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ERRATA ON "ON THE VARIETY OF HEYTING ALGEBRAS WITH SUCCESSOR GENERATED BY ALL FINITE CHAINS"

In [3] we have claimed that finite Heyting algebras with successor only generate a proper subvariety of that of all Heyting algebras with successor, and in particular all finite chains generate a proper subvariety of the latter. As Xavier Caicedo made us notice, this claim is not true. He proved, using techniques of Kripke models, that the intuitionistic calculus with S has finite model property and from this result he concluded that the variety of Heyting algebras with successor is generated by its finite members [2].

This fact particularly affects Section 3.2 of our article. Concretely, in Remark 3.3, our claim *"Let $\mathcal{K}$ be a class of S-Heyting algebras of height less or equal to a fixed ordinal ξ. Using the categorical duality between S-Heyting algebras and S-Heyting spaces, it can be shown that the elements of classes $\mathbf{H}(\mathcal{K})$, $\mathbf{S}(\mathcal{K})$ and $\mathbf{P}(\mathcal{K})$ have also height less or equal to ξ. Here $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ are the class operators of universal algebra. Hence for each ordinal ξ, the class of S-Heyting algebras of height less or equal to ξ is a variety"* is not true as stated. It remains valid only if ξ is a finite ordinal.

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In particular, the class of S-Heyting algebras of height ω is not a variety and the variety generated by all finite chains is exactly the variety of linear S-Heyting algebras.

In what follows, instead of using the proof given in [2], which is not published, we shall give a simple algebraic proof that the variety of linear Heyting algebras with S is generated by the finite chains.

Let T be the type of Heyting algebras with successor built in the usual way from the operation symbols $\wedge$, $\vee$, $\rightarrow$, and S corresponding to meet, join, implication and successor, respectively. Write $T(X)$ for the term algebra of type T with variables in the set X. It is well known that any function $v : X \rightarrow H$, with H a S-Heyting algebra, may be extend to a unique homomorphism $v : T(X) \rightarrow H$.

Write SLH for the variety of linear S-Hetying algebras. Recall that SLH is said to have the *finite model property* (FMP) if for every $\varphi \in T(X)$ there is a linear S-Heyting algebra H and a homomorphism $v : T(X) \rightarrow H$ such that if $v(\varphi) \neq 1$ then there is a finite linear S-Heyting algebra L and a homomorphism $w : T(X) \rightarrow \mathrm{L}$ such that $w(\varphi) \neq 1$. Let us prove that SLH has the FMP. In so doing we shall use the two following well known Lemmata.

Lemma 1. *(Lemma 1.1 of [4]) If P is a prime filter in a linear algebra H, then H/P is a chain.*

Lemma 2. *Let C be a S-Heyting algebra which is a chain and* L *a bounded sublattice of C, endowed with its implication $\rightarrow_{\mathrm{L}}$ and successor S_{L}, as finite lattice. Then, we have that,*

1. *If $x, y \in \mathrm{L}$ then $x \rightarrow y = x \rightarrow_{\mathrm{L}} y$.*

2. *If $x, S(x) \in \mathrm{L}$ then $S_{\mathrm{L}}(x) = S(x)$.*

Take α and β in $T(X)$. Note that an equation $\alpha \approx \beta$ holds in a S-Heyting algebra H if and only if $\alpha \rightarrow \beta \approx 1$ holds in H; and the latter is equivalent to ask that for any homomorphism $v : T(X) \rightarrow H$, $v(\alpha \rightarrow \beta) = 1$.

We are now ready to prove the main result.

Proposition 3. *The variety* SLH *has the FMP.*

Proof. Let $\psi \in T(X)$, H be a linear S-Heyting algebra and $v : T(X) \to H$ be a homomorphism such that $v(\psi) \neq 1$. Let $\to$ and S be the implication and the successor of H respectively. We will find a finite chain L and a homomorphism $t : T(X) \to \text{L}$ such that $t(\psi) \neq 1$.

By the Prime Filter Theorem there is a prime filter P of H such that $v(\psi) \notin P$, so $v(\psi)/P \neq 1$. Write C in place of H/P. Hence, by Lemma 1, we have that C is a chain. On the other hand, using that the successor operator is compatible [1] we have that the quotient function $\rho : H \to C$ is a homomorphism. Hence $w = \rho v : T(X) \to C$ is a homomorphism. Note that $w(\psi) = v(\psi)/P \neq 1$.

Let $Sub_\psi = \{\psi_1, ..., \psi_n\}$ be the set of subformulas of ψ and L the subset of C given by $\{0, 1\} \cup \{w(\alpha) : \alpha \in Sub_\psi\}$. If V is the set of propositional variables which appear in ψ, we can define the function $t : X \to \text{L}$ in the following way:

$$t(x_i) = \begin{cases} w(x_i) & \text{if } x_i \in V \\ 0 & \text{if } x_i \notin V. \end{cases}$$

We know that t may be uniquely extended to a homomorphism $t : T(X) \to \text{L}$. Using Lemma 2, we can prove, by an easy induction on sentences, that : $t(\psi_i) = w(\psi_i)$ for $i = 1, ..., n$. Therefore we have that $t(\psi) = w(\psi) \neq 1$. □

In particular, we have that the following corollary holds.

Corollary 4. *The variety* SLH *is generated by all finite chains.*

Finally, we want to call the attention to a typos in Proposition 5.6. We wrote $\mathbf{SH}_S$ in place of SLH.

References

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