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ON EVERYWHERE STRONGLY LOGIFIABLE ALGEBRAS

A b s t r a c t. We introduce the notion of an everywhere strongly
logifiable algebra: a finite non-trivial algebra $\boldsymbol{A}$ such that for every
$F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$ the logic determined by the matrix $\langle \boldsymbol{A}, F \rangle$
is a strongly algebraizable logic with equivalent algebraic seman-
tics the variety generated by $\boldsymbol{A}$. Then we show that everywhere
strongly logifiable algebras belong to the field of universal alge-
bra as well as to the one of logic by characterizing them as the
finite non-trivial simple algebras that are constantive and generate
a congruence distributive and n-permutable variety for some $n \geqslant 2$.
This result sets everywhere strongly logifiable algebras surprisingly
close to primal algebras. Nevertheless we shall provide examples
of everywhere strongly logifiable algebras that are not primal.
Finally, some conclusion on the problem of determining whether
the equivalent algebraic semantics of an algebraizable logic is
a variety is obtained.

1. Introduction

In the late 80's Blok and Pigozzi introduced the theory of algebraizability as a uniform framework for the algebraic approach to the analysis of propositional logics [4]. In general the equivalent algebraic semantics $\mathsf{Alg}^*\mathcal{L}$ of an algebraizable logic $\mathcal{L}$ is a generalized quasi-variety. However, most of the well-known algebraizable logics have an equivalent algebraic semantics that is a variety. This posed the natural question, sometimes called in the literature *the variety problem* (see for example [10]), of explaining this phenomenon by finding some meaningful sufficient conditions under which the equivalent algebraic semantics of an algebraizable logic is a variety. Accordingly, a logic $\mathcal{L}$ is called *strongly algebraizable* if it is algebraizable and $\mathsf{Alg}^*\mathcal{L}$ is a variety. Several advances in the study of the variety problem and strongly algebraizable logics have been done by Czelakowski, Pigozzi and Jansana [7, 8, 20]. In fact the variety problem can be formulated also outside the landscape of algebraizable logics, by requiring that the *algebraic counterpart* (instead of the equivalent algebraic semantics) of a certain logic is a variety, and in this second version it has been studied by Font and Jansana in [11, 18, 19].

This paper is a contribution to the study of the variety problem from the other side of the bridge: building on the intuition that a well-behaved algebraizable logic must be strongly algebraizable, we propose to investigate the finite algebras that behave in the best possible way according to this criterion. More precisely, we say that a finite non-trivial algebra $\boldsymbol{A}$ is *strongly logifiable* when there is a matrix $\langle \boldsymbol{A}, F \rangle$ that determines a strongly algebraizable logic with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$. Observe that in this case $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$, since $\mathbb{V}(\boldsymbol{A})$ is a non-trivial variety. The notion of a strongly logifiable algebra can be strengthened as follows: a finite non-trivial algebra $\boldsymbol{A}$ is *everywhere strongly logifiable* when the logic determined by the matrix $\langle \boldsymbol{A}, F \rangle$ is strongly algebraizable with equivalent algebraic semantics the variety $\mathbb{V}(\boldsymbol{A})$, for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$. The main goal of this paper is to show that the concept of an everywhere strongly logifiable algebra can be characterized by means of purely algebraic conditions. In particular, the central result (Theorem 3.6) states that a finite non-trivial algebra is everywhere strongly logifiable if and only if it is simple, constantive and generates a congruence distributive and n-permutable variety for some $n \geqslant 2$. Drawing consequences from this

result we obtain a solution to the variety problem for logics determined by matrices of the form $\langle \boldsymbol{A}, \{a\} \rangle$, where $\boldsymbol{A}$ is a finite non-trivial constantive algebra (Corollary 3.9).

Due to the very demanding definition of everywhere strongly logifiable algebras, it is natural to wonder whether they exist or not. This is indeed the case, since from the above algebraic characterization it follows that primal algebras are always everywhere strongly logifiable (Lemma 3.2). This fact may not surprise since, loosely speaking, primal algebras are the finite algebras that behave in the best possible way from the point of view of universal algebra, while everywhere strongly logifiable algebras do the same from the point of view of algebraizability theory. In fact, in congruence permutable varieties these two concepts coincide (Lemma 4.1). Nevertheless, this is not true in general: we will see that for every $n \geqslant 3$ there is an everywhere strongly logifiable algebra of n elements that is not primal (Example 4.3).

Since the characterization of everywhere strongly logifiable algebras relies both on results from abstract algebraic logic and universal algebra, we found useful to organize the paper as follows. In Section 2 we introduce the necessary machinery from algebraizability theory, universal algebra and tame congruence theory. Then in Section 3 we develop the algebraic characterization of everywhere strongly logifiable algebras. Finally in Section 4 we investigate the relations that hold between primal and everywhere strongly logifiable algebras and conclude with an open question (Problem 4.1).

2. Preliminaries

All definitions and results mentioned in this section are standard and can be found in the literature. In particular for abstract algebraic logic and algebraizability theory we refer the reader to [3, 4, 6, 12, 14, 15] and for universal algebra to [1, 5, 16, 17, 21, 23]. We begin by algebraizability theory. Fixed an algebraic type $\mathscr{L}$, we denote by Fm the set of formulas over it built up with countably many variables and by $\boldsymbol{Fm}$ the corresponding term algebra. Then a **logic** $\mathcal{L}$ is a closure operator $C_{\mathcal{L}} \colon \mathcal{P}(Fm) \to \mathcal{P}(Fm)$ such that $\sigma C_{\mathcal{L}}(\Gamma) \subseteq C_{\mathcal{L}}\sigma(\Gamma)$ for every $\Gamma \subseteq Fm$ and every endomorphism (or, equivalently, substitution) $\sigma \colon \boldsymbol{Fm} \to \boldsymbol{Fm}$. Given $\Gamma \cup \{\varphi\} \subseteq Fm$, we

write $\Gamma \vdash_{\mathcal{L}} \varphi$ in case $\varphi \in C_{\mathcal{L}}(\Gamma)$. A formula $\varphi \in Fm$ is a **theorem** of $\mathcal{L}$ if $\emptyset \vdash_{\mathcal{L}} \varphi$. From now on we will assume that we are working within a fixed algebraic type, unless explicitly warned. Moreover, we denote by Eq the set of equations over this language (equations are just pairs of formulas but are written in the more suggestive notation $\alpha \approx \beta$). We will denote algebras with italic boldface capital letters $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, etc. (with universes A, B, C, etc. respectively). Sometimes we will skip assumptions like "let $\boldsymbol{A}$ be an algebra" in the formulation of our results.

A natural way of constructing algebra-based semantics for a given logic is to consider the elements of the algebras as truth values and select some of them as representing logical truth. More precisely, given a logic $\mathcal{L}$ and an algebra $\boldsymbol{A}$, we say that a set $F \subseteq A$ is a **deductive filter** of $\mathcal{L}$ over $\boldsymbol{A}$ when

$$\text{if } \Gamma \vdash_{\mathcal{L}} \varphi, \text{then for every homomorphism } h \colon \boldsymbol{Fm} \to \boldsymbol{A}$$
$$\text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. We denote by $\mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$ the set of deductive filters of $\mathcal{L}$ over $\boldsymbol{A}$, which turns out to be a complete lattice when ordered under the inclusion relation. A pair $\langle \boldsymbol{A}, F \rangle$ is a **matrix** when $\boldsymbol{A}$ is an algebra and $F \subseteq A$, and a matrix $\langle \boldsymbol{A}, F \rangle$ is a **model** of $\mathcal{L}$ when $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.

Congruences of $\boldsymbol{A}$ are associated to matrices of the form $\langle \boldsymbol{A}, F \rangle$ in a way independent from any logic. Given an algebra $\boldsymbol{A}$ we will denote by $\mathrm{Con}\boldsymbol{A}$ its lattice of congruences. Also we will denote by Δ_A and ∇_A respectively the identity and the total congruences on $\boldsymbol{A}$ (when no confusion shall occur, we will omit the indexes). Then $\theta \in \mathrm{Con}\boldsymbol{A}$ is **compatible** with the set F when

$$\text{if } a \in F \text{ and } \langle a, b \rangle \in \theta, \text{ then } b \in F$$

for every $a, b \in A$. It is easy to prove that given any $F \subseteq A$, the largest congruence of $\boldsymbol{A}$ compatible with F exists. This congruence is denoted by $\boldsymbol{\Omega^A} F$ and called the **Leibniz congruence** of F over $\boldsymbol{A}$. A map $f \colon A^n \to A$ is a **polynomial function of** A if there are a natural number m, a term $\varphi(x_1, \ldots, x_{n+m})$ and elements $b_1, \ldots, b_m \in A$ such that

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n, b_1, \ldots, b_m) = f(a_1, \ldots, a_n)$$

for every $a_1, \ldots, a_n \in A$. Observe that the notation $\varphi(x_1, \ldots, x_{n+m})$ means just that the variables really occurring in φ are among (but do not necessarily

exhaust) $x_1, \ldots, x_{n+m}$. We denote by $\mathrm{Pol}(\boldsymbol{A})$ the set of polynomial functions of $\boldsymbol{A}$ and by $\mathrm{Pol}_n(\boldsymbol{A})$ its set of n-ary polynomial functions. It turns out that the Leibniz congruence can be characterized by means of polynomial functions. More precisely, given an algebra $\boldsymbol{A}$ and a set $F \subseteq A$, we have that for every $a, b \in A$:

$$\langle a, b \rangle \in \boldsymbol{\Omega^A} F \Longleftrightarrow (p(a) \in F \text{ iff } p(b) \in F) \text{ for every } p \in \mathrm{Pol}_1(\boldsymbol{A}).$$

A matrix $\langle \boldsymbol{A}, F \rangle$ is **reduced** when $\boldsymbol{\Omega^A} F = \Delta$. Observe that if $\boldsymbol{A}$ is simple and non-trivial, then the matrix $\langle \boldsymbol{A}, F \rangle$ is reduced if and only if $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$. The Leibniz congruence commutes with inverse images of epimorphisms in the sense that if $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is an epimorphism and $F \subseteq B$, then $\boldsymbol{\Omega^A} h^{-1}[F] = h^{-1} \boldsymbol{\Omega^B} F$. Given a matrix $\langle \boldsymbol{A}, F \rangle$, we define $\langle \boldsymbol{A}, F \rangle^* := \langle \boldsymbol{A}/\boldsymbol{\Omega^A} F, F/\boldsymbol{\Omega^A} F \rangle$. The commutation property of the Leibniz congruence implies that the matrix $\langle \boldsymbol{A}, F \rangle^*$ is always reduced. Remarkably, the matrices $\langle \boldsymbol{A}, F \rangle$ and $\langle \boldsymbol{A}, F \rangle^*$ determine the same logic.

The definition of the Leibniz congruence gives rise to a map $\boldsymbol{\Omega^A} \colon \mathcal{P}(A) \to \mathrm{Con}\boldsymbol{A}$, called the **Leibniz operator**, whose behaviour over deductive filters of the logic captures interesting facts concerning the definability of truth and that of equivalence in selected classes of logics; this is one of the central topics studied in abstract algebraic logic and has given rise to the so-called **Leibniz hierarchy**. However, for the moment, it is enough to keep in mind the fact that the Leibniz congruence allows to associate with a logic $\mathcal{L}$ a special class of models and a special class of algebras:

$$\mathsf{Mod}^* \mathcal{L} := \big\{ \langle \boldsymbol{A}, F \rangle : F \in \mathcal{F}i_{\mathcal{L}} \boldsymbol{A} \text{ and } \boldsymbol{\Omega^A} F = \Delta \big\}$$
$$\mathsf{Alg}^* \mathcal{L} := \big\{ \boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{L}} \boldsymbol{A} \text{ such that } \boldsymbol{\Omega^A} F = \Delta \big\}.$$

That is, $\mathsf{Alg}^* \mathcal{L}$ is the collection of the algebraic reducts of the matrices in $\mathsf{Mod}^* \mathcal{L}$. It is worth remarking that $\mathcal{L}$ is always complete with respect to $\mathsf{Mod}^* \mathcal{L}$. Moreover, if $\mathcal{L}$ is the logic determined by a single matrix $\langle \boldsymbol{A}, F \rangle$, then $\mathsf{Alg}^* \mathcal{L} \subseteq \mathbb{V}(\boldsymbol{A})$.

Along the paper we will be particularly interested in the notion of an algebraizable logic, due to Blok and Pigozzi [4]. In order to introduce it, we need to recall some preliminary definitions. A map $\boldsymbol{\tau} \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ is a **structural transformer** (from formulas to equations) when there is a set $E(x)$ of equations in a single variable x such that for all $\Gamma \subseteq Fm$,

$$\boldsymbol{\tau}(\Gamma) = \big\{ \sigma_\varphi \alpha \approx \sigma_\varphi \beta : \alpha \approx \beta \in E(x) \text{ and } \varphi \in \Gamma \big\}$$

where $\sigma_\varphi \colon \boldsymbol{Fm} \to \boldsymbol{Fm}$ is any substitution sending the variable x to φ. It is easy to see that this is equivalent to requiring that $\boldsymbol{\tau} \colon \mathcal{P}(Fm) \to \mathcal{P}(Eq)$ commutes with arbitrary unions and with substitutions, that is, it is residuated and commutes with substitutions. Thus, symmetrically, we say that a map $\boldsymbol{\rho} \colon \mathcal{P}(Eq) \to \mathcal{P}(Fm)$ is a structural transformer (from equations into formulas) when it is residuated and commutes with substitutions.

A class of algebras K is a ***generalized quasi-variety*** if it is axiomatized by generalised quasi-equations, i.e., quasi-equations where the antecedent is the conjunction of a possibly infinite set of equations. Let $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, $\mathbb{P}_\mathrm{U}$ and $\mathbb{P}_\mathrm{SD}$ be respectively the usual isomorphic images, subalgebras, direct products, ultraproducts and subdirect products operators. Moreover, given a class of algebras K, define

$$\mathbb{U}(\mathsf{K}) \coloneqq \{\boldsymbol{A} : \text{Every countably generated subalgebra } \boldsymbol{B}$$
$$\text{of } \boldsymbol{A} \text{ belongs to } \mathsf{K}\}.$$

The restriction to *countable* sets in the definition of $\mathbb{U}(\cdot)$ depends on our choice to work with a countable set (rather than a proper class) of variables. It turns out [2] that a class of algebras K is a generalized quasi-variety if and only if it is closed under $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{U}$. In particular, this implies that generalized quasi-varieties are closed under $\mathbb{P}_\mathrm{SD}$. Moreover, given a class of algebras K, the generalized quasi-variety $\mathbb{GQ}(\mathsf{K})$ generated by K coincides with the class $\mathbb{UISP}(\mathsf{K})$. It is useful to observe that if K is a finite set of finite algebras, then $\mathbb{GQ}(\mathsf{K})$ coincides with the quasi-variety $\mathbb{Q}(\mathsf{K})$ generated by K. Generalized quasi-varieties need not to be closed under the formation of homomorphic images. This makes the following device useful. Given a class of algebras K and an algebra $\boldsymbol{A}$, the set of congruences of $\boldsymbol{A}$ that yield a quotient in K is denoted by $\mathrm{Con}_\mathsf{K}\boldsymbol{A}$. It is easy to prove that if K is a generalized quasi-variety, then $\mathrm{Con}_\mathsf{K}\boldsymbol{A}$ is a complete lattice, whose arbitrary meets coincide with those of $\mathrm{Con}\boldsymbol{A}$. If K is a (generalized) quasi-variety, we denote by K_rfsi the class of relatively finitely subdirectly irreducible members of K. It is well known that

$$\mathbb{Q}(\mathsf{K})_\mathrm{rfsi} \subseteq \mathbb{ISP}_\mathrm{U}(\boldsymbol{A}).$$

Even if the theory of algebraizability is in general concerned with generalized quasi-varieties, we will focus on varieties as well. Therefore some more notation will be useful. Given a class of algebras K, we denote by K_si

the class of subdirectly irreducible members of K and by $\mathbb{V}(\mathsf{K})$ the variety generated by K. Finally, the **equational consequence relative to** K is the consequence relation over the set of equations defined as follows:

$$\Theta \vDash_\mathsf{K} \varphi \approx \psi \Longleftrightarrow \text{ for every } \boldsymbol{A} \in \mathsf{K} \text{ and homomorphism } h\colon \boldsymbol{Fm} \to \boldsymbol{A}$$
$$\text{if } h(\alpha) = h(\beta) \text{ for every } \alpha \approx \beta \in \Theta, \text{ then } h(\varphi) = h(\psi)$$

for every $\Theta \cup \{\varphi \approx \psi\} \subseteq Eq$. Since $\vDash_\mathsf{K}$ corresponds to the validity of generalised quasi-equations in K, it is easy to see that the relations $\vDash_\mathsf{K}$ and $\vDash_{\mathbb{GQ}(\mathsf{K})}$ coincide.

A logic $\mathcal{L}$ is **algebraizable** with **equivalent algebraic semantics** the generalised quasi-variety K when there are two structural transformers $\boldsymbol{\tau}\colon \mathcal{P}(Fm) \longleftrightarrow \mathcal{P}(Eq)\colon \boldsymbol{\rho}$ satisfying the following conditions:

A1. $\Gamma \vdash_\mathcal{L} \varphi$ if and only if $\boldsymbol{\tau}(\Gamma) \vDash_\mathsf{K} \boldsymbol{\tau}(\varphi)$;

A2. $x \approx y \vDash\!\vdash_\mathsf{K} \boldsymbol{\tau\rho}\,(x \approx y)$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$ and for $x \approx y \in Eq$. It is worth to remark that the notion of an algebraizable logic could be defined equivalently by means of conditions dual to A1 and A2. More precisely, we have that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics K if and only if

A3. $\Theta \vDash_\mathsf{K} \varphi \approx \psi$ if and only if $\boldsymbol{\rho}(\Theta) \vdash_\mathcal{L} \boldsymbol{\rho}(\varphi, \psi)$;

A4. $x \dashv\vdash_\mathcal{L} \boldsymbol{\rho\tau}(x)$

for every $\Theta \cup \{\varphi \approx \psi\} \subseteq Eq$. For example, the intuitionistic propositional calculus is algebraizable with equivalent algebraic semantics the variety of Heyting algebras through the structural transformers $\boldsymbol{\tau}(x) = \{x \approx 1\}$ and $\boldsymbol{\rho}(x, y) = \{x \to y, y \to x\}$.

It is possible to see that if $\mathcal{L}$ is algebraizable, then $\mathsf{K} = \mathsf{Alg}^*\mathcal{L}$ and therefore $\mathsf{Alg}^*\mathcal{L}$ is a generalized quasi-variety. This is not true for arbitrary logics: for example $\mathsf{Alg}^*\mathcal{CPC}_{\wedge\vee}$, where $\mathcal{CPC}_{\wedge\vee}$ is the $\{\wedge, \vee\}$-fragment of classical propositional logic, is not a generalized quasi-variety [13]. Another remarkable feature of algebraizable logics is that they have theorems. Keep in mind these two properties of algebraizable logics, since we will use them in the proof of Theorem 3.6.

The heart of the theory of algebraizability lies in a correspondence between deductive filters and congruences over arbitrary algebras. More precisely, we have the following:

Theorem 2.1. *Let $\mathcal{L}$ be a logic and K a generalized quasi-variety. The following conditions are equivalent:*

(i) *$\mathcal{L}$ is algebraizable with equivalent algebraic semantics K.*

(ii) *For every algebra $\boldsymbol{A}$ there is an isomorphism of complete lattices $\Phi^{\boldsymbol{A}}\colon \mathcal{F}i_{\mathcal{L}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ such that $\Phi^{\boldsymbol{A}}g^{-1}[F] = g^{-1}\Phi^{\boldsymbol{A}-1}(F)$ for every endomorphism g of $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{L}}\boldsymbol{A}$.*

It is worth remarking that the isomorphism $\Phi^{\boldsymbol{A}}$ of condition (ii) can be taken to be the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ restricted to the deductive filters over $\boldsymbol{A}$. The correspondence between deductive filters and congruences typical of algebraizable logics $\mathcal{L}$ is actually equivalent to the fact that the Leibniz congruence and the truth predicates, i.e., the filter components of the matrices, in $\mathsf{Mod}^{*}\mathcal{L}$ admit a particularly nice description, as we remark below.

Theorem 2.2. *A logic $\mathcal{L}$ is algebraizable through the structural transformers $\boldsymbol{\tau}(x)$ and $\boldsymbol{\rho}(x,y)$ if and only if for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{*}\mathcal{L}$:*

1. $a = b \Longleftrightarrow \boldsymbol{\rho}^{\boldsymbol{A}}(a,b) \subseteq F$, *for every $a,b \in A$ and*

2. $F = \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(x)[\![a]\!]\}$.

Now we turn to review the tools from universal algebra we will make use of. Along the paper we will consider several properties of congruence lattices of algebras. Let us briefly recall some definitions and basic properties. Let V be a variety. We say that V is ***congruence distributive*** (resp. ***modular***) if for every $\boldsymbol{A} \in \mathsf{V}$, the lattice $\mathrm{Con}\boldsymbol{A}$ is distributive (resp. modular). Moreover, given $n \geqslant 2$, we say that V is n-***permutable*** if for every $\boldsymbol{A} \in \mathsf{V}$ and every $\phi, \eta \in \mathrm{Con}\boldsymbol{A}$

$$\phi \vee \eta = \theta_1 \circ \cdots \circ \theta_n \text{ where } \theta_i = \begin{cases} \phi & \text{if } i \text{ is even} \\ \eta & \text{otherwise.} \end{cases}$$

Here $\circ$ denotes the usual relational product. Observe that for $n = 2$ the n-permutability coincides with the usual permutability of congruences. We say that V is ***arithmetical*** if it is congruence distributive and permutable. Finally, we say that V is ***point regular*** if there is a term 1 that is constant in V and for every $\boldsymbol{A} \in \mathsf{V}$ and $\theta, \phi \in \mathrm{Con}\boldsymbol{A}$

$$\text{if } 1/\theta = 1/\phi, \text{ then } \theta = \phi.$$

These concepts are related as follows:

Theorem 2.3. *Point regular varieties are congruence modular and n-permutable for some $n \geqslant 2$.*

A finite algebra $\boldsymbol{A}$ is ***primal*** if for every finitary function $f \colon A^n \to A$ with $n \geqslant 1$, there is a term $\varphi(x_1, \ldots, x_n)$ which represents f in the sense that

$$f(a_1, \ldots, a_n) = \varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$$

for every $a_1, \ldots, a_n \in A$. Primal algebras can be characterized by means of familiar algebraic conditions (see for example [1]):

Theorem 2.4. *A finite algebra is primal if and only if it is simple, has no proper subalgebras, has no automorphism except the identity map, and generates an arithmetical variety.*

We will be interested in algebras that have a name for each of their elements. Formally speaking, an algebra $\boldsymbol{A}$ is ***constantive*** if for every $a \in A$ there is an (at most unary) term $c_a(x)$ that represents the map constantly equal to a. Primal algebras are examples of constantive algebras. Moreover, constantive algebras enjoy at least two of the four conditions that characterize primality in Theorem 2.4, since they have no proper subalgebra and no automorphism except the identity map. Dealing with special constantive algebras, we will need the following result of commutator theory [16].

Theorem 2.5. *If $\boldsymbol{A}$ is finite, simple, without proper subalgebras and $\mathbb{V}(\boldsymbol{A})$ is minimal and congruence modular, then $\mathbb{V}(\boldsymbol{A})$ is congruence distributive.*

Now we turn to review the basic concepts of tame congruence theory we will make use of. The reader familiar with the topic may safely choose to proceed directly to next section. For a systematic exposition of tame congruence theory we refer to the monograph of Hobby and McKenzie [17] in which most of the techniques of tame congruence theory were introduced for the first time. A finite non-trivial algebra $\boldsymbol{A}$ is ***minimal*** if every member of $\mathrm{Pol}_1(\boldsymbol{A})$ is either a constant map or a bijection. Prototypical examples of minimal algebras are described in the following example.

Example 2.6. It is straightforward that every two-element algebra is minimal. In order to introduce other kinds of minimal algebras, let us recall some definitions. A **permutation group** over a set A is an algebra of the form $\langle A, G \rangle$, where G is the universe of a subgroup of the symmetric group on A. In other words the basic operations of $\langle A, G \rangle$ are unary bijections that contain the identity map and are closed under composition and inverse. It is easy to see that finite permutation groups are minimal.

Finally, recall that a **vector space** over a field $\boldsymbol{F}$ can be regarded as an algebra $\boldsymbol{A} = \langle A, +, -, 0, \langle \lambda_r : r \in F \rangle \rangle$ such that $\langle A, +, -, 0 \rangle$ is an Abelian group and for every $r, s \in F$ and $a, b \in A$ the following conditions hold (where 1 is the multiplicative identity of the field):

$$\lambda_r(a + b) = \lambda_r(a) + \lambda_r(b)$$
$$\lambda_{r+s}(a) = \lambda_r(a) + \lambda_s(a)$$
$$\lambda_r \lambda_s(a) = \lambda_{r \cdot s}(a)$$
$$\lambda_1(a) = a.$$

Then it is possible to see that every finite vector space over a finite field $\boldsymbol{A}$ is a minimal algebra. This is an easy consequence of the fact that every unary polynomial function of $\boldsymbol{A}$ is of the form $\lambda_r(x) + a$ for some $r \in F$ and $a \in A$. $\square$

Surprisingly the inventory of the previous example is exhaustive and in fact even redundant. To clarify this point, recall that two algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ with the same universe (but possibly different similarity type) are **polynomially equivalent** if $\mathrm{Pol}(\boldsymbol{A}) = \mathrm{Pol}(\boldsymbol{B})$. Then it turns out that a finite non-trivial algebra is minimal if and only if it is polynomially equivalent to *exactly one* of the following kinds of algebras:

(1) A permutation group.

(2) A finite vector space over a finite field.

(3) The two-element Boolean algebra.

(4) The two element lattice.

(5) The two-element semilattice.

Accordingly to the above classification, each kind of minimal algebra is associated with a natural number from 1 to 5.

One of the main discoveries of tame congruence theory is that minimal algebras can be used to describe the local behaviour of finite algebras and that from this blow-up process, which focuses on the local components, one can deduce global properties of the algebra, i.e., properties that hold in the whole algebra. In order to describe how, we shall present a general localization construction. Suppose that we are given an arbitrary algebra $\boldsymbol{A}$ and a non-empty set $U \subseteq A$. We set

$$\mathrm{Pol}(\boldsymbol{A})|_U := \{f\!\restriction_U\colon f \in \mathrm{Pol}(\boldsymbol{A}) \text{ and } f[U^n] \subseteq U\}$$
$$\boldsymbol{A}|_U := \langle U, \mathrm{Pol}(\boldsymbol{A})|_U \rangle$$
$$\theta|_U := \theta \cap U^2 \text{ for every } \theta \in \mathrm{Con}\boldsymbol{A}.$$

The algebra $\boldsymbol{A}|_U$ is obtained by equipping U with the polynomial functions that can be reasonably restricted to it, and is called the algebra ***induced on U by $\boldsymbol{A}$***. Observe that in general $\boldsymbol{A}|_U$ is not of the same similarity type of $\boldsymbol{A}$. It is clear that if $\theta \in \mathrm{Con}\boldsymbol{A}$, then $\theta|_U \in \mathrm{Con}(\boldsymbol{A}|_U)$.

Minimal algebras hide inside arbitrary finite algebras. To see how we need to introduce some more concepts. Given a finite algebra $\boldsymbol{A}$, we say that a pair $\langle \theta, \phi \rangle$ of congruences $\theta, \phi \in \mathrm{Con}\boldsymbol{A}$ is a ***prime quotient*** if ϕ covers θ (i.e., $\theta \subsetneq \phi$ and for every $\eta \in \mathrm{Con}\boldsymbol{A}$ if $\theta \subseteq \eta \subseteq \phi$, then either $\eta = \theta$ or $\eta = \phi$). Then, given a finite algebra $\boldsymbol{A}$ and a prime quotient $\langle \theta, \phi \rangle$, we define

$$\mathrm{Sep}_{\boldsymbol{A}}(\theta, \phi) := \{f \in \mathrm{Pol}_1(\boldsymbol{A}) : f[\phi] \nsubseteq \theta\}.$$

We denote by $\mathrm{Min}_{\boldsymbol{A}}(\theta, \phi)$ the set of minimal elements of the non-empty poset

$$\langle \{f[A] : f \in \mathrm{Sep}_{\boldsymbol{A}}(\theta, \phi)\}, \subseteq \rangle.$$

The elements of $\mathrm{Min}_{\boldsymbol{A}}(\theta, \phi)$ are the $\langle \theta, \phi \rangle$-***minimal sets*** of $\boldsymbol{A}$. Finally $N \subseteq A$ is a $\langle \theta, \phi \rangle$-***trace*** if there are $U \in \mathrm{Min}_{\boldsymbol{A}}(\theta, \phi)$ and $a \in U$ such that

$$N = a/\phi|_U \quad \text{and } a/\theta|_U \subsetneq a/\phi|_U .$$

It is not difficult to see that each prime quotient has at least one trace. Moreover, each prime quotient of $\boldsymbol{A}$ is in correspondence with exactly one kind of minimal algebra. More precisely, given a finite non-trivial algebra $\boldsymbol{A}$ and a prime quotient $\langle \theta, \phi \rangle$, we have that:

P1. $\boldsymbol{A}|_N /\theta|_N$ is a minimal algebra for every $\langle \theta, \phi \rangle$-trace N.

P2. If N and M are two $\langle \theta, \phi \rangle$-traces, then $\boldsymbol{A}|_N /\theta|_N$ and $\boldsymbol{A}|_M /\theta|_M$ are isomorphic as non-indexed algebras.

In point P2 we mean that there is a bijection $h\colon N/\theta|_N \to M/\theta|_M$ and a way of indexing the operations of $\boldsymbol{A}|_N /\theta|_N$ and $\boldsymbol{A}|_M /\theta|_M$ in a way such that h becomes a real isomorphism.

It is worth remarking that the process that associates a minimal algebra to a prime quotient highly shortens for simple algebras. Suppose that $\boldsymbol{A}$ is a finite non-trivial simple algebra. Then the only prime quotient of $\boldsymbol{A}$ is $\langle \Delta, \nabla \rangle$. First observe that in this case

$$\mathrm{Sep}_{\boldsymbol{A}}(\Delta, \nabla) = \{f \in \mathrm{Pol}_1(\boldsymbol{A}) : |f[A]| > 1\}.$$

Then observe that every $U \in \mathrm{Min}_{\boldsymbol{A}}(\Delta, \nabla)$ is a $\langle \Delta, \nabla \rangle$-trace. Therefore the algebra $\boldsymbol{A}|_U$ (which is isomorphic to $\boldsymbol{A}|_U /\Delta|_U$) is up to isomorphism of non-indexed algebras the minimal algebra associated to $\langle \Delta, \nabla \rangle$. In other words, when dealing with simple algebras, one can forget about traces and work directly with minimal sets. We will make use of the following two properties of simple algebras, which can be thought respectively as *connection* and *separation* conditions:

Theorem 2.7. *Let $\boldsymbol{A}$ be a finite non-trivial simple algebra.*

1. *For every $a, b \in A$ there are $U_1, \ldots, U_n \in \mathrm{Min}_{\boldsymbol{A}}(\Delta, \nabla)$ such that $a \in U_1$, $b \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $1 \leqslant i \leqslant n - 1$.*

2. *For every different $a, b \in A$ and $U \in \mathrm{Min}_{\boldsymbol{A}}(\Delta, \nabla)$ there is $p(x) \in \mathrm{Pol}_1(\boldsymbol{A})$ such that $p[A] = U$ and $p(a) \neq p(b)$.*

Leaving aside the restricted case of simple algebras, observe that conditions P1 and P2 imply that each prime quotient $\langle \theta, \phi \rangle$ of a finite non-trivial algebra $\boldsymbol{A}$ is in correspondence with exactly one of the five types of minimal algebras and, therefore, with a natural number from 1 to 5. We call this number the **type** of $\langle \theta, \phi \rangle$ and we denote it by $\mathrm{typ}(\theta, \phi)$. This allows to associate a type also to the algebra $\boldsymbol{A}$ as follows:

$$\mathrm{typ}\{\boldsymbol{A}\} \coloneqq \{\mathrm{typ}(\theta, \phi) : \langle \theta, \phi \rangle \text{ is a prime quotient of } \boldsymbol{A}\}.$$

Finally, given a variety V, the **_type_** of the variety V is defined as:

$$\mathrm{typ}\{\mathsf{V}\} \coloneqq \bigcup\{\mathrm{typ}\{\boldsymbol{A}\} : \boldsymbol{A} \in \mathsf{V} \text{ and } \boldsymbol{A} \text{ finite}\}.$$

A lattice $\boldsymbol{L}$ is **_meet semi-distributive_** if for every $a, b, c \in L$:

$$\text{if } a \wedge b = a \wedge c, \text{ then } a \wedge b = a \wedge (b \vee c).$$

A variety V is **_congruence meet semi-distributive_** if for every $\boldsymbol{A} \in \mathsf{V}$, the lattice $\mathrm{Con}\boldsymbol{A}$ is meet semi-distributive. We will make use of the following result:

Theorem 2.8. *Let V be a locally finite variety. Then $\mathrm{typ}\{\mathsf{V}\} = \{3\}$ if and only if V is congruence meet semi-distributive and n-permutable for some $n \geqslant 2$.*

3.   The characterization result

In this section we address the main goal of the paper, as explained in the Introduction. To this end we introduce our new concept:

Definition 3.1. A finite non-trivial algebra $\boldsymbol{A}$ is **_everywhere strongly logifiable_** when the logic determined by the matrix $\langle \boldsymbol{A}, F \rangle$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$, for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$.

Before turning to characterize everywhere strongly logifiable algebras, let us observe that they exist. In fact it is easy to see that primal algebras are always of this kind, as we remark below.

Lemma 3.2. *Non-trivial primal algebras are everywhere strongly logifiable.*

Proof. Let $\boldsymbol{A}$ be a non-trivial primal algebra. Then consider $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$. We need to prove that the logic $\mathcal{L}$ determined by the matrix $\langle \boldsymbol{A}, F \rangle$ determines an algebraizable logic, whose equivalent algebraic semantics is $\mathbb{V}(\boldsymbol{A})$. In order to do this, choose any $1 \in F$ and $0 \in A \smallsetminus F$. We consider the functions $\square \colon A \to A$ and $\vartriangleleft\vartriangleright \colon A^2 \to A$ defined as

$$\square(a) \coloneqq \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{otherwise} \end{cases} \qquad a \vartriangleleft\vartriangleright b \coloneqq \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

for every $a, b \in A$. Moreover, let $1(x)$ be the map constantly equal to 1. From the primality of $\boldsymbol{A}$ it follows that $\square(x)$, $1(x)$ and $x \lhd\rhd y$ are represented by some terms (which we will denote by the same symbols). Keeping in mind that $\mathcal{L}$ is the logic of $\langle \boldsymbol{A}, F \rangle$, it is easy to check that

$$\Gamma \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \{\square(\gamma) \approx 1(\gamma) : \gamma \in \Gamma\} \vDash_{\boldsymbol{A}} \square(\varphi) \approx 1(\varphi) \tag{1}$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$. Moreover, we have that:

$$x \approx y =\!\|\!\vDash_{\boldsymbol{A}} \square(x \lhd\rhd y) \approx 1(x \lhd\rhd y). \tag{2}$$

But recall that the equational consequence relative to $\{\boldsymbol{A}\}$ coincides with the one relative to $\mathbb{GQ}(\boldsymbol{A})$. Therefore conditions (1) and (2) are stating that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics the generalized quasi-variety $\mathbb{GQ}(\boldsymbol{A})$ through the structural transformers

$$\boldsymbol{\tau}(x) = \{\square(x) \approx 1(x)\} \text{ and } \boldsymbol{\rho}(x, y) = \{x \lhd\rhd y\}.$$

Finally observe that generalized quasi-varieties are closed under the formation of subdirect products and that $\mathbb{P}_{\mathrm{SD}}(\boldsymbol{A}) = \mathbb{V}(\boldsymbol{A})$, since $\boldsymbol{A}$ is primal. Therefore we have that

$$\mathbb{V}(\boldsymbol{A}) = \mathbb{P}_{\mathrm{SD}}(\boldsymbol{A}) \subseteq \mathbb{GQ}(\boldsymbol{A}) \subseteq \mathbb{V}(\boldsymbol{A}).$$

Hence we conclude that $\mathbb{GQ}(\boldsymbol{A})$ and $\mathbb{V}(\boldsymbol{A})$ coincide and, therefore, that $\mathbb{V}(\boldsymbol{A})$ is the equivalent algebraic semantics of $\mathcal{L}$. $\square$

In order to provide a characterization of everywhere strongly logifiable algebras we will need to go through some technical results. The first one is a very general fact about algebraizable logics:

Lemma 3.3. *Let $\mathcal{L}$ be the logic determined by a reduced matrix $\langle \boldsymbol{A}, F \rangle$. If $\mathcal{L}$ is algebraizable, then its equivalent algebraic semantics is $\mathbb{GQ}(\boldsymbol{A})$.*

Proof. Suppose that $\mathcal{L}$ is algebraizable through two structural transformers $\boldsymbol{\tau}(x)$ and $\boldsymbol{\rho}(x, y)$. From the theory of algebraizable logics we know that $\mathsf{Alg}^*\mathcal{L}$ is a generalized quasi-variety. Together with the fact that $\boldsymbol{A} \in \mathsf{Alg}^*\mathcal{L}$, this implies that $\mathbb{GQ}(\boldsymbol{A}) \subseteq \mathsf{Alg}^*\mathcal{L}$. Therefore we turn to prove the other inclusion. Consider any generalized quasi-equation $\&\, \Theta \to \varphi \approx \psi$ that holds in $\boldsymbol{A}$. Recall that $\mathsf{Alg}^*\mathcal{L}$ is the equivalent algebraic semantics

of $\mathcal{L}$. Therefore we can safely apply condition A2 of the definition of algebraizability yielding that $x \approx y =\|\vDash_{\boldsymbol{A}} \boldsymbol{\tau\rho}(x, y)$ and, therefore, that

$$\boldsymbol{\tau\rho}(\Theta) \vDash_{\boldsymbol{A}} \boldsymbol{\tau\rho}(\varphi, \psi). \tag{3}$$

Now observe that $\langle \boldsymbol{A}, F \rangle \in \mathsf{Mod}^{*}\mathcal{L}$. By point 2 of Theorem 2.2, $F = \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(x)[\![a]\!]\}$. Together with the fact that $\mathcal{L}$ is the logic determined by $\langle \boldsymbol{A}, F \rangle$ and (3), this implies that

$$\boldsymbol{\rho}(\Theta) \vdash_{\mathcal{L}} \boldsymbol{\rho}(\varphi, \psi). \tag{4}$$

Applying condition A3 of the definition of algebraizability to (4), we conclude that $\Theta \vDash_{\mathsf{Alg}^{*}\mathcal{L}} \varphi \approx \psi$. But this means that $\mathsf{Alg}^{*}\mathcal{L} \subseteq \mathbb{GQ}(\boldsymbol{A})$ and, therefore, that $\mathsf{Alg}^{*}\mathcal{L} = \mathbb{GQ}(\boldsymbol{A})$. $\square$

As we are interested in *strongly* algebraizable logics, the following observation will be very useful since it applies to the case where the generalized quasi-variety and the variety generated by a given algebra coincide.

Lemma 3.4. *If $\boldsymbol{A}$ is finite without proper subalgebras and $\mathbb{V}(\boldsymbol{A}) = \mathbb{GQ}(\boldsymbol{A})$, then $\mathbb{V}(\boldsymbol{A})$ is a minimal variety.*

Proof. Observe that $\mathbb{Q}(\boldsymbol{A}) = \mathbb{V}(\boldsymbol{A})$, since $\mathbb{V}(\boldsymbol{A}) = \mathbb{GQ}(\boldsymbol{A}) \subseteq \mathbb{Q}(\boldsymbol{A}) \subseteq \mathbb{V}(\boldsymbol{A})$. Then we have that $\mathbb{V}(\boldsymbol{A})_{\mathrm{si}} \subseteq \mathbb{Q}(\boldsymbol{A})_{\mathrm{rfsi}}$. But recall that in general $\mathbb{Q}(\boldsymbol{A})_{\mathrm{rfsi}} \subseteq \mathbb{ISP}_{\mathrm{U}}(\boldsymbol{A})$. Applying in succession the fact that $\boldsymbol{A}$ is finite and that it has no proper subalgebras we obtain that

$$\mathbb{V}(\boldsymbol{A})_{\mathrm{si}} \subseteq \mathbb{Q}(\boldsymbol{A})_{\mathrm{rfsi}} \subseteq \mathbb{ISP}_{\mathrm{U}}(\boldsymbol{A}) = \mathbb{IS}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A}).$$

In particular, this implies that $\mathbb{V}(\boldsymbol{A})_{\mathrm{si}} = \mathbb{I}(\boldsymbol{A})$. Therefore $\boldsymbol{A}$ is up to isomorphism the only subdirectly irreducible member of $\mathbb{V}(\boldsymbol{A})$. Hence $\mathbb{V}(\boldsymbol{A})$ is a minimal variety. $\square$

Before moving to the main result, it is worth to remark that point regular varieties are naturally related to algebraizable logics. In particular, we will make use of the following observation.

Lemma 3.5. *Let $\mathcal{L}$ be the logic determined by a matrix of the form $\langle \boldsymbol{A}, \{1\} \rangle$, where 1 is a constant of $\boldsymbol{A}$. If $\mathsf{Alg}^{*}\mathcal{L}$ is a variety, then:*

1. $\mathsf{Alg}^{*}\mathcal{L} = \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\})$.

2. $\mathsf{Alg}^*\mathcal{L}$ *is point regular.*

3. $\mathcal{L}$ *is algebraizable.*

Proof. 1. Recall that the logic determined by $\langle \boldsymbol{A}, \{1\} \rangle^*$ coincides with $\mathcal{L}$. But this implies that $\mathsf{Alg}^*\mathcal{L} \subseteq \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\})$. Moreover $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} \in$ $\mathsf{Alg}^*\mathcal{L}$, since $\langle \boldsymbol{A}, \{1\} \rangle^*$ is reduced. In particular this means that $\mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\})$ $\subseteq \mathsf{Alg}^*\mathcal{L}$, since $\mathsf{Alg}^*\mathcal{L}$ is a variety. Therefore we are done.

2. We claim that if $\langle \boldsymbol{B}, F \rangle \in \mathsf{Mod}^*\mathcal{L}$, then $F = \{1\}$. To check this, observe that from the fact that $\mathcal{L}$ is the logic determined by $\langle \boldsymbol{A}, \{1\} \rangle$ it follows that $x, y, \varphi(x, \overline{z}) \vdash_{\mathcal{L}} \varphi(y, \overline{z})$ for every formula $\varphi(v, \overline{z})$. Then consider $\langle \boldsymbol{B}, F \rangle \in \mathsf{Mod}^*\mathcal{L}$. Clearly we have that $1^{\boldsymbol{B}} \in F$, since 1 is a theorem. Then consider any $a \in F$. We have that $\varphi(a, \overline{c}) \in F$ if and only if $\varphi(1, \overline{c}) \in F$ for every formula $\varphi(v, \overline{z})$ and sequence $\overline{c} \in B$. But this means that $p(a) \in F$ if and only if $p(1) \in F$ for every $p \in \mathrm{Pol}_1(\boldsymbol{B})$. As we mentioned, this implies that $\langle a, 1 \rangle \in \boldsymbol{\Omega}^{\boldsymbol{B}}F$. Since $\langle \boldsymbol{B}, F \rangle$ is reduced, we conclude that $a = 1$. This establishes our claim.

Now consider any algebra $\boldsymbol{B} \in \mathsf{Alg}^*\mathcal{L}$ and $\theta, \phi \in \mathrm{Con}\boldsymbol{B}$ and suppose that $1/\theta = 1/\phi$. Together with our claim, the fact that $\mathsf{Alg}^*\mathcal{L}$ is a variety implies that the matrices $\langle \boldsymbol{B}/\theta, 1/\theta \rangle$ and $\langle \boldsymbol{B}/\phi, 1/\phi \rangle$ are reduced. Then consider the projections on the quotients $\pi_\theta \colon \boldsymbol{B} \to \boldsymbol{B}/\theta$ and $\pi_\phi \colon \boldsymbol{B} \to \boldsymbol{B}/\phi$. We have that:

$$\begin{aligned} \theta &= \pi_\theta^{-1}\Delta_{B/\theta} = \pi_\theta^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}/\theta}1/\theta = \boldsymbol{\Omega}^{\boldsymbol{B}}\pi_\theta^{-1}1/\theta \\ &= \boldsymbol{\Omega}^{\boldsymbol{B}}\pi_\phi^{-1}1/\phi = \pi_\phi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}/\phi}1/\phi = \pi_\phi^{-1}\Delta_{B/\phi} \\ &= \phi. \end{aligned}$$

Hence the constant 1 witnesses the point regularity of $\mathsf{Alg}^*\mathcal{L}$.

3. Czelakowski proved in [6, Corollary 5.2.8] that if V is a point regular variety, then the logic determined by the class of matrices $\{\langle \boldsymbol{B}, \{1\} \rangle : \boldsymbol{B} \in \mathsf{V}\}$, where 1 is the constant term that witnesses the point regularity of V, is algebraizable. Now, observe that in the proof of point 2 we showed that $\mathsf{Mod}^*\mathcal{L} = \{\langle \boldsymbol{B}, \{1\} \rangle : \boldsymbol{B} \in \mathsf{Alg}^*\mathcal{L}\}$ and that $\mathsf{Alg}^*\mathcal{L}$ is a point regular variety. Therefore we conclude that $\mathcal{L}$ is algebraizable. $\square$

Now we provide the announced characterization of the notion of everywhere strongly logifiable algebra. As it will become evident from the proof, one of its central constructions is a generalization of that made in the proof of Lemma 3.2.

Theorem 3.6. *Let A be a non-trivial finite algebra. The following conditions are equivalent:*

(i) *A is everywhere strongly logifiable.*

(ii) *The logic of $\langle A, \{a\} \rangle$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(A)$, for every $a \in A$.*

(iii) *A is simple, without proper subalgebras and generates a minimal and point regular variety.*

(iv) *A is simple, constantive and generates a congruence distributive and n-permutable variety for some $n \geqslant 2$.*

Proof. (i)$\Rightarrow$(ii): Straightforward. (ii)$\Rightarrow$(iii): Consider an arbitrary element $a \in A$. Let $\mathcal{L}$ be the logic determined by $\langle A, \{a\} \rangle$ and $\boldsymbol{\tau}(x)$ and $\boldsymbol{\rho}(x,y)$ be two structural transformers the witness its algebraization. Since $\mathcal{L}$ is algebraizable, it has theorems. Therefore there is a term $a(x)$ (in at most one variable x) that represents in A the constant function with value a. Since this argument can be repeated for every element of A, we conclude that A is constantive and therefore that it has no proper subalgebras.

Now we turn to prove that A is simple. First we claim that $\mathcal{F}i_{\mathcal{L}}A = \{\{a\}, A\}$. In order to prove this, observe that clearly $\{a\}, A \in \mathcal{F}i_{\mathcal{L}}A$. Now we check the other inclusion. Let $F \in \mathcal{F}i_{\mathcal{L}}A$ such that $F \neq \{a\}$. Then there is $b \in F$ such that $a \neq b$. Since A is constantive, there is a term $b(x)$, that represents the constant function with value b. But from the fact that $\mathcal{L}$ is the logic determined by the matrix $\langle A, \{a\} \rangle$, it follows that $b(x) \vdash_{\mathcal{L}} y$. In particular, this implies that $F = A$. This concludes the proof that $\mathcal{F}i_{\mathcal{L}}A = \{\{a\}, A\}$. Now, from condition (ii) of Theorem 2.1 we know that the lattices $\mathcal{F}i_{\mathcal{L}}A$ and $\mathrm{Con}A$ are isomorphic. Therefore we conclude that A is simple.

Moreover, recall from point 2 of Lemma 3.5 that $\mathbb{V}(A)$ is point regular. Now observe that the matrix $\langle A, \{a\} \rangle$ is reduced, since A is simple. Then we can apply Lemma 3.3 yielding that $\mathbb{V}(A) = \mathsf{Alg}^*\mathcal{L} = \mathbb{GQ}(A)$. But A is finite without proper subalgebras. Therefore $\mathbb{V}(A)$ is a minimal variety by Lemma 3.4.

(iii)$\Rightarrow$(iv): From Theorem 2.3 we know that $\mathbb{V}(A)$ is congruence modular and n-permutable for some $n \geqslant 2$. Moreover, applying Theorem 2.5 to the fact that $\mathbb{V}(A)$ is congruence modular and to the assumptions, we conclude

that $\mathbb{V}(\boldsymbol{A})$ is also congruence distributive. Finally observe that $\boldsymbol{A}$ has a constant term 1, which witnesses the point regularity of $\mathbb{V}(\boldsymbol{A})$. Together with the fact that $\boldsymbol{A}$ has no proper subalgebras, this implies that $\boldsymbol{A}$ is constantive.

(iv)$\Rightarrow$(i): First observe that $\mathbb{V}(\boldsymbol{A})$ is locally finite, since $\boldsymbol{A}$ is finite. Moreover, from the assumptions we know that $\mathbb{V}(\boldsymbol{A})$ is congruence distributive and n-permutable for some $n \geqslant 2$. Since congruence distributivity implies congruence meet semi-distributivity, we can apply Theorem 2.8 obtaining that $\mathrm{typ}\{\mathbb{V}(\boldsymbol{A})\} = \{3\}$. Together with the fact that $\boldsymbol{A}$ is simple by assumption, this implies that $\mathrm{typ}(\Delta, \nabla) = \mathrm{typ}(\boldsymbol{A}) = 3$, i.e., the minimal algebra associated to $\boldsymbol{A}$ is the two-element Boolean algebra. This means that $\boldsymbol{A}|_U$ is polynomially equivalent to the two-element Boolean algebra for every $\langle \Delta, \nabla \rangle$-minimal set U and, in particular, that every $\langle \Delta, \nabla \rangle$-minimal set has exactly two elements.

Then pick an arbitrary set $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$ and let $\mathcal{L}$ be the logic determined by the matrix $\langle \boldsymbol{A}, F \rangle$. We have to prove that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$. We begin by showing that $\mathcal{L}$ is algebraizable. In order to do this, choose $a, b \in A$ such that $a \in F$ and $b \in A \smallsetminus F$. From *connection* (point 1 of Theorem 2.7) it follows that there is a finite sequence of overlapping $\langle \Delta, \nabla \rangle$-minimal sets of $\boldsymbol{A}$ that connects a to b. In particular, this implies that there are $0, 1 \in A$ such that $1 \in F$, $0 \in A \smallsetminus F$ and $\{0, 1\}$ is a $\langle \Delta, \nabla \rangle$-minimal set. Hence $\boldsymbol{A}|_{\{0,1\}}$ is polynomially equivalent to the two-element Boolean algebra. In particular, this means that there are $\wedge, \vee, \leftrightarrow \in \mathrm{Pol}_2(\boldsymbol{A})$ that, when restricted to $\{0, 1\}$, coincide with the usual Boolean connectives for this set ordered with $0 < 1$.

Now, from *separation* (point 2 of Theorem 2.7) it follows that for every different $a, b \in A$ there is $p_{ab}(x) \in \mathrm{Pol}_1(\boldsymbol{A})$ such that $p_{ab}[A] = \{0, 1\}$ and $p_{ab}(a) \neq p_{ab}(b)$. Then fix an enumeration $F = \{a_1, \ldots, a_n\}$. Recall that $\boldsymbol{A}$ is constantive and therefore that we have a term $a_i(x)$ that represents the constant function with value a_i for every $i \leqslant n$. Then we define

$$\Box(x) := \bigvee_{i \leqslant n} \Big(\bigwedge \Big\{ p_{a_i b}(x) \leftrightarrow p_{a_i b}(a_i(x)) : b \in A \smallsetminus \{a_i\} \Big\} \Big).$$

In principle $\Box(x)$ is a polynomial function of $\boldsymbol{A}$. But since $\boldsymbol{A}$ is constantive we can assume that it is also a term function. It is easy to prove that for every $a \in A$:

$$a \in F \Longleftrightarrow \Box(a) = 1 \text{ and } a \notin F \Longleftrightarrow \Box(a) = 0.$$

As before, we keep denoting by $1(x)$ the constant function with value 1. Since $\mathcal{L}$ is the logic determined by the matrix $\langle \boldsymbol{A}, F \rangle$ we have that

$$\Gamma \vdash_{\mathcal{L}} \varphi \Longleftrightarrow \{\Box(\gamma) \approx 1(\gamma) : \gamma \in \Gamma\} \vDash_{\boldsymbol{A}} \Box(\varphi) \approx 1(\varphi) \tag{5}$$

for every $\Gamma \cup \{\varphi\} \subseteq Fm$.

Now we define another map as follows:

$$x \lhd\rhd y := \bigwedge \{p_{ab}(x) \leftrightarrow p_{ab}(y) : a, b \in A \text{ and } a \neq b\}.$$

As before, the function $x \lhd\rhd y$ turns out to be a term function. It is easy to prove that for every $a, b \in A$:

$$a = b \Longleftrightarrow a \lhd\rhd b = 1 \text{ and } a \neq b \Longleftrightarrow a \lhd\rhd b = 0.$$

In particular this implies that

$$x \approx y =\!\vDash\!\vDash_{\boldsymbol{A}} \Box(x \lhd\rhd y) \approx 1(x \lhd\rhd y). \tag{6}$$

Observe that conditions (5) and (6) are saying exactly that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{GQ}(\boldsymbol{A})$ via the structural transformers

$$\boldsymbol{\tau}(x) := \{\Box(x) \approx 1(x)\} \text{ and } \boldsymbol{\rho}(x, y) := \{x \lhd\rhd y\}.$$

It only remains to prove that $\mathbb{GQ}(\boldsymbol{A}) = \mathbb{V}(\boldsymbol{A})$, i.e., that $\mathbb{V}(\boldsymbol{A})$ is the equivalent algebraic semantics of $\mathcal{L}$. It will be enough to show that $\mathbb{V}(\boldsymbol{A}) \subseteq \mathbb{GQ}(\boldsymbol{A})$. To do this, we reason as follows. Since $\mathbb{V}(\boldsymbol{A})$ is congruence distributive, we can apply Jónsson's Lemma, obtaining that $\mathbb{V}(\boldsymbol{A})_{\mathrm{si}} \subseteq \mathbb{HS}(\boldsymbol{A})$. But from the fact that $\boldsymbol{A}$ is constantive and simple it follows that $\mathbb{HS}(\boldsymbol{A}) = \mathbb{I}(\boldsymbol{A})$. Therefore $\boldsymbol{A}$ is up to isomorphism the only subdirectly irreducible algebra of $\mathbb{V}(\boldsymbol{A})$ and $\mathbb{V}(\boldsymbol{A}) = \mathbb{P}_{\mathrm{SD}}(\boldsymbol{A})$. Since generalized quasi-varieties are closed under the formation of subdirect products, we conclude that $\mathbb{V}(\boldsymbol{A}) \subseteq \mathbb{GQ}(\boldsymbol{A})$. Therefore we are done. $\square$

Along the proof of Theorem 3.6 some features of everywhere strongly logifiable algebras were highlighted. One of them is the following:

Corollary 3.7. *If $\boldsymbol{A}$ is everywhere strongly logifiable, then it is (up to isomorphism) the only subdirectly irreducible member of $\mathbb{V}(\boldsymbol{A})$.*

Another property of everywhere strongly logifiable algebras, which was not evident from their original definition, is the following one:

Corollary 3.8. *If $\boldsymbol{A}$ is everywhere strongly logifiable, then for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$ the algebraization of the logic of $\langle \boldsymbol{A}, F \rangle$ is witnessed by two one-element structural transformers.*

Proof. It is enough to take a look to the proof of direction (iv)$\Rightarrow$(i) of Theorem 3.6. The structural transformers $\boldsymbol{\tau}(x)$ and $\boldsymbol{\rho}(x,y)$ constructed there are in fact a pair of singletons. $\qquad\square$

From Theorem 3.6 one can draw also some consequences on the "variety problem" in the case of logics determined by matrices of the form $\langle \boldsymbol{A}, \{a\} \rangle$, where $\boldsymbol{A}$ is a finite non-trivial constantive algebra.

Corollary 3.9. *Let $\boldsymbol{A}$ be a finite non-trivial constantive algebra and $\mathcal{L}$ be the logic of a matrix of the form $\langle \boldsymbol{A}, \{a\} \rangle$ with $a \in A$. The following conditions are equivalent:*

(i) *The class $\mathsf{Alg}^*\mathcal{L}$ is a variety.*

(ii) *The algebra $\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}$ is everywhere strongly logifiable.*

Moreover, in this case $\mathsf{Alg}^*\mathcal{L} = \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$.

Proof. (i)$\Rightarrow$(ii): First observe that Lemma 3.5 can be applied here, because $\boldsymbol{A}$ is constantive, and therefore $\mathcal{L}$ is algebraizable with equivalent algebraic semantics the point regular variety $\mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$. The point regularity of $\mathsf{Alg}^*\mathcal{L}$ is witnessed by the constant a.

Now, the fact that $\mathcal{L}$ is algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$ allows us to repeat exactly the same argument used in the proof of part (ii)$\Rightarrow$(iii) of Theorem 3.6 to show that $\mathbb{Q}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}) = \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$ and, therefore, to deduce that $\mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$ is a minimal variety. The only difference is that in the proof of Theorem 3.6 we were working with the matrix $\langle \boldsymbol{A}, \{a\} \rangle$, while now one needs to use its reduction.

Then we turn to prove that that $\boldsymbol{A}$ is simple. An argument analogous to the one used to prove simplicity in part (ii)$\Rightarrow$(iii) of Theorem 3.6 yields that $\mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}) = \{\{a\}/\boldsymbol{\Omega^A}\{a\}, A/\boldsymbol{\Omega^A}\{a\}\}$. Together with the fact that $\mathsf{Alg}^*\mathcal{L} = \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$, Theorem 2.1 implies that the lattices $\mathcal{F}i_{\mathcal{L}}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$ and $\mathrm{Con}(\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\})$ are isomorphic. Therefore we conclude that $\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}$ is simple.

Hence $\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}$ is a finite non-trivial simple and constantive algebra that generates a minimal point regular variety. From Theorem 3.6 it follows that $\boldsymbol{A}/\boldsymbol{\Omega^A}\{a\}$ is everywhere strongly logifiable.

(ii)$\Rightarrow$(i): Observe that $\mathcal{L}$ coincides with the logic determined by the matrix $\langle \boldsymbol{A}, \{a\} \rangle^{\boldsymbol{*}}$. Together with the assumption, this implies that $\mathsf{Alg}^{\boldsymbol{*}}\mathcal{L} = \mathbb{V}(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}\{a\})$, a variety. $\qquad \square$

4.   Relations with primal algebras

The only examples of everywhere strongly logifiable algebras we met until now are primal algebras (Lemma 3.2). It is natural to ask whether there are everywhere strongly logifiable algebras that fail to be primal and, more generally, what is the relation between these two concepts. The first fact that it is worth to remark is that in several cases they coincide. For example, this is the case in congruence permutable varieties.

Lemma 4.1. *A a finite non-trivial algebra in a congruence permutable variety is everywhere strongly logifiable if and only if it is primal.*

Proof. Let $\boldsymbol{A}$ be an everywhere strongly logifiable algebra in a congruence permutable variety. From condition (iv) of Theorem 3.6 we know that $\boldsymbol{A}$ is simple, has no proper subalgebra, no automorphism except the identity map and generates a congruence distributive variety. Moreover we assumed that $\boldsymbol{A}$ generates a congruence permutable variety. Therefore, applying Theorem 2.4, we conclude that $\boldsymbol{A}$ is primal. $\qquad \square$

In particular, Lemma 4.1 implies that within the landscape of residuated lattices the notion of an everywhere strongly logifiable algebra and that of a primal algebra coincide. The same thing happens if we restrict our attention to two-element algebras, as we remark below.

Lemma 4.2. *A two-element algebra is everywhere strongly logifiable if and only if it is primal.*

Proof. Let $\boldsymbol{A}$ be a two-element everywhere strongly logifiable algebra. Taking a look at the proof of direction (iv)$\Rightarrow$(i) of Theorem 3.6, one sees that the fact that $\boldsymbol{A}$ has just two elements implies that it is polynomially equivalent to the two-element Boolean algebra. But we know that $\boldsymbol{A}$ is constantive and, therefore, that its polynomial functions are term functions too. In particular, this implies that $\boldsymbol{A}$ is term equivalent to the two-element Boolean algebra which is primal. $\qquad \square$

Remarkably, Lemma 4.2 cannot be strengthened as far as the cardinality issue is regarded. In the following example we show that for every $n \geqslant 3$ there is an everywhere strongly logifiable algebra of n elements that fails to be primal. In order to do this, we make use of the characterization of everywhere strongly logifiable algebras given in Theorem 3.6.

Example 4.3. Let $\langle A, \leqslant \rangle$ be a finite bounded poset with top 1 and bottom 0 such that $|A| \geqslant 3$. We can convert $\langle A, \leqslant \rangle$ into an algebra $\boldsymbol{A}$ equipping its universe with a name for each element and two operations $\rightarrow$ and $\square$ defined as follows:

$$x \to y := \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases} \qquad \square x := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $x, y \in A$. Observe that the basic operations of $\boldsymbol{A}$ have a logical flavour: the $\{\rightarrow\}$-reduct of $\boldsymbol{A}$ is a Hilbert algebra.

We shall prove that $\boldsymbol{A}$ is everywhere strongly logifiable but not primal. In order to do this, we begin by showing that it is everywhere strongly logifiable. To see that $\boldsymbol{A}$ is simple, consider $\theta \in \mathrm{Con}\boldsymbol{A} \smallsetminus \{\Delta\}$. Then there are two different $a, b \in A$ such that $\langle a, b \rangle \in \theta$. We can assume without loss of generality that $a \nleqslant b$ and, therefore, that $b \neq 1$. We have that $1 = a \to a \equiv_\theta a \to b = b$ and then $1 = \square 1 \equiv_\theta \square b = 0$. Now consider an arbitrary element $c \in A$. We have that $1 = 0 \to c \equiv_\theta 1 \to c = c$. Therefore we conclude that $\theta = \nabla$ and that $\boldsymbol{A}$ is simple. Moreover, $\boldsymbol{A}$ is constantive by definition. Now, it is well known that Hilbert algebras form a point regular variety HiA. Therefore, applying Theorem 2.3, we obtain that HiA is n-permutable for some $n \geqslant 2$. Moreover, HiA is congruence distributive [9, 22]. Now, recall that congruence distributivity and n-permutability are Maltsev conditions. Therefore there are Maltsev terms, in the signature $\langle \rightarrow \rangle$, which witness these conditions for Hilbert algebras. Since the $\{\rightarrow\}$-reduct of $\boldsymbol{A}$ is a Hilbert algebra, the same Maltsev terms witness the congruence distributivity and the n-permutability of $\mathbb{V}(\boldsymbol{A})$. Therefore, applying condition (iv) of Theorem 3.6, we conclude that $\boldsymbol{A}$ is everywhere strongly logifiable.

It only remains to check that $\boldsymbol{A}$ is not primal. But this is very easy. First observe that

$$X := \Delta \cup (\{1\} \times A) \cup (A \times \{1\})$$

is the universe of a subalgebra of $\boldsymbol{A} \times \boldsymbol{A}$. Since $|A| \geqslant 3$, there are two different elements $a, b \in A \smallsetminus \{1\}$. Clearly $\langle a, b \rangle \notin X$. Then consider the unary function $f \colon A \to A$ defined as

$$f(x) := \begin{cases} a & \text{if } x = a \\ b & \text{otherwise} \end{cases}$$

for every $x \in A$. The fact that $\langle a, 1 \rangle \in X$ and $\langle f(a), f(1) \rangle = \langle a, b \rangle \notin X$ implies that f is not represented by any term of $\boldsymbol{A}$. In particular, this means that $\boldsymbol{A}$ is not primal. $\qquad\square$

We conclude by posing the following question: is it possible to characterize everywhere strongly logifiable algebras in a way analogous the one given in Theorem 2.4 for primal algebras? More precisely:

Problem 4.1. Prove or disprove that a finite non-trivial algebra is everywhere strongly logifiable if and only if it is simple, with no proper subalgebras, has no automorphism except the identity map, and generates a congruence distributive and n-permutable variety for some $n \geqslant 2$.

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