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Grzegorz JAGIELLA and Ludomir NEWELSKI

ON ORDERED MINIMAL STRUCTURES

A b s t r a c t. We investigate minimal first-order structures and consider interpretability and definability of orderings on them. We also prove the minimality of their canonical substructures.

Recall that a structure M is minimal if every definable (with parameters) subset of M is either finite or cofinite. The study of minimal partially ordered structures was initiated by Tanović in [4, 5]. The results from [4, 5] were essential in the proof by Tanović of the Pillay's conjecture [1], which states that every countable structure in a countable language has infinitely many nonisomorphic, countable extensions. The analysis of minimal ordered structures continued in [6] and produced further results. An example is a partial answer to Kueker's Conjecture, which states that if a theory T is not $\aleph_0$-categorical and its every uncountable model is $\aleph_0$-saturated, then T is $\aleph_1$-categorical. In [3] the conjecture is proved under the additional assumption that T has the NIP property and $\mathrm{dcl}(\emptyset)$ is infinite. Another example is a reduction of Podewski's Conjecture which states that every minimal field is algebraically closed. The paper [7] proves that this reduces

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to showing that there are no minimal partially ordered fields of characteristic 0 such that the order has an infinite chain.

The results of Tanović justify our interest in minimal ordered structures as an independent object of study. In this paper we deal with pure minimal partially ordered structures, that is minimal structures of the form $(M,<)$ where $<$ is a partial order. In this situation it is natural to expect that in the nontrivial case, where $<$ has arbitrarily long chains (subsets of M linearly ordered by $<$), $(M,<)$ should resemble a linear ordering. Despite the simple setup it is not easy to obtain a full description of such structures. We obtained some partial results in this direction. In Theorem 6 we prove that if $(M,<)$ is minimal then both its lower and upper parts $\mathbb{L} = (L(M),<), \mathbb{U} = (U(M),<)$ (defined below) are minimal and stably embedded in M. In Theorem 9 we show that M is almost linear if and only if $\mathbb{L}$ or $\mathbb{U}$ is almost linear.

We begin by setting up the notation and recalling the fundamental results from [4, 5]. All structures in the paper are models of an arbitrary first-order theory in the language $\mathcal{L} = \{<\}$ where $<$ is interpreted as an ordering. Given a set of parameters A we write $\mathcal{L}(A)$ for the set of formulas in $\mathcal{L}$ with parameters from A. By definable sets we mean sets definable with parameters. Likewise, an interpretable structure also means interpretable with parameters.

Definition 1. Let $(M,<)$ be a minimal ordered structure. Let p be the unique non-algebraic type over M. We define

$$\begin{aligned} L(M) &= \{m \in M : (m < x) \in p\}, \\ U(M) &= \{m \in M : (m > x) \in p\}, \\ I(M) &= \{m \in M : (m \perp x) \in p\}. \end{aligned}$$

The following result appears in [4, Proposition 1.1, Proposition 1.2] and [5, Theorem 2].

Theorem 2. *Let $(M,<)$ be a minimal ordered structure with an infinite chain. Then*

(1) M is countable.

(2) $I(M)$ is finite.

(3) Assuming $(M,<)$ *has an ascending infinite chain, it falls into one of the following two types:*

$$\begin{array}{lll} \text{Type}(\omega) & \textit{where} & M = L(M) \cup \textit{ a finite set;} \\ \text{Type}(\omega+\omega^*) & \textit{where} & M = L(M) \cup U(M) \cup \textit{ a finite set} \end{array}$$

(here both $L(M)$ *and* $U(M)$ *are infinite).*

The theorem gives further description of the structure: if $(M,<)$ is of Type(ω) then $(L(M),<)$ has no maximal elements, is directed upwards and does not contain chains of order type $\omega+1$. If $(M,<)$ is of Type($\omega+\omega^*$) then $(L(M),<)$ has no maximal elements, is directed upwards and does not contain chains of order type $\omega+1$ while $(U(M),<)$ has no minimal elements, is directed downwards and does not contain chains of order type $1+\omega^*$, where ω^* is the ω with reverse order relation.

The next result shows that the type of a structure is well defined in a class of interdefinable minimal orders with infinite, ascending chains:

Proposition 3 ([5], Theorem 3)**.** *Let* $\mathbb{M}=(M,\ldots)$ *be a minimal structure. The following are equivalent:*

(1) There exists a definable $<$ *such that* $(M,<)$ *is of* Type(ω)*.*

(2) $\mathbb{M}$ *is ordered of* Type(ω) *with respect to any definable ordering with an infinite ascending chain.*

The following is similar to König's Lemma:

Fact 4. *Let* $(M,<)$ *be minimal. Then* $<$ *has an infinite chain iff* $<$ *has finite chains of arbitrary length.*

Proof. Assume that $<$ has finite chains of arbitrary length. Let $L_0=\emptyset$ and for $n>0$ write L_n for the set of minimal elements of $M\setminus\bigcup_{i<n}L_i$. Then every L_n is a definable antichain. If some L_n is infinite, then its complement is finite and we get a contradiction. So it is easy to see that we only have two cases:

(i) For some $n\in\omega, L_n=\emptyset$. Then $M\setminus\bigcup_{i<n}L_i$ is infinite and has no minimal elements, so it must have a descending, infinite chain.

(ii) Each L_n is nonempty. Then $\bigcup_n L_n$ contains an ascending, infinite chain. □

From now on, we assume that all considered structures $(M, <)$ have an infinite chain.

By Theorem 2(2), given a minimal $(M, <)$ it is always possible to define an order $<'$ such that $I_{<'}(M) = \emptyset$. Thus for the remainder of this paper we additionaly assume $I_<(M) = \emptyset$.

Now assume that $(M, <)$ is minimal of Type(ω). Then it is easy to see that $<$ is the transitive closure of the successor relation S induced by it: for every $a < b$ there are only finitely many elements c such that $a < c < b$. Given that every definable antichain must be finite, we can further modify $<$ so it has the smallest element. Now $(M, <)$ can be described as a rooted, directed acyclic graph: we put an edge between a and b whenever aSb.

Note that for any $n \in \omega$ the set L_n of elements that are (in the sense of the graph) n edges away from the root is $\emptyset$-definable. This gives the following:

Fact 5. $M \subset \mathrm{acl}(\emptyset)$.

It is easy to see that a similar description (as well as the fact) holds for Type$(\omega + \omega^*)$ structures as well, in which case $<$ is the transitive closure of the induced successor relation together with "$x < y$" for all $x \in L(M), y \in U(M)$.

So at glance, a minimal $(M, <)$ of Type$(\omega + \omega^*)$ consists of two "halves" that have no other obvious relations between them (other than $L(M) < U(M)$). Given such a "half", say $L(M)$, it is natural to ask about the fashion in which it (together with the order relation restricted to it) is embedded in the whole structure. The main results of the paper are the following Theorem 6, 7 and Theorem 9.

Theorem 6. *Let* $\mathbb{M} = (M, <)$ *be minimal and* $\mathbb{L} = (L(M), <\restriction_{L(M)})$. *Then*

(1) For any $A \subset M^n$ *definable in* $\mathbb{M}$, *the set* $A \cap L(M)^n$ *is definable in* $\mathbb{L}$,

(2) For any $B \subset L(M)^n$ *definable in* $\mathbb{L}$, *there is an* $A \subset M^n$ *definable in* $\mathbb{M}$ *such that* $A \cap L(M)^n = B$,

(3) $\mathbb{L}$ *is minimal.*

In particular the theorem asserts that $\mathbb{L}$ is "stably embedded" in $\mathbb{M}$.

In the context of interpretable orderings, having established Theorem 6(3), we will also prove:

Theorem 7. *Let $\mathbb{M} = (M, <)$ be minimal with $L(M)$ infinite and $\mathbb{L} = (L(M), <\upharpoonright_{L(M)})$. If $\mathbb{M}$ interprets an infinite order, then $\mathbb{L}$ interprets an infinite suborder of it.*

Note that we do not assume the linearity of the interpreted order.

An analogue of Theorem 7 exists when we consider "lifting" of definable orders from $L(M)$ to M. We recall a definition from [7]:

Definition 8. Let $(M, <)$ be a minimal structure. We say that M is almost linear if it has a definable order $\preceq$ with infinite chains such that the relation of incomparability with respect to $\preceq$ is an equivalence relation.

It is shown that for $(M, <, \preceq)$ as in the definition, and $\sim$ the equivalence relation induced by $\preceq$, the quotient structure $(M/\sim, \preceq/\sim)$ is isomorphic to either $(\omega, <)$ or $(\omega + \omega^*, <)$. We prove:

Theorem 9. *Let $(M, <)$ be a minimal structure with $L(M)$ infinite. Then $(M, <)$ is almost linear if and only if $\mathbb{L} = (L(M), \lhd\upharpoonright_{L(M)})$ is almost linear.*

In order to prove the theorems, we begin with a crucial lemma. Before we state it, we need to set up some additional notation.

Definition 10. (i) Given an $n \in \omega$ and any $A \subset M$, an (n, A)-type is a complete type over A restricted to the formulas of quantifier rank $\leq n$.

(ii) With n, A as above and $\bar{a}, \bar{b} \in M^m$, we write $\bar{a} \equiv_{n,A} \bar{b}$ whenever $\bar{a}$ and $\bar{b}$ satisfy the same (n, A)-type.

Note that for any n and any finite A, the set of all (n, A)-types is finite.

Lemma 11. *Let $(M, <)$ be minimal. We write L for $L(M)$ and U for $U(M)$. For any $n \in \omega$,*

F_n) For each $k, l \in \omega$, $\bar{a} \in L^k$ and $\phi(x, \bar{y}, \bar{z}) \in \mathcal{L}(U)$ with $\mathrm{qr}(\phi) \leq n, |\bar{y}| = k, |\bar{z}| = l$, there is a finite $S \subset L$ such that for each $\bar{b} \in U^l$,

$$\phi^M(M, \bar{a}, \bar{b}) \subset L \Rightarrow \phi^M(M, \bar{a}, \bar{b}) \subset S.$$

S_n) For each $k, l \in \omega$ there is a finite $S_{n,k,l} \subset L$ such that for each $\bar{b}_1, \bar{b}_2 \in U^l$ and $\bar{a} \in L^k$,

$$\bar{b}_1 \equiv_{n,S_{n,k,l}} \bar{b}_2 \Rightarrow \bar{b}_1\bar{a} \equiv_n \bar{b}_2\bar{a}.$$

Proof. Without loss of generality we assume that $(M, <)$ is of Type($\omega + \omega^*$). First, we note that it is enough to prove the Lemma with $\phi(x, \bar{y}, \bar{z}) \in \mathcal{L}$ in the statement of F_n as the full statement follows from it. So we replace $\mathcal{L}(U)$ there with $\mathcal{L}$ for the remainder of the proof. We proceed by simultaneous induction, proving the following:

(1) S_0 holds,

(2) S_n implies F_n,

(3) $S_n + F_n$ imply S_{n+1}.

(1) Let $S_{0,k,l} = \emptyset$. Assume $\bar{b}_1 \equiv_{0,\emptyset} \bar{b}_2$. For any $b_1 \in \bar{b}_1, b_2 \in \bar{b}_2, a \in \bar{a}$ we have $a < b_1, a < b_2$. So $\bar{b}_1\bar{a} \equiv_{0,\emptyset} \bar{b}_2\bar{a}$.
(2) Assume S_n and fix $\phi(x, \bar{y}, \bar{z})$ and $\bar{a} \in L^k$. By S_n there is a finite $S_{n,k+1,l}$ such that for each $a \in L$ and $\bar{b}_1, \bar{b}_2 \in U^l$ with $\bar{b}_1 \equiv_{n,S_{n,k+1,l}} \bar{b}_2$ we have

$$\models \phi(a, \bar{a}, \bar{b}_1) \iff \phi(a, \bar{a}, \bar{b}_2).$$

So whenever the set $\phi(M, \bar{a}, \bar{b})$ is contained in L (and therefore is finite), that set depends only on the $(n, S_{n,k+1,l})$-type of $\bar{b}$. Since there are only finitely many such types, we conclude that

$$\bigcup \{\phi(M, \bar{a}, \bar{b}) : \bar{b} \in U^l, \phi(M, \bar{a}, \bar{b}) \subset L\}$$

is a finite union of finite sets. We define S to be this union.
(3) We construct $S_{n+1,k,l}$, stipulating that it contains the already defined sets $S_{n,k+1,l} \cup S_{n,k,l+1}$ and enlarging it as needed. We aim to show

$$\bar{b}_1 \equiv_{n+1,S_{n+1,k,l}} \bar{b}_2 \Rightarrow \bar{b}_1\bar{a} \equiv_{n+1} \bar{b}_2\bar{a}.$$

Assuming the left side of the implication, we prove that for each $c \in M$ there is a $c' \in M$ such that $\bar{b}_1\bar{a}c \equiv_n \bar{b}_2\bar{a}c'$. We distinguish three cases.

Case 1. $c \in L$. Assume $\bar{b}_1 \equiv_{n+1,S_{n+1,k,l}} \bar{b}_2$. In particular, $\bar{b}_1 \equiv_{n,S_{n,k+1,l}} \bar{b}_2$ and by the induction hypothesis, $\bar{b}_1 c\bar{a} \equiv_n \bar{b}_2 c\bar{a}$.

Case 2. $c \in U$ and there is a $d \in U$ such that $\bar{b}_1 c \equiv_{n,S_{n+1,k,l}} \bar{b}_2 d$. Then, as $S_{n+1,k,l} \supset S_{n,k,l+1}$, by the induction hypothesis again we get $\bar{a}\bar{b}_1 c \equiv_n \bar{a}\bar{b}_2 d$.

Case 3. $c \in U$ and for each $d \in M$,

$$\bar{b}_1 c \equiv_{n,S_{n+1,k,l}} \bar{b}_2 d \Rightarrow d \in L.$$

Here, we prove that for a sufficiently large (yet finite) $S_{n+1,k,l}$, this case leads to a contradiction. So, observe that the assumption implies that $d\bar{b}_2$ satisfies some $(n, S_{n,k,l+1})$-type p.
Take any $\phi(x, \bar{y}) \in p$ such that $\phi(M, \bar{b}_2) \subset L$. By F_n there is a finite $S = S_\phi$ such that whenever $\phi(M, \bar{b}) \subset L$ for some $\bar{b} \in U^l$, we also have $\phi(M, \bar{b}) \subset S$. There are only finitely many formulas in p, so let $S_{n+1,k,l} \supset \bigcup_\phi S_\phi$, where the union ranges over all formulas $\phi \in p$ with realizations contained in L. The union depends only on $S_{n,k,l+1}$, so $S_{n+1,k,l}$ is well-defined. As d certainly belongs to this union, $d \in S_{n+1,k,l}$ and $\bar{b}_1 c \not\equiv_{n,S_{n+1,k,l}} \bar{b}_2 d$, a contradiction. □

Proof of Theorem 6. Without loss of generality we assume $\mathbb{M}$ to be of Type$(\omega+\omega^*)$. (1) Write L for $L(M)$ and U for $U(M)$. Let $\Sigma \subset \mathcal{L}(M)$ be the set of formulas $\phi(\bar{x})$ such that there is $\phi'(\bar{y}) \in \mathcal{L}(L)$ with $\phi^{\mathbb{M}}(L^n) = \phi'^{\mathbb{L}}(L^n)$. It is easy to see that Σ contains all atomic formulas (recall that $a < b$ whenever $a \in L, b \in M$). It is clearly closed under taking disjunctions and negations. We prove that $\Sigma = \mathcal{L}(M)$ by induction on quantifier rank of formulas. So let

$$\phi(\bar{x}) = \exists y \psi(\bar{x}, y),$$

with $|\bar{x}| = n$ and $\psi(\bar{x}, y)$ belonging to Σ along with all formulas of the same quantifier rank. In particular, all instances of ψ belong to Σ.

Consider all tuples $\bar{l} \in L^n$ such that $\psi(\bar{l}, M) \subset U$. By Lemma 11 (with U and L swapped), there is a finite U_0 such that $\psi(\bar{l}, M) \subset U \Rightarrow \psi(\bar{l}, M) \subset U_0$. Let

$$\chi(\bar{x}, y) = \bigvee_{u \in U_0} \psi(\bar{x}, u) \vee \psi(\bar{x}, y).$$

One checks that for any $\bar{l} \in L^n$,

$$\mathbb{M} \models \phi(\bar{l}) \iff \exists a \in L\ \mathbb{M} \models \chi(\bar{l}, a).$$

As $\chi(\bar{x}, y)$ is a disjunction of formulas from Σ, there is $\chi'(\bar{x}, y) \in \mathcal{L}(L)$ such that $\chi^{\mathbb{M}}(L^n, L) = \chi'^{\mathbb{L}}(L^n, L)$. Now clearly $\exists y \chi'(\bar{x}, y)$ defines in $\mathbb{L}$ the set $\phi^{\mathbb{M}}(L^n)$. (2) Again, write L for $L(M)$ and U for $U(M)$. We proceed in a similar manner as before. So let $\Sigma \subset \mathcal{L}(L)$ be the set of formulas such that their realizations in $\mathbb{L}$ are traces of sets definable in $\mathbb{M}$. Again it is easy to see that Σ contains atomic formulas and is closed under boolean combinations. Let

$$\phi(\bar{x}) = \exists y \psi(\bar{x}, y),$$

with $|\bar{x}| = n$ and $\psi(\bar{x}, y)$ belonging to Σ. So there is $\psi'(\bar{x}, y) \in \mathcal{L}(M)$ such that $\psi'^{\mathbb{M}}(L^n, L) = \psi^{\mathbb{L}}(L^n, L)$. By Lemma 11 there is a finite U_0 such that for each $\bar{l} \in L^n$ we have $\psi'^{\mathbb{M}}(\bar{l}, M) \subset U \Rightarrow \psi'^{\mathbb{M}}(\bar{l}, M) \subset U_0$. Let

$$\chi'(\bar{x}, y) = \psi'(\bar{x}, y) \wedge \bigwedge_{u \in U_0} y \neq u.$$

Then $\exists y \chi'(\bar{x}, y)$ witnesses $\phi(\bar{x}) \in \Sigma$.
(3) Follows from (2). □

Turning attention to Theorem 7, assume without loss of generality that $\mathbb{M}$ is of Type$(\omega + \omega^*)$ and consider any order interpretable in $\mathbb{M}$, that is a definable equivalence relation on M^n and a definable ordering on its classes. By Theorem 6, their restrictions to $L(M)^n$ are definable in $\mathbb{L}$, giving an interpretable order in $\mathbb{L}$. The following lemma guarantees that this order has infinitely many elements, proving Theorem 7.

Lemma 12. *Let $(M, <)$ be minimal of* Type$(\omega + \omega^*)$ *and E be a definable equivalence relation on M^n with infinitely many classes. Write U for $U(M)$ and L for $L(M)$. Then at least one of the following holds:*

(1) $E{\restriction}_{U^{2n}}$ and $E{\restriction}_{L^{2n}}$ have infinitely many classes.

(2) There is an $a \in M$ and $i < n$ such that E restricted to

$$(\underbrace{M \times \ldots \times M \times \{a\} \times M \times \ldots \times M}_{n \text{ times}})^2,$$

where $\{a\}$ is on the i-th axis, has infinitely many classes.

Proof. Assume that (2) does not hold and (aiming for a contradiction) that $E{\restriction}_{U^{2n}}$ has finitely many classes. We will show that E also has finitely many classes.
M^n is the union of sets of the form $H_0 \times \ldots \times H_{n-1}$ where each of H_i's is either U or L. For a set X of this form, let $|X|$ be number of L's that appear in the product. We show the following by induction on k:

($*_k$) Let X be of the form as above and $|X| = k$. Then $E\restriction_{X^2}$ has finitely many classes.

We assumed $*_0$. Now take any X with $|X| = k$ and assume $*_{k'}$ for all $k' < k$. Without loss of generality $X = U^{n-k} \times L^k$. There is a single formula $\phi(\bar{x})$ with $|\bar{x}| = n$ saying "$\bar{x}$ is not E-equivalent to any $\bar{m} \in X$ with $|X| < k$". Consider the formula

$$\psi(x, \bar{y}) = \exists x_{n-k}, x_{n-k+1}, \dots, x_{n-1}\phi(\bar{y}, x, x_{n-k}, \dots x_{n-1}).$$

Whenever we consider an instance $\psi(x, \bar{u})$ for some $\bar{u} \in U^{n-k-1}$, all x's satisfying this instance must belong to L, since the tuples satisfying $\phi(\bar{x})$ have at most $n-k-1$ elements of U. So we can apply Lemma 11 to $\psi(x, \bar{y})$: there is a finite $S \subset L$ such that for any $\bar{u} \in U^{n-k-1}$, $\psi(M, \bar{u}) \subset S$. Unrolling the definition of ψ we have that all elements of $U^{n-k} \times (L \setminus S) \times L^{k-1}$ satisfy ϕ, i.e. they are contained in finitely many of the E-classes. By the assumption that (2) does not hold, for each $a \in S$ the set $U^{n-k} \times \{a\} \times L^{k-1}$ is also contained in a finite union of E-classes. But

$$X = \bigcup_{a \in S} U^{n-k} \times \{a\} \times L^{k-1} \cup U^{n-k} \times (L \setminus S) \times L^{k-1},$$

and we are done. □

We now give a prove of the remaining theorem (this proof is in part due to an anonymous reviewer of a previous version of this paper):

Proof of Theorem 9. The forward direction follows from Theorem 7. For the other direction, without loss of generatility assume $(M, <)$ to be of Type($\omega + \omega^*$). Write L for $L(M)$ and U for $U(M)$. Let $\preceq$ be a definable, almost linear order on L. By Theorem 6(2) there is a definable relation R on M such that R restricted to L^2 is $\preceq$. For $a \in M$, let $I_a = \{x \in M : x \preceq a\}$. We see that for every $a \in L$, the relation R restricted to I_a defines a linear order. By minimality, this is also true for almost all $a \in U$ and the result follows. □

Theorem 7 can be viewed as an attempt to deal with a question by Tanović whether all ordered minimal structures with infinite chains interpret an infinite linear ordering, particularly in dimension 1. In dealing with Podewski's conjecture, only structures of Type(ω) are concerned [7].

In both Podewski's and Kueker's conjectures, only definable orders are of the matter [3, 7]. It would be beneficial to have equivalences of the kind "no order interpretable in $\mathbb{L}$ in dimension 1 iff no order interpretable in $\mathbb{M}$" which would restrict the area in which any counterexample has to be found, or provide a way to produce definable orders having only an interpretable one in a structure of different type. An example of such equivalence can be proven (here we do not make any assumptions on M other than stated in the Proposition):

Proposition 13. *Let M be a first-order structure that interprets $(\omega, <)$. Then there exists a definable equivalence relation E on M and a definable order $\prec$ on M/E such that $(M/E, \prec) \cong (\omega, <)$.*

Proof. Let $n \in \mathbb{N}$ be the least such that there is a definable equivalence relation E' on M^n and a definable order $<'$ on M^n/E' with $(M^n/E', <') \cong (\omega, <)$. Assume that $n > 1$. We head for a contradiction. For each $m \in M$ let

$$C(m) = \{(m, m_2, m_3, \dots, m_n)_{E'} \in M^n/E' : m_2, m_3, \dots, m_n \in M\}.$$

Under the isomorphism $(M^n/E', <') \cong (\omega, <)$ the set $C(m)$ corresponds to a subset of ω.

Case 1. For each $m \in M$, $C(m)$ is finite. Let k_m be the greatest element of such a subset. We show that a desired definable relation and a definable ordering on classes can be found in M itself.

Let $\preceq$ be defined on M by

$$a \preceq b \iff \exists \bar{y} \forall \bar{x} \forall \bar{y}' \; ((a, \bar{x})_{E'} \leq' (b, \bar{y})_{E'} \wedge (b, \bar{y}')_{E'} \leq' (b, \bar{y})_{E'}),$$

and let E be defined by

$$a \, E \, b \iff \exists \bar{y} \exists \bar{x} \forall \bar{y}' \; ((a, \bar{x}) \, E' \, (b, \bar{y}) \wedge (b, \bar{y}') \leq' (b, \bar{y}) \wedge (a, \bar{y}') \leq' (a, \bar{x})).$$

Immediately from the definitions, we have $a \preceq b$ iff the greatest element of $C(b)$ is larger than any element of $C(a)$ and we have aEb iff the greatest elements of $C(a)$ and $C(b)$ are equal. We have

$$\begin{aligned} a \preceq b &\iff k_a \leq k_b, \\ a \, E \, b &\iff k_a = k_b. \end{aligned}$$

It is now sufficient to prove that $\{k_m \in \omega : m \in M\}$ is an infinite subset of ω. But this follows from the fact that $\bigcup\{C(m) : m \in M\} = M^n/E$: there can be no uniform bound on the greatest element of $C(m)$.

Case 2. There is $m_0 \in M$ such that $C(m_0)$ is infinite. We construct the desired relations on M^{n-1}, contradicting the minimality of n.

Let $\preceq$ be defined on M^{n-1} by

$$\bar{a} \preceq \bar{b} \iff (m_0, \bar{a}) \leq' (m_0, \bar{b}),$$

and let E be defined by

$$\bar{a}\, E\, \bar{b} \iff (m_0, \bar{a})\, E'\, (m_0, \bar{b}).$$

It is easy to see that $(M^{n-1}/E, \prec) \cong (\omega, <)$. □

We conclude the paper with a remark regarding minimal structures of theories with the strict order property (SOP). Recall that given a theory T, a formula $\phi(\bar{x}, \bar{y})$ with $|\bar{x}| = |\bar{y}| = n$ is said to have the strict order property for T if for a sufficiently saturated model $\mathfrak{C} \models T$, ϕ defines a partial order on the elements of $\mathfrak{C}^n$ with arbitrarily long finite chains. We say that T has the strict order property if there is a formula with the strict order property for T. Given an SOP theory T and a formula $\phi(\bar{x}, \bar{y})$ (without parameters) with the strict order property, it is natural to ask whether any minimal $M \models T$ can be ordered with infinite chains in a definable way, possibly using the formula $\phi(\bar{x}, \bar{y})$ which defines an ordering on a power of M. It is true when $|x| = 1$: $\phi(x, y)$ defines an order on M and since it has chains of arbitrary length in a monster model, it also has chains of arbitrary length in M. Thus by Fact 4 it has an infinite chain there.

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Instytut Matematyczny Uniwersytetu Wrocławskiego,
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

grzegorz.jagiella@math.uni.wroc.pl

ludomir.newelski@math.uni.wroc.pl