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JANKOV-STYLE FORMULAS AND REFUTATION SYSTEMS

A b s t r a c t. The paper studies the logics which algebraic semantics comprises of the Hilbert algebras endowed with additional operations - the regular algebras. With any finite subdirectly irreducible regular algebra one can associate a Jankov formula. In its turn, the Jankov formulas can be used as anti-axioms for a refutation system. It is proven that a logic has a complete refutation system based on Jankov formulas if and only if this logic enjoys finite model property. Also, such a refutation system is finite, that is, it contains a finite number of axioms and anti-axioms, if and and only if the logic is tabular.

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1. Introduction

In 1952 J. Łukasiewicz suggested [8] the following refutation system: antiaxiom $\dashv p$ and rules modus tollens

$$\dashv B, \vdash (A \to B) / \dashv B \tag{MT}$$

and reverse substitution

$$\dashv \sigma(A) / \dashv A. \tag{RS}$$

He proved that all classically invalid formulas are refutable by means of this refutation system (the system is complete) and no classically valid formula can be refuted (the system is consistent). The sketch of the proof is as follows.

1. Completeness: if a formula A is not derivable in classical propositional calculus (CPC) then there exists such a substitution σ of formulas $(p \to p)$ and $(p \wedge \neg p)$ for propositional variables that $\vdash \sigma(A) \to (p \wedge \neg p)$ and, hence, $\vdash \sigma(A) \to p$. Application of rules MT and RS completes the proof.
2. Consistency: since classical logic is closed under rules Modus Ponens and substitution, it is impossible to refute any classically valid formula by rules MT and RS.

In order to use a similar approach for intermediate logics (or normal extensions of **S4**) we need to find "a replacement" for formulas representing logical constants. In the case when logic enjoys the finite model property (f.m.p.) instead of formulas-constants one can use the Jankov formulas. Let say L is an intermediate (or normal modal) logic and $\mathcal{A} = \{\mathsf{A}_1, \mathsf{A}_2, ...\}$ is a characteristic set of finite subdirectly irreducible (f.s.i.) algebras, i.e. every formula from L is valid in every algebra from $\mathcal{A}$, while if $A \notin L$ then A is invalid in at least one algebra from $\mathcal{A}$ [1]. Let A be a formula refutable in A_i and C_i be a Jankov formula of A_i. Instead of substituting variables with $(p \to p)$ and $(p \wedge \neg p)$ as we did for CPC, we will find (see Theorem 3.1) such a substitution σ that $\vdash \sigma(A) \to C_i$. If $\dashv C_i$ is an antiaxiom, then $\dashv A$ can be derived from $\vdash \sigma(A) \to C_i$ by (MT) and

[1] The most famous such a set for intuitionistic propositional calculus (IPC) is, of course, the set of Jaśkowski matrices.

(RS). This means that the system $\langle\{\dashv C_i \,; i\in I\}\,;\ MT, RS\rangle$ is a complete consistent refutation system [10, 12, 11]. Moreover, if L is tabular, that is L has a finite characteristic set of f.s.i. algebras, there is a finite complete set of antiaxioms. For instance, characteristic set for CPC consists of one algebra, namely, 2-element Boolean algebra whose characteristic formula is $(\neg p \to p)$, which is classically equivalent to p.

In order to be able to construct Jankov formulas for finite s.i. algebras we need only implication properly coordinated with congruences. In the Section 2 we introduce and study the class of algebras (that we call "'regular"') which are Hilbert algebras [3] with additional compatible[2] operations. In Section 3 we will see how one can construct Jankov formulas for regular algebras and we will prove the Jankov theorem for Jankov formulas of finite s.i. regular algebras. And in the last section we will show how the Jankov formulas can be used for constructing refutation systems.

2. Regular Logics and Algebras

We will consider algebras in the signature $\{f_0, f_1, \dots, f_n\}$, where f_0 is $\to$ and f_1 is $\mathbf{1}$. Let us recall (e.g. [3]) the following definition of *Hilbert algebra*[3].

Definition 2.1. An algebra $\mathsf{A} = \langle \mathsf{A}, \to, \mathbf{1}\rangle$ is called Hilbert algebra if $\to$ satisfies the regular axioms for implication:

1. $x \to (y \to x) = \mathbf{1}$;
2. $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = \mathbf{1}$;
3. if $x \to y = y \to x = \mathbf{1}$, then $x = y$;
4. $x \to \mathbf{1} = \mathbf{1}$.

Since in any Hilbert algebra the identity $x \to x = \mathbf{1}$ holds (e.g. [9, Ch.2 (3)]), condition 3 of the above Definition is equivalent to the following:

$$x = y \text{ if and only if } x \to y = y \to x = \mathbf{1}$$

[2]See [1, 2].

[3]In [9] Hilbert algebras are called "positive implication algebras".

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A subset F of the elements of Hilbert algebra A is called *implicative filter* if

1. $\mathbf{1} \in \mathsf{F}$
2. $\mathsf{a}, \mathsf{a} \to \mathsf{b} \in \mathsf{F}$ yields $\mathsf{b} \in \mathsf{F}$.

If $\mathsf{a} \in \mathsf{A}$ is an element, by $[\mathsf{a})$ we denote an implicative filter generated by element a, that is, $[\mathsf{a}) = \{\mathsf{b}; \mathsf{a} \to \mathsf{b} = \mathbf{1}\}$. If A is an algebra and θ is a congruence on A then by $[\![\mathsf{a}]\!]_\theta$ we denote a congruence class containing element a (and often we will omit index when no confusion arises).

All facts regarding Hilbert algebras that we will be using can be found in [3, 9]. We will need the following properties of Hilbert algebras.

Proposition 2.1. *Let* A *be a Hilbert algebra and* $\mathsf{F} \subseteq \mathsf{A}$ *be an implicative filter. Then the following holds*

(a) $\mathbf{1} \to \mathsf{a} = \mathsf{a}$ *(e.g. [9, 2.3 (10)]);*

(b) The relation θ *such that* $\mathsf{a} \equiv \mathsf{b}$ *if and only if* $\mathsf{a} \to \mathsf{b}, \mathsf{b} \to \mathsf{a} \in \mathsf{F}$ *is a congruence on* A *(e.g. [9, 3.2]);*

(c) If θ *is a congruence on* A *then* $\mathsf{F}(\theta) = [\![\mathbf{1}]\!]_\theta$ *is an implicative filter (e.g. [9, 3.1]);*

(d) A relation $\mathsf{a} \leq \mathsf{b}$ *if and only if* $\mathsf{a} \to \mathsf{b} = \mathbf{1}$ *is a partial order on* A *(e.g. [9, 2.2]) and* $\mathbf{1}$ *is the greatest relative to this order element.*

Let A be an algebra and θ be a congruence on A. Let us observe that $[\![\mathsf{c}]\!] = [\![\mathsf{d}]\!]$ if and only if $\mathsf{c} \to \mathsf{d}, \mathsf{d} \to \mathsf{c} \in \mathsf{F}(\theta)$. Indeed, by Definition 2.1(3) $[\![\mathsf{c}]\!] = [\![\mathsf{d}]\!]$ if and only if $[\![\mathsf{c}]\!] \to [\![\mathsf{d}]\!] = [\![\mathbf{1}]\!]$ and $[\![\mathsf{d}]\!] \to [\![\mathsf{c}]\!] = [\![\mathbf{1}]\!]$. The latter is equivalent to $[\![\mathsf{c} \to \mathsf{d}]\!] = [\![\mathbf{1}]\!] = \mathsf{F}(\theta)$ and $[\![\mathsf{d} \to \mathsf{c}]\!] = \mathsf{F}(\theta)$, which is equivalent to $\mathsf{c} \to \mathsf{d}, \mathsf{d} \to \mathsf{c} \in \mathsf{F}(\theta)$. Thus, a congruence class $\mathsf{F}(\theta)$ uniquely defines the congruence. If F is a filter, then by $\theta(\mathsf{F})$ we will denote a congruence induced by filter F. Clearly $\theta(\mathsf{F}(\theta)) = \theta$ and $\mathsf{F}(\theta(\mathsf{F})) = \mathsf{F}$. If $\mathsf{a} \in \mathsf{A}$ then by $\mathsf{F}(\mathsf{a})$ we denote a filter F of algebra A generated by a. A filter generated by one element is called *principal*.

Definition 2.2. Let $\mathcal{K}$ be a class of algebras. We say that $\mathcal{K}$ is a class of *regular algebras* if

(a) the $\{\to, \mathbf{1}\}$-reduct of each algebra $\mathsf{A} \in \mathcal{K}$ is a Hilbert algebra;

(b) there is a formula $R(p)$ such that $\mathsf{F}(\mathsf{a}) = [R(\mathsf{a}))$ for every $\mathsf{A} \in \mathcal{K}$ and every $\mathsf{a} \in \mathsf{A}$.

We will say that an element a is R-stable if $\mathsf{a} = R(\mathsf{a})$. The condition (b) of above definition means that every principal filter is a principal implicative filter generated by some R-stable element.

The following propositions provide a natural and intrinsic characterization of formula R.

Proposition 2.2. *Let* A *be a (regular) algebra,* $\mathsf{a}, \mathsf{b} \in \mathsf{A}$ *and* $R(\mathsf{a}) \to \mathsf{b} \neq \mathbf{1}_{\mathsf{A}}$*. Then there is a congruence* θ *such that* $[\![\mathsf{a}]\!]_\theta = \mathbf{1}_{\mathsf{A}/\theta}$*, while* $[\![\mathsf{b}]\!]_\theta \neq \mathbf{1}_{\mathsf{A}/\theta}$*.*

Proof. Let $\theta = \theta(R(\mathsf{a}))$. Then by Definition 2.2(b) we have $F(\theta) = [R(\mathsf{a}))$, hence, $\mathsf{b} \notin F(\theta))$, that is $[\![\mathsf{b}]\!]_\theta \neq \mathbf{1}_{\mathsf{A}/\theta}$. □

To simplify notation, if $(A \to (B \to C)) = \mathbf{1}$ and $(A \to (C \to B)) = \mathbf{1}$ we will write $(A \to (B \leftrightarrow C)) = \mathbf{1}$.

Proposition 2.3. *Suppose* $\mathcal{V}$ *is a variety and* $R(p)$ *is a formula. Then the following conditions are equivalent*

1. *Condition (b) of Definition 2.2 holds;*
2. *For each algebra* $\mathsf{A} \in \mathcal{V}$ *the following hold*
   - *(a)* $R(\mathbf{1}) = \mathbf{1}$*;*
   - *(b)* $R(p) \to p = \mathbf{1}$*;*
   - *(c)* $R(p) \to R(R(p)) = \mathbf{1}$*;*
   - *(d)* $R(p \to q) \to (R(p) \to R(q)) = \mathbf{1}$*;*
   - *(e) for each fundamental operation* $f(p_1, \dots, p_n)$ *if for all* $i = 1, \dots, n$ $R(p) \to (p_i \leftrightarrow q_i) = \mathbf{1}$*, then* $R(p) \to (f(p_1, \dots, p_n) \leftrightarrow f(q_1, \dots, q_n)) = \mathbf{1}$*;*
3. *For each algebra* $\mathsf{A} \in \mathcal{V}$ *an implicative filter* F *of* A *is a filter of* A *if and only if* F *is closed under* R*, that is,* $\mathsf{a} \in \mathsf{F}$ *yields* $R(\mathsf{a}) \in \mathsf{F}$*.*

Proof. $1 \Rightarrow 2$(a). $\{\mathbf{1}\} = F(\mathbf{1})$, hence, by Definition 2.2(b) $R(\mathbf{1}) \in \{\mathbf{1}\}$.
$1 \Rightarrow 2$(b). Straight from the definition of R.

$1 \Rightarrow 2$(c). $[R(\mathsf{a}))$ is a filter of algebra A, hence, by Definition 2.2(b), we have $[R(R(\mathsf{a}))) = [R(\mathsf{a}))$.

$1 \Rightarrow 2$(d). For contradiction, assume $R(p \to q) \to (R(p) \to R(q)) \neq \mathbf{1}$. Then, by virtue of Proposition 2.2 (applied twice), we conclude that for some congruence θ we have $[\![R(\mathsf{a} \to \mathsf{b})]\!]_\theta = \mathbf{1}_{\mathsf{a}/\theta}$ and $[\![R(\mathsf{a})]\!]_\theta = \mathbf{1}_{\mathsf{a}/\theta}$, but $[\![\mathsf{b}]\!]_\theta \neq \mathbf{1}_{\mathsf{A}/\theta}$. On the other hand, by Definition 2.2(b) we get $[\![\mathsf{a} \to \mathsf{b}]\!]_\theta = \mathbf{1}_{\mathsf{A}/\theta}$ and $[\![\mathsf{a}]\!]_\theta = \mathbf{1}_{\mathsf{A}/\theta}$. Thus, $[\![\mathsf{b}]\!]_\theta = \mathbf{1}_{\mathsf{A}/\theta}$.

$1 \Rightarrow 2$(e). Immediately from the Definition 2.2: $[R(\mathsf{a}))$ is a filter, therefore $[R(\mathsf{a}))$ defines such a congruence θ that $\mathsf{b} \equiv \mathsf{c}(\theta)$ if and only if $(\mathsf{b} \to \mathsf{c}), (\mathsf{c} \to \mathsf{b}) \in [R(\mathsf{a}))$.

$2 \Rightarrow 3$. Assume that F is a filter of A and $\mathsf{a} \in \mathsf{F}$. Then $\mathsf{a} \equiv \mathbf{1}(\theta(\mathsf{F}))$. Hence, $R(\mathsf{a}) \equiv R(\mathbf{1})(\theta(\mathsf{F}))$. By 2.(a), $R(\mathbf{1}) = \mathbf{1}$, therefore, we have $R(\mathsf{a}) \equiv \mathbf{1}(\theta(\mathsf{F}))$, that is $R(\mathsf{a}) \in \mathsf{F}$.

Conversely, let F be an implicative filter and F is closed under R. Then by 2(e) F is a filter.

$3 \Rightarrow 1$. Straightforward. □

Remark 1. Let us point out that the regular algebras are different from pseudo-interior algebras (e.g. [1]) and the formula $R(p)$, as we will see from the examples, does not necessarily define a pseudo-interior operator p° (see examples of pseudo-interior algebras and pseudo-interior operators in [2]). On the other hand, the formula $R(x \to y) \to (R(y \to x) \to z)$ is a ternary deductive term [1], thus, any variety of regular algebras has equationally definable principal congruences ([1][Corollary 2.5]).

Example 1. *The following is a (not exhaustive) list of varieties of algebras that have a formula satisfying* (2.2).

- *Hilbert algebras:* $R(p) = p$;
- *Brouwerian semilattices:* $R(p) = p$;
- *Brouwerian lattices:* $R(p) = p$;
- *Heyting algebras:* $R(p) = p$;
- *interior (S4) algebras:* $R(p) = \Box p$;
- *monadic Heyting algebras:* $R(p) = \Box p$
- *n-transitive algebras:* $R(p) = p \wedge \Box p \wedge \Box^2 p \wedge \cdots \wedge \Box^{n-1} p$.

Let us also observe that the properties of regular s.i. algebras are similar to those of interior algebras. First, let us recall [5] that an algebra is *subdirectly irreducible* if and only if it has either the only congruence, or the smallest non-trivial congruence.

Proposition 2.4. *A non-trivial regular algebra* A *is s.i. if and only if it has the greatest distinct from* $\mathbf{1}$ *R-stable element.*

Proof. By virtue of Proposition 2.3.3, the meet of any set of filters is a filter. If an algebra A is s.i., the meet of all non-trivial (that is, distinct from $\{\mathbf{1}\}$) filters is a filter $\mathsf{F} \subseteq \mathsf{A}$. Since F is the smallest proper filter, F is a principal filter. Assume that $\mathsf{F} = [\mathsf{a})$. Let us check that $R(\mathsf{a})$ is the greatest distinct from $\mathbf{1}$ R-stable element of A. From Proposition 2.3 it follows that element a is R-stable. Let $\mathsf{b} \in \mathsf{A}$ be a R-stable element and $\mathsf{b} \neq \mathbf{1}$. Then $[R(\mathsf{b}))$ is a filter and $[R(\mathsf{a})) \subseteq [R(\mathsf{b}))$. Hence, $R(\mathsf{b}) \leq R(\mathsf{a})$. Thus, $R(\mathsf{a})$ is the greatest distinct from $\mathbf{1}$ R-stable element of A.

Conversely, if a is the greatest distinct from $\mathbf{1}$ R-stable element of A then, by virtue of Proposition 2.3, filter $[R(\mathsf{a}))$ is the smallest proper filter of A. □

If A is an s.i. algebra, the element that generates the smallest non-trivial filter, that is the greatest distinct from $\mathbf{1}$ R-stable element, we will call an *opremum* and denote it by $op(\mathsf{A})$.

The Proposition 2.3 suggests the following definition.

Definition 2.3. A logic L in signature $\{\to, f_1, \dots, f_m\}$ we call *regular* if the following axioms

A1. $R(p) \to p$;

A2. $R(p) \to R(R(p))$;

A3. $R(p \to q) \to (R(p) \to R(q))$;

and the rules

$$\frac{A}{R(A)} \tag{RG}$$

$$\frac{R(p) \to (q_1 \leftrightarrow r_1), \dots, R(p) \to (q_k \leftrightarrow r_k)}{R(p) \to (f_i(q_1, \dots, q_k) \to f_i(r_1, \dots, r_k))} \tag{RE}$$

hold for all $i = 1, \dots, m$.

Let us note that (RE) is the requirement of compatibility of additional operations [1]. Clearly, regular algebras are models for regular logics.

3. Characteristic Formulas of Finite Regular Algebras

From this point forward we consider an arbitrary but fixed variety $\mathcal{V}$ of regular algebras. And let L be a logic, corresponding to this variety, that is, L is the set of all formulas valid in every algebra of $\mathcal{V}$. If A is a formula and $A \in \mathsf{L}$ we also will write $\vdash A$. If $D = \{A_1, \dots, A_k\}$ is a set of formulas and B is a formula by $D \Rightarrow B$ we denote a formula $R(A_1) \to (R(A_2) \to (\dots (R(A_k) \to R(B)\dots)))$. If A is a finite algebra then by $Dg(\mathsf{A})$ we denote a *diagram set* (cf. with diagram formula in [4, p. 442]): with each element $\mathsf{a} \in \mathsf{A}$ we associate a variable p_{a} and we let $Dg(\mathsf{A})$ be a set of all formulas

$$f(p_{\mathsf{a}_1}, \dots, p_{\mathsf{a}_k}) \to p_{f(p_{\mathsf{a}_1}, \dots, p_{\mathsf{a}_k})}$$

and

$$p_{f(p_{\mathsf{a}_1}, \dots, p_{\mathsf{a}_k})} \to f(p_{\mathsf{a}_1}, \dots, p_{\mathsf{a}_k})$$

for all fundamental operations $f(p_1, \dots, p_k)$.

With each finite s.i. (regular) algebra A we associate a *Jankov formula* in the following way (cf. [6]):

$$J(\mathsf{A}) = Dg(\mathsf{A}) \Rightarrow p_{op(\mathsf{A})}.$$

As we will see from the following theorem, the Jankov formulas of regular algebras enjoy the same properties as Jankov formulas of Heyting algebras (cf. [7]).

Theorem 3.1. *Let A be a f.s.i. algebra, B be an algebra and B be a formula. Then*

(Hom) if $\mathsf{B} \not\models J(\mathsf{A})$, then A is (isomorphically) embeddable in some homomorphic image of B;

(Ded) if $\mathsf{A} \not\models B$, then there is such a substitution σ that $\vdash \sigma(B) \Rightarrow J(\mathsf{A})$.

Proof. (Hom) Let $\mathsf{B} \not\models J(\mathsf{A})$ and ν be a refuting valuation. Then applying multiple times Proposition 2.2, we can conclude that there is such a congruence θ that $[\![\nu(D)]\!] = [\![\mathbf{1}_{\mathsf{B}}]\!]$ for all $D \in Dg(\mathsf{A})$, while $[\![\nu(p_{op(\mathsf{A})}]\!] \neq [\![\mathbf{1}_{\mathsf{B}}]\!]$. Let us consider $\mathsf{B}' = \mathsf{B}/\theta$ and let $\overline{\nu}$ be a natural extension of ν. Then we have

$$\overline{\nu}(D) = \mathbf{1}_{\mathsf{B}'} \text{ for all } D \in Dg(\mathsf{A}),$$

while

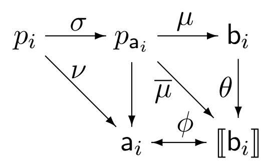
$$\overline{\nu}(p_{op(\mathsf{A})}) \neq \mathbf{1}_{\mathsf{B}'}.$$

It is not hard to check that the mapping $\phi : \mathsf{a} \mapsto \overline{\nu}(p_{\mathsf{a}})$ is a homomorphism. Let us observe that $\phi(op(\mathsf{A})) \neq \mathbf{1}_{\mathsf{B}'}$, hence, $\mathsf{F}(\phi) = \{\mathbf{1}_{\mathsf{A}}\}$, thus, ϕ is an isomorphism.

(Ded) Let $\mathsf{A} \not\models B(p_1, \ldots, p_k)$ and ν is a refuting valuation. Let $\sigma : p_i \mapsto p_{\nu(p_i)}$ be a substitution and let $B' = \sigma(B)$. We want to prove that $\vdash \sigma(B) \to J(\mathsf{A})$. Assume the contrary: $\nvdash \sigma(B) \to J(\mathsf{A})$. Then for some algebra B there is such a valuation μ that

$$\mu(\sigma(B)) = \mathbf{1}_{\mathsf{B}} \text{ and } \mu(J(\mathsf{A})) \neq \mathbf{1}_{\mathsf{B}}. \tag{3.1}$$

By virtue of (Hom), algebra A is embeddable in some homomorphic image B' of algebra B. Let $\phi : \mathsf{A} \to \mathsf{B}'$ be the embedding. Then if $\overline{\mu}$ is a natural extension of the valuation μ to B', we have $\overline{\mu}(\sigma(B)) \neq \mathbf{1}_{\mathsf{B}'}$ and this contradicts (3.1) (see Diag. 1).



Diag. 1:

□

4. Refutation Systems

In this section we study how Jankov formulas can be used in constructing the refutation systems for regular logics.

Definition 4.1. If A is a formula then A^+ and A^- are *meta-statements*. We will say that a meta-statement A^+ is valid in logic L is $A \in \mathsf{L}$. Accordingly, a meta-statement A^- is valid in L if $A \notin \mathsf{L}$.

We will use the following inference rules for refutation:

$$\frac{(A \to B)^+, B^-}{A^-}, \tag{MT}$$

$$\frac{(\sigma(A))^-}{A^-}, \tag{RS}$$

where σ is any substitution, and

$$\frac{(R(A))^-}{A^-}. \tag{RG}$$

Definition 4.2. System $\langle Ant; MT, RS, RG\rangle$, where Ant is a set of the negative meta-statements $\left\{A_i^-; i \in I\right\}$ (set of *antiaxioms*), is called a *primitive*[4] *refutation system for* L. Refutation system is called *finite* if it contains only a finite number of antiaxioms.

Definition 4.3. (cf. [13, 14]) Let L be a logic, R be a refutation system. The sequence of the meta-statements $A_1^{\iota_1}, ..., A_k^{\iota_k}$, where $\iota_j \in \{+, -\}$ for all $j = 1, ..., k$ is called R-*inference of the meta-statement* $A_k^{\iota_k}$ *over* L if for each $j = 1, ..., k$ one of the following hold

1. ι_j is $+$ and A_j^+ is valid in logic L;
2. ι_j is - and A_j^- is an antiaxiom;
3. ι_j is - and A_j^- can be derived from the preceding meta-statements by MT,RS or RG.

If there is a R-inference that ends with meta-statement A^- we will say that A^- is R-derivable over L.

Refutation system R *is complete* for logic L if A^- is R-derivable over L for any $A \notin \mathsf{L}$. Refutation system R is *consistent* for logic L if for every $A \in \mathsf{L}$ meta-statement A^- is not R-derivable over L. If each antiaxiom of refutations system R is (interderivable with) a Jankov formula, we will say that R is a *Jankov refutation system.*

If L is a logic by $Mod(\mathsf{L})$ we denote the class of all algebras that are models for L, that is, all algebras in which each formula from L is valid. A logic L is said to have a *finite model property (f.m.p.)* if for any formula $A \notin \mathsf{L}$ there is a finite algebra from $Mod(\mathsf{L})$ in which formula A is not valid.

In this paper we consider only consistent refutation systems.

Proposition 4.1. *If* $\mathsf{R} = \langle\{J(\mathsf{A}_i)^-; i \in I\}; MT, RS, RG\rangle$ *is a complete Jankov refutation system for* L*, then* $\mathsf{A}_i \in Mod(\mathsf{L})$ *for all* $i \in I$.

[4]Since here we consider only primitive refutation system the word “primitive” often will be omitted.

Proof. For contradiction, assume $\mathsf{A}_i \notin Mod(\mathsf{L})$. Then there is such a formula A that

$$A \in \mathsf{L} \text{ and } \mathsf{A}_i \not\models A. \tag{4.1}$$

By virtue of Theorem 3.1 (Ded), there is a substitution σ such that

$$\vdash \sigma(A) \Rightarrow J(\mathsf{A}_i). \tag{4.2}$$

Thus, the following sequence of meta-statements is a R-inference of A^- (we can apply consequently (MT), (RG) and (RS) to the first two statements):

$$(\sigma(A) \Rightarrow J(\mathsf{A}_i))^+, J(\mathsf{A}_i)^-, R(\sigma(A))^-, \sigma(A)^-, A^-$$

From the consistency of R it follows that $A \notin \mathsf{L}$. And the latter contradicts (4.1). □

Lemma 4.2. *Let logic L has a complete Jankov refutation system*

$$\mathsf{R} = \langle \{J(\mathsf{A}_i); i \in I\}; MT, RS, RG \rangle.$$

If meta-statement A^- is R-derivable over L, then for some $i \in I$ formula A is invalid in algebra A_i.

Proof. Assume A^- is R-derivable over L. Then there is a R-inference of A^- over L. We will prove our claim by induction on the length of this R-inference.

Let $A_1^{\iota_1}, ..., A_k^{\iota_k}$ be a R-inference of A^- over L. Thus, $A_k^{\iota_k} = A^-$.

Basis. If $k = 1$ then A^- is an antiaxiom, hence, $A = J(\mathsf{A}_i)$ for some $i \in I$ and, therefore, A is invalid in A_i.

Let us assume that for all R-inferences of A^- of length $< k$ the statement is true. Now let us consider a R-inference of length k. In this case the meta-statement A^- can be either antiaxiom, or be obtained from the preceding meta-statements by (MT),(RS) or (RG). If A^- is an antiaxiom then, as we saw, A is not valid in one of the algebras A_i. Let us consider three remaining possibilities.

Case of (MT). Assume A^- is derived by (MT) and we need to demonstrate that formula A is invalid in one of the algebras A_i. The fact that A^- was derived by (MT) means that for some $1 \leq r, s < k$ we have $A_r^+ = (A \to B)^+$ and $A_s^- = B^-$. Due to consistency of R, we get

$$(A \to B) \in L. \tag{4.3}$$

According to the assumption formula B is invalid in some algebra A_i, that is

$$\mathsf{A}_i \not\models B. \tag{4.4}$$

Since $(A \to B) \in L$ and, by virtue of Proposition 4.1, $\mathsf{A}_i \in Mod(\mathsf{L})$, we have

$$\mathsf{A}_i \models (A \to B). \tag{4.5}$$

From (4.4) and (4.5) it follows that $\mathsf{A}_i \not\models A$.

Case of RS. Assume A^- is derived by (RS). It means that for some $1 \leq r < k$ and some substitution σ we have $A_r^- = \sigma(A)$. By assumption A_r, and therefore $\sigma(A)$, is invalid in some algebra A_i and, obviously, formula A cannot be valid in A_i too.

Case of RG. Assume A^- is derived by (RG). It means that for some $1 \leq r < k$ we have $A_r^- = R(A)$. By assumption $R(A)$ is invalid in some algebra A_i and, since $R(\mathbf{1}_{\mathsf{A}_i}) = \mathbf{1}_{\mathsf{A}_i}$, formula A cannot be valid in A_j too. □

Theorem 4.3. *A logic* L *has a complete Jankov refutation system if and only if* L *enjoys f.m.p.*

Proof. Let logic L has a complete Jankov refutation system

$$\mathsf{R} = \langle \{ J(\mathsf{A}_i)^-; i \in I \} ; MT, RS, RG \rangle.$$

Assume A is a formula and $A \notin \mathsf{L}$. From the completeness of R it follows that there is a R-inference of the meta-statement A^- over L. By Lemma 4.2, for some $i \in I$ formula A is invalid in A_i. Recall, that for all $i \in I$ algebra A_i is finite. Hence, logic L enjoys f.m.p.

Conversely, assume L enjoys f.m.p.. Then from f.m.p. and Theorem 3.1(Ded) it immediately follows that the negative meta-statements obtained from the Jankov formulas of all f.s.i. algebras from $Mod(\mathsf{L})$ form a set of antiaxioms for a complete consistent refutation system for L. □

Remark 2. For intermediate logics cf. [10].

Proposition 4.4. *If* L *is a logic that enjoys f.m.p. and has a finite complete refutation system then* L *has a finite complete Jankov refutation system.*

Proof. Assume $A_1^-, ..., A_n^-$ are all antiaxioms of a complete refutation system. Since refutation system is consistent we have $A_j \notin \mathsf{L}$ for all $j = 1, \ldots, n$. Because L enjoys f.m.p. there are such finite algebras $\mathsf{A}_j \in Mod(\mathsf{L}); j = 1, \ldots, n$, that formula A_j is refutable in algebra $\mathsf{A}_j; j = 1, ..., n$. Since $Mod(\mathsf{L})$ forms a variety, we can safely assume that algebras A_j are subdirectly irreducible. By virtue of Theorem 3.1(Ded), for each $j = 1, \ldots, n$ we can replace each antiaxiom A_j^- with the metastatement $J(\mathsf{A}_j)^-$ and get a new consistent complete refutation system for L. □

Let us recall that a logic L is *tabular* if for some finite algebra A we have $A \in \mathsf{L}$ if and only if $\mathsf{A} \models A$ for all formulas A. The set of algebras $\{\mathsf{A}_i; i \in I\}$ is a *characteristic set for* L if $\mathsf{L} = \cap_{i \in I} \{A; \mathsf{A}_i \models A\}$. It is easy to see that if for logic L there is a finite characteristic set of algebras, then L is tabular: the direct product of all algebras from this set defines logic L.

Theorem 4.5. *Let* L *be a logic that enjoys f.m.p.. Then* L *has a complete finite refutation system if and only if it is tabular.*

Proof. Assume L has a finite complete refutation system. Then by virtue of Proposition 4.4, logic L has a finite complete Jankov refutation system. Let

$$\mathsf{R} = \langle \left\{ J(\mathsf{A}_i)^-; i = 1, \ldots, n \right\}; MT, RS, RG \rangle$$

be a complete refutation system for L. Let us check that the set of algebras $\mathcal{A} = \{\mathsf{A}_i; i = 1, \ldots, n\}$ is a characteristic set for logic L. Indeed, assume A is a formula and $A \notin \mathsf{L}$. From completeness of R it follows that the metastatement A^- is R-derivable over L. Hence, by virtue of Lemma 4.2, formula A is invalid in some algebra from $\mathcal{A}$. Thus, the set $\mathcal{A}$ is a characteristic set for L and L is tabular.

Conversely, suppose L is tabular and A is a finite algebra such that for any formula A

$$\mathsf{A} \models A \text{ if and only if } A \in \mathsf{L}.$$

Let

$$\mathcal{A} = \{\mathsf{A}_1, \ldots, \mathsf{A}_n\}$$

be the set of all (modulo isomorphism) s.i. homomorphic images of A. Note, that any formula A is valid in A (or in L for this matter) if and only if formula A is valid in all algebras from $\mathcal{A}$. From Theorem 3.1 (Ded) it follows that the refutation system $\langle \{J(\mathsf{A}_i)^-; i = 1, \dots, n\}; MT, RS, RG\rangle$ is complete for logic L. □

References

[1] W. J. Blok and D. Pigozzi, *On the structure of varieties with equationally definable principal congruences. III*, Algebra Universalis **32**:4 (1994), pp. 545–608.

[2] W. J. Blok and Don Pigozzi, *On the structure of varieties with equationally definable principal congruences. IV*, Algebra Universalis **31**:1 (1994), pp. 1–35.

[3] A. Diego, Sur les algèbres de Hilbert, Translated from the Spanish by Luisa Iturrioz. Collection de Logique Mathématique, Sér. A, Fasc. XXI. Gauthier-Villars, Paris, 1966.

[4] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated lattices: an algebraic glimpse at substructural logics, volume 151 of Studies in Logic and the Foundations of Mathematics, Elsevier B. V., Amsterdam, 2007.

[5] G. Grätzer, Universal algebra, Springer, New York, second edition, 2008. With appendices by Grätzer, Bjarni Jónsson, Walter Taylor, Robert W. Quackenbush, Günter H. Wenzel, and Grätzer and W. A. Lampe.

[6] V. A. Jankov, *On the relation between deducibility in intuitionistic propositional calculus and finite implicative structures*, Dokl. Akad. Nauk SSSR **151** (1963), pp. 1293–1294.

[7] V. A. Jankov, *Conjunctively irresolvable formulae in propositional calculi*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), pp. 18–38.

[8] Jan Łukasiewicz, *On the intuitionistic theory of deduction*, Nederl. Akad. Wetensch. Proc. Ser. A. **55** = Indagationes Math., 14 (1952), pp. 202–212.

[9] H. Rasiowa, An algebraic approach to non-classical logics, Studies in Logic and the Foundations of Mathematics, Vol. 78, North-Holland Publishing Co., Amsterdam, 1974.

[10] T. Skura. *Refutation calculi for certain intermediate logics*, Notre Dame Journal of Formal Logic **33**:4 (1992), pp. 552–560.

[11] T. Skura, Aspects of Refutation Procedures in the Intuitionistic Logic and Related Modal Systems, Acta Universitatis Wratislaviensis N 2190, Wroclaw, 1998.

[12] T. Skura. *Syntactic refutations against finite models in modal logic*, Notre Dame J. Formal Logic **35**:4 (1994), pp. 595–605.

[13] W. Staszek, *On proofs of rejection*, Studia Logica **29** (1971), pp. 17–25.

[14] W. Staszek, *A certain interpretation of the theory of rejected propositions*, Studia Logica **30** (1972), pp. 147–152.

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