t(x, y, z) = [z \wedge d^n((x \vee y) \to (x \wedge y))] \vee [x \wedge \neg d^n((x \vee y) \to (x \wedge y))]$$

is the discriminator on $\mathcal{H}_n^+$.

(1) $\implies$ (5): If $t(x, y, z)$ is a discriminator term for $\mathcal{V}$ then for all $\mathbf{A} \in \mathcal{J}$ and all $a, b, c, d \in \mathbf{A}$ we have $(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$ if and only if $t(a, b, c) = t(a, b, d)$. See [1, p. 201].

(5) $\implies$ (4): This follows immediately.

(4) $\implies$ (3): Let $\mathcal{V}$ have DPC and suppose $\mathcal{V} \nsubseteq \mathcal{H}_n^+$ for all $n \in \omega$. Then there exists a sequence $\{\mathbf{A}_i\}_{i \in \mathbb{N}} \subseteq \mathcal{V}$ such that each $\mathbf{A}_i$ is subdirectly irreducible, but $\mathbf{A}_i \notin \mathcal{H}_i^+$. Let $\mathbf{A} = \prod_{i \in \mathbb{N}} \mathbf{A}_i/U$ be an ultraproduct for some non-principal ultrafilter U on $\mathbb{N}$.

Since $\mathcal{V}$ has DPC, subdirect irreducibility is a first-order property and so $\mathbf{A}$ is subdirectly irreducible. Let μ be its monolith. By Lemma 2.6, for all $a \in A \backslash \{1\}$ and all $b \in 1/\mu$, there is some $k \in \omega$ with $d^k a \leq b$.

Let μ_n denote the monolith for $\mathbf{A}_n$ and consider any sequence $\{b_n\}_{n \in \mathbb{N}}$ such that each $b_n \in (1/\mu_n) \backslash \{1\}$. Let $\bar{b} = \langle b_n \mid n \in \mathbb{N} \rangle/U$. It follows from

DPC and properties of ultraproducts that $\bar{b} \in 1/\mu$ and so $\bar{b}$ satisfies the property of Lemma 2.6(2):

$$(\forall a \in A\backslash\{1\})(\exists k \in \omega)\ d^k a \leq \bar{b}.$$

We now construct an $a \in A\backslash\{1\}$ that does not satisfy this inequality. As $\mathbf{A}_n \notin \mathcal{H}_n^+$, by Lemma 3.3 there exists $a_n \in A_n$ such that $d^{\lfloor \frac{n-1}{2} \rfloor} a_n \nleq b_n$. By construction, $\bar{a} \neq 1$ and so by Lemma 2.6 there exists some $k \in \omega$ with $d^{k+1}\bar{a} \leq \bar{b}$. But for every $m > 2k+1$, we have $d^k a_m \nleq b_m$ as otherwise $d^{\lceil \frac{m-1}{2} \rceil} a_m \leq b_m$. So we must have in the ultraproduct that $d^k \bar{a} \nleq \bar{b}$, contradicting Lemma 2.6. Thus there exists $n \in \omega$ such that $\mathcal{V} \subseteq \mathcal{H}_n^+$. □

Our main result now follows from Theorem 2.2.

Corollary 3.7. *Let $\mathcal{V}$ be a variety of double-Heyting algebras. Then the following are equivalent.*

(1) *$\mathcal{V}$ is a discriminator variety,*

(2) *$\mathcal{V}$ is semisimple,*

(3) *$\mathcal{V} \subseteq \mathcal{DH}_n$ for some $n \in \omega$,*

(4) *$\mathcal{V}$ has DPC,*

(5) *$\mathcal{V}$ has EDPC.*

4. Concluding remarks

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \neg, \sim, 0, 1\rangle$ is called a (*distributive*) *double p-algebra* if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded (distributive) lattice and $\neg$ and $\sim$ are pseudocomplement and dual pseudocomplement operations respectively. Recall that an algebra is *regular* if, whenever two congruences share a class, they are in fact the same congruence. Varlet [7] has given an equational characterisation of regular double p-algebras. Furthermore, a result of Katriňák [3] has shown that every regular double p-algebra is term-equivalent to a double-Heyting algebra via the term

$$x \to y = \neg\neg(\neg x \vee \neg\neg y) \wedge (\sim(x \vee \neg x) \vee \neg x \vee y \vee \neg y),$$

and its dual. Let $\mathcal{R}$ denote the variety of regular double p-algebras.

Corollary 4.1. *Let $\mathcal{V}$ be a variety of regular double p-algebras. Then the following are equivalent.*

(1) *$\mathcal{V}$ is a discriminator variety,*

(2) *$\mathcal{V}$ is semisimple,*

(3) *$\mathcal{V} \subseteq \mathcal{R}_n$ for some $n \in \omega$,*

(4) *$\mathcal{V}$ has DPC,*

(5) *$\mathcal{V}$ has EDPC.*

Bearing in mind that regular double p-algebras can be treated as double-Heyting algebras, and double-Heyting algebras can be treated as H^+-algebras, we conclude by observing that, for $n \in N$, the classes $\mathcal{R}_n$ are not finitely generated, which then extends to $\mathcal{DH}_n$ and $\mathcal{H}_n^+$. Note that $\mathcal{R}_0$ is the class of boolean algebras and is therefore finitely generated. Let $\mathbf{B}$ be any infinite boolean algebra and let $\mathbf{B}^\top$ denote the double p-algebra obtained by affixing a new top element to $\mathbf{B}$, say $\top$. In [7], Varlet proved that a double p-algebra $\mathbf{A}$ is regular if and only if for all $x, y \in A$ the inequality ${\sim}x \wedge x \leq y \vee \neg y$ is satisfied. It is routine to verify that $\mathbf{B}^\top$ satisfies this inequality, and thus $\mathbf{B}^\top$ forms a regular double p-algebra. Moreover, the new top element is join-irreducible and so for all $x \in B^\top \backslash \{\top\}$ we have ${\sim}x = \top$, and hence $dx = 0$. It then follows that $\mathbf{B}^\top$ is simple, and that, for all $n \geq 1$, we have $\mathbf{B}^\top \in \mathcal{R}_1 \subseteq \mathcal{R}_n$. Since $\mathbf{R}$ is congruence distributive, it follows that the classes $\mathcal{R}_n$ are not finitely generated for each $n \geq 1$.

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