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# KRONECKER IN CONTEMPORARY MATHEMATICS GENERAL ARITHMETIC AS A FOUNDATIONAL PROGRAMME

A b s t r a c t. Kronecker called his programme of arithmetization "General Arithmetic" (*Allgemeine Arithmetik*). In his view, arithmetic is the building block of the whole edifice of mathematics. The aim of this paper is to show that Kronecker's arithmetical philosophy and mathematical practice have exerted a permanent influence on a long tradition of mathematicians from Hilbert to Weil, Grothendieck and Langlands. The conclusion hints at a constructivist finitist stance in contemporary mathematical logic, especially proof theory, beyond Hilbert's programme of finitist foundations which can be seen as the continuation of Kronecker's arithmetization programme by metamathematical or logical means. It is finally argued that the introduction of higher-degree polynomials by Kronecker inspired Hilbert's notion of functionals, which in turn influenced Gödel's functional *Dialectica* interpretation for his intuitionistic proof of the consistency of arithmetic.

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## 1. Introduction. Arithmetical philosophy

I understand arithmetical philosophy on the model of Russell's mathematical philosophy as an internal examination of arithmetical concepts – in the case of Russell, the internal examination of logical and general mathematical concepts (Russell [1919]). The two texts "On the concept of number" (Kronecker [1987b]) and his last lectures in Berlin "On the concept of number in mathematics" (see the German text edited by Boniface and Schappacher [2001]) summarize Kronecker's conception of number or whole number (integer). Kronecker shares with Gauss the idea that the concept of number is in the mind or *a priori* while space is a property or relation in the external world; geometry and mechanics do not belong to the realm of pure mathematics since they have to represent and picture natural processes by using the concept of continuity whereas number inhabits the discrete universe of ordinals. Cardinals are invariants for the counting of groups of objects and equivalence is an intensional relation. The concrete combinatorial procedures (*Verfahren*) of addition, multiplication, congruence, etc. join with the general concepts of forms or homogeneous polynomials and their properties in the process of arithmetization.

There is a Kantian background to Kronecker's conception of number and Kronecker could not help but mock the philosophy of mathematics of post-Kantian philosophers like Schelling and Hegel[1]. Philosophical definitions of number are useless and one must start with the basic facts of a science (arithmetic here) and then fully elaborate the conceptual determinations (*Begriffsbestimmungen*) of the subject matter. In that sense, pure mathematics was for Kronecker an experimental science in the construction of concepts in accord with Kant's dictum "Mathematics constructs concepts, philosophy analyzes them". Beyond this motto, Kronecker has hoped for a thorough arithmetization of mathematics, especially algebra; arithmetization of algebra has been the main task of his mathematical life as Kronecker confesses in a letter to Lipschitz ([1986] : 181-182)

> On that occasion [the publication of his 1882 paper], I have

[1] Kronecker does not reject all of Hegel and he quotes him approvingly on certain occasions, but he has not taken Hegel's conception of number seriously. One should mention however that contemporary mathematicians, like Lawvere in category theory, logicians in non-standard analysis and philosophers of logic (dialetheism and paraconsistent logic) have tried to make good of some of Hegel's ideas.

> found the long-sought foundations of my entire theory of forms which somehow brings to completion "the arithmetization of algebra" which has been the goal of my whole mathematical life; it is evident to me that at the same time arithmetic cannot dispense with the "association of forms" and that without them, it can only go astray in meandering thoughts (*Gedankengespinste*) as is the case with Dedekind, where the true nature of the matter is obscured rather than illuminated.
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> (my translation)

Beyond the polemical tone, one sees the central role of his 1882 formulation and it is especially in that connection that Hermann Weyl has asserted the superiority of Kronecker's algorithmic approach in algebraic number theory with his domains of rationality (*Rationalitätsbereiche*) over Dedekind's concept of field (*Körper*). Association of forms means in that context homogeneous polynomials with integer coefficients and indeterminates, the central topic of Kronecker's major work (1882) "*Die Grundzüge einer arithmetischen Theorie der algebraischen Grössen*" ("On the Fundamental Features of an Arithmetical Theory of Algebraic Quantities").

As far as analysis is concerned, Kronecker has sought arithmetical invariants in the theory of elliptic functions and Weil has granted him the status of the pioneer of algebraic-arithmetic geometry. In those lectures of 1891, Kronecker comes back to the approximation method which he calls localization (*Isolierung*) of real roots of an algebraic equation in well-defined intervals of values for algebraic equalities and inequalities. In his criticism of Bolzano's theorem on intermediate values, Kronecker villifies Bolzano for having used the crudest means (*mit den rohesten Mitteln*) to obtain an analytical result which cannot be applied to the roots of an entire function. He also mentions Dirichlet's celebrated analytical proof on the infinity of primes in any arithmetical progression which he has discussed in his *Vorlesungen über Zahlentheorie* (Kronecker [1901]). As K. Hensel puts it in the Preface, the methods of arithmetic and algebra rest on a finite number of steps, (*eine endliche Anzahl von Versuchen*), while analysis is built upon the concepts of continuity and limit. Here Kronecker tackles Dirichlet's transcendental proof on the infinity of primes in any arithmetical progression and introduces an arithmetical extension on a finite interval $(\mu \ldots \nu)$

for two integers $\mu, \nu$ where one must find at least one prime of the form $hm + r$ for $m$ and $r$ with no common divisors. Kronecker says that it is one case among others where arithmetic can do more than analysis and go beyond analytical methods. Dirichlet had used infinitesimal analysis (infinite series) in his proof and had confessed that what was still lacking were the right principles or conditions under which transcendental relations between indeterminate integers could be removed.

Kronecker defines arithmetic as pure mathematics free from space and time (see Boniface and Shappacher [2001] : 227) and pays tribute to Gauss for having defined the true nature of arithmetic with the introduction of the concepts of composition (and decomposition into roots) of algebraic systems, in this case quadratic forms (*ibid.*, 262), and he credits him also with the introduction of the notion of indeterminates (*indeterminatae*). In his opposition to the analytical concepts of continuity and limit, Kronecker is echoing Gauss who in a 1831 letter to Schumacher did denounce the use of completed infinite quantities (*vollendete unendliche Grösse*) with the maxim "The infinite is only a manner of speech", (*Das Unendliche ist nur eine Façon de parler*). Kronecker could have made that maxim his own, but Leibniz had already qualified those infinitesimal quantities as useful fictions in the calculus. Kronecker would have been surprised though, had he studied more seriously Hegel's conception of the mathematical infinite, to learn that Hegel espoused the Leibnizian-Gaussian idea of a differential calculus dealing with the relative character of quantities rather than with the absolute limits of an infinite iterative process. In any case, Kronecker's view of the matter is in total agreement with Gauss' arithmetical philosophy and it is no surprise this time if he has opposed Cantor's practice of transfinite arithmetic which he has discarded as mathematical sophistry.

A few important mathematicians have emphasized Kronecker's influence on contemporary mathematics : among them, first and foremost Weil ([1976], [1979]) has stressed the fact that Kronecker is the founder of modern algebraic geometry and Edwards [1990], [1992] after Weyl [1940] has insisted on Kronecker's pioneering work in algebraic number theory (divisor theory). Bishop [1970] has admitted in his work on the computational (or numerical) content of classical analysis that his enterprise was more in line with Kronecker than with Brouwer. Brouwer himself paid tribute to Kronecker – as did Poincaré and Hadamard – for his contribution to the fixed point theorems (see Gauthier [2009]). Poincaré for one among many others like Skolem or van der Waerden repeated Hensel's catch phrase in the Preface of Kronecker [1901] "a finite number of trials" (*eine endliche Anzahl von Versuchen*) to characterize Kronecker's finitist stand; Poincaré used the phrase "finite number of hypotheses" (*nombre fini d'hypothèses*) in his work on the arithmetical properties of algebraic curves (Poincaré [1951]) which was the starting point of contemporary algebraic geometry, from Mordell to Weil and Faltings. I want to concentrate in the following on contemporary algebraic-arithmetic geometry and the two main programmes in the field, Langlands' programme and Grothendieck's programme as they are motivated to a large extent by Weil's own work in algebraic geometry (see Weil [1979a]). Both programmes invoke Kronecker's dream of youth, his theory of forms (homogeneous polynomials) and modular systems which consist in sums and products of polynomials in a general divisor theory that was to become a theory of moduli spaces by successive generalisations and enlargments. In my view, these programmes share some measure of Kronecker's arithmetical philosophy which sees arithmetic as the building block of mathematics.

## 2. Grothendieck's Programme

In *SGA 1*, that is *Séminaire de Géométrie algébrique du Bois-Marie* of 1961, Grothendieck [1971] starts his inquiry into what will be called Grothendieck's programme of the new foundation of algebraic geometry by taking a Kroneckerian point of view :

> The present volume introduces the foundations of a theory of the fundamental group in algebraic geometry from a "Kroneckerian" point of view which allows to deal on the same footing with an algebraic variety (of current usage) and with the ring of integers over a number field, for example. This point of view is best expressed in the language of schemes [...].
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> (my translation)

The Kroneckerian point of view implies that function fields are the analogue of number fields in the sense that an algebraic function field in one variable over the field of rational numbers $\mathbb{Q}$ is an extension of finite degree of the ring of polynomials in one indeterminate $\mathbb{Q}[x]$; function fields behave in $\mathbb{Q}[x]$ as algebraic number fields in $\mathbb{Z}$, the ring of integers, while the field of rational functions is the field of quotients $\mathbb{Q}(x)$ of $\mathbb{Q}[x]$. Kronecker [1883] had sketched in his paper "On the Theory of Higher-Level Forms" (*Zur Theorie der Formen höherer Stufen*) a notion of content or inclusion (*Enthalten-Sein*) for forms or homogeneous polynomials with sums and products of rational functions in a domain of rationality – *Rationalitätsbereich* is the term used by Kronecker instead of Dedekind's term *Körper*, *corps* in French and field in English (see Gauthier [2002]). The notion of *Enthalten-Sein* or "being contained in" is not perfectly clear in Kronecker [1882]. Molk [1885] and Vandiver [1936] have shown how to give a meaning to Kronecker's construction. Molk had insisted on the divisibility theory of polynomials and Vandiver has exhibited an explicit construction of decomposition or devolution (as opposed to convolution) for polynomial ideals. I give here a brief description of Kronecker's construction of these higher-level forms – Kronecker's terminology is in various contexts *Stufe*, *Rang*, *Ordnung* or even *Dimension*. Kronecker had outlined [1882] the most general setting for the decomposition (*Zerlegung*) of polynomial content. I propose here my own interpretation in terms of the convolution (Cauchy) product for polynomials. The general form of the convolution product of two polynomials (forms) encloses (includes) or contains higher-order forms and the substitution-elimination method enables one to remain within the confines of integral forms. Let us start with the convolution or Cauchy product of two polynomials

$$f \cdot g = (\sum_m f_m x^m) \cdot (\sum_n g_n x^n) = (\sum_m \sum_n f_m g_n x^{m+n})$$

with addition of their coefficients $m$ and $n$. In his major work, Kronecker ([1882] : 343) states that a form $M$ is contained in another form $M'$ when the coefficients of the one are contained in the second. He then goes on to formulate propositions on the equivalence of forms like :

> Linear homogeneous forms that are equivalent can be transformed into one another through substitution with integer coefficients.
>
> (Proposition $X$ in Kronecker [1882] : 345.)

and

> Two forms are absolutely equivalent, when they contain each other.
>
> (Proposition $X^0$ in Kronecker [1882] : 351.)

Kronecker states then what he calls a principal result (*Hauptresultat*),

> Every entire algebraic form in the sense of the absolute equivalence of Proposition $X^0$ is representable as a product of irreducible (prime) forms in a unique way.
>
> (Proposition $XIII^0$ in Kronecker [1882] : 352.)

Here Kronecker declares that this result shows that the association of entire algebraic forms by the method of indeterminate coefficients conserves the conceptual determinations of the elementary laws (of arithmetic) in the passage from the rational domain or the domain of entire rational functions to the domain of algebraic functions. But Kronecker is not satisfied and comes back the following year (Kronecker [1883] : 422) to the question and introduces the product

$$\sum_{h=0}^{m} M_h U^h \cdot \sum_{i=1}^{m+1} M_{m+2} U_{m+1}$$

(where $M = M_0, M_1, M_2, \ldots, M_{n+1}$ are integral quantities of successive domains of rationality $R$ and the $U$'s are indeterminates) which defines a form of power $r$ containing the product of forms

$$\prod_{h}^{r} \sum_{k} M_k' V_{hk}$$

which he maintains is still more general than the 1882 formulation. "To be contained" here means only that the polynomials in the domains of rationality are included or contained in a higher rank (order) of their coefficients. A modular system will then decompose this construction into irreducible polynomials. Hence, the notions of inclusion and of equivalence (reciprocal inclusion) of forms are valid generally, *i.e.* for both forms and divisors and factor decomposition is a descending technique perfectly similar to the division algorithm for integers or the Euclidean algorithm for polynomials. Dedekind's Prague Theorem is in the continuity of Kronecker's construction; it says[2] that if all coefficients of the product *fg* of two polynomials *f* and *g* (in one indeterminate) are algebraic, then the product of any coefficient of *f* and any coefficient of *g* is an algebraic integer.

For this unique decomposition of polynomials, descent is used to arrive at irreducible polynomials, much in the same way as in Euclid's proof of the divisibility of composite numbers by primes. Now the fact (Gauss lemma) that the product of two primitive polynomials (with the g.c.d. of their respective coefficients = 1) is primitive can also be had with infinite descent and *reductio ad absurdum*. From this fact combined with the fact that there is unique decomposition into irreducible (prime) polynomials, we obtain unique prime factorization. Kronecker's version of unique decomposition rests on the formula quoted above

$$\prod_{h=1}^{r} M_k U_{hk}$$

and

$$\prod_{i=j+k} c_i = \sum_{j+k=i} a_j b_k$$

with $j = (0, \ldots, m)$ and $k = (0, \ldots, n)$. We shall read it in the form – remembering that $a^{p-1} \equiv 1 (mod\ p)$ from a divisibility point of view –

$$\prod_{i=1}^{m+n} (1 + c_i x_i) = \sum_{i=0}^{m+n} (c_i x^{m+n-1}) = \sum_{m+n=1} (a_m b_n).$$

Kronecker's generalization uses the convolution product for polynomials

$$\sum_h M_h U^h \cdot \sum_i M_{m+i} U^{i-1} = \sum_k M'_k U^k$$

where $k = 0, 1, \ldots, n$ and the equation defines an $n+1$ order system containing $n$ order forms. I would call those forms *polynomial functionals*; they are the entire integer-valued functions that fill up the sphere of forms (Kronecker [1883] : 423). Here the $M$'s are integral forms and the $U$'s indeterminates so that the product mentioned above

[2]See Kronecker ([1883] : 421). Edwards ([1990 : 2] rightly suggests that Dedekind's Prague Theorem – a generalization of Gauss Lemma to the algebraic case – is but a consequence of Kronecker's result.

$$\prod_h \sum_k M_{k'} U_{hk}$$

is "contained" in the resulting form and the product can then be expressed as

$$\sum_k M'_k U^k = (M_k M_{m+1})^k + (M_k M_{m+1})^{k-1} + (M_k M_{m+1})^{k-2} + ... + (M_k M_{m+1})$$

in the decreasing order of the rank $k$ of the polynomial sum. This linear combination obtained by the convolution product and the finite descent on powers shows simply that integral rational forms generate integral algebraic forms, *i.e.* algebraic integers. What we find in the 1883 text is simply a generalisation of Kronecker's 1882 theory of forms which encompasses both the theory of modular systems and the theory of polynomials. The equivalence principle for forms stated in 1882 is valid in full generality and the generalised notion of content or being contained in (*Enthaltensein*) shows that in the construction (*Bildung*) of entire or integral functions the sphere of forms finds its fullest expression (Kronecker[1883] : 423). This is not to say that Kronecker has fully realized his dream, only that he has conceived of an ambitious project that could possibly be fulfilled by a long list of successors.

Kronecker ([1883] : 422) refers explicitely to his 1882 text for the narrower concept of content in his text. As a matter of fact, Hurwitz (see [1895], vol. 2 : 198-207) obtained a proof of Kronecker's theorem by using Lagrange's interpolation (rather than Cauchy's convolution product) and the Euclidean algorithm which is also the original form of the descent method – Hurwitz speaks of the elimination of composite powers. Here again the ring of polynomials is the proper arena (with the largest area!) for Kronecker's general arithmetic of forms and their divisors. It is at this point that Dieudonné ([1974], vol.1 : 200) speaks of Kronecker's old dream (*vieux rêve*) as being realized by Grothendieck's notion of scheme (*schéma*). It is of course Kronecker's *Jugendtraum* that Dieudonné evokes here and he describes Kronecker's ambitious project as encompassing both number theory and algebraic geometry in the polynomial theory of modular systems (see Dieudonné [1974] : vol. 1, 59-61). Kronecker's forms or homogeneous polynomials become algebraic varieties and his notion of level (*Stufe*) means dimension or codimension in algebraic geometry – Kronecker's dream of youth in that context is translated into Hilbert's 12th problem on the extension of Kronecker's proposition on Abelian fields in an arbitrary algebraic domain of rationality.

The decomposition process or division algorithm is thus a descent to irreducible (linear) polynomials and Kronecker ([1884] : 336) in a later paper "On Some Uses of Modular Systems in Elementary Algebraic Questions" (*Über einige Anwendungen der Modulsysteme auf elementare algebraische Fragen*) makes it clear that his theory of higher forms or modular systems makes it unnecessary to have recourse to infinite series as in formal power series and that finite series, that is polynomials, suffice or are up to the task of extracting the arithmetic-algebraic content of general arithmetic, as Kronecker says. The content in question amounts to substructures and extensions in model-theoretic terms and the function field appears then naturally as an extension of the number field, but Kronecker's way was algorithmic in the combinatorial build-up of the hierarchy of forms. As Edwards points out in his *Divisor Theory* [1990], such extensions of finite degree are not couched in an algebraic closed field in modern usage and Kronecker avoids the transfinite setting by simply introducing new algebraic numbers to $\mathbb{Q}$ in a finite process of adjunction (see Gauthier [2002] : chap. 4 for details). Weil has insisted on the deep connection between function fields and number fields without avoiding entirely the transfinite construction and Grothendieck in his pursuit of Weil's conjectures has enlarged the geometric landscape with his notion of scheme. What we call today an algebraic variety was essentially a divisor sytem or a modular system (*Modulsystem*) for the polynomial ring in the hands of Kronecker; when it changes hands it becomes a locally ringed space in the functorial category-theoretic style of Grothendieck. Here functors transport arrows (functions) and their objects by making room for the larger topological or toposical (topos-theoretic) structures. I would call this approach structuralist as it is in line with the Bourbaki School to contrast it with the constructivist approach of Kroneckerian ascent and one could consider algebraic geometry *à la* Grothendieck as a tension between two mother-structures, algebraic structures and topological structures, as defined by Bourbaki. Grothendieck [1961] in any case refers indifferently to descent techniques or construction techniques in his 1958-1961 exposés in the Bourbaki Seminar. Algebraic geometry could be seen more accurately as pulled between a purely arithmetical internal logic and an external geometrical logic. I would put Langlands' programme more on the side of arithmetic geometry with Grothendieck's programme on the other, geometric side of algebraic geometry. In Langlands' case as we shall see, the "principle of functoriality" puts the emphasis on the correspondence between arithmetic objects and analytic data while for Grothendieck the question is : is algebraic geometry more algebraic than geometric? For example in Grothendieck's theory of motives (*motifs*), correspondences are sought between arithmetic objects and algebraic-geometric structures. One answer could be found, I believe, in the massive work of Jacob Lurie on *Higher Topos Theory* [2009]. Lurie's work is certainly of Grothendieck's lineage and I would like to concentrate my short analysis of the matter in the chapter "Descent versus Hyperdescent" of Lurie's treatise.

## 3. Descent

Descent is a central topic in algebraic geometry. It is of course of arithmetic inspiration having its origin in Fermat's notion of infinite or indefinite descent. It has been practised in number theory by Fermat, Euler, Gauss, Lagrange, Legendre, Dirichlet, Kummer and Kronecker (see [1901]) and in modern times by Hilbert, Poincaré, Mordell, Weil, Faltings, Serre and many others. Since I have explored infinite descent extensively elsewhere (see Gauthier [2002] and [2010]), I shall simply quote André Weil's version :

> Infinite descent *à la* Fermat depends ordinarily upon no more than the following simple observation : if the product $\alpha\beta$ of two ordinary integers (resp. two integers in an algebraic number-field) is equal to an $m$-th power, and if the g.c.d. of $\alpha$ and $\beta$ can take its values only in a given finite set of integers (resp. of ideals), then both $\alpha$ and $\beta$ are $m$-th powers, up to factors which take their values only in some assignable finite set. For ordinary integers this is obvious; it is so for algebraic number-fields provided one takes for granted the finiteness of the number of ideal-classes and Dirichlet's theorem about units. In the case of a quadratic number-field $\mathbb{Q}(\sqrt{N})$, this can be replaced by equivalent statements about binary quadratic forms of discriminant $N$.
>
> (see Weil [1983] : 335-336)

For algebraic number theory, descent in Weil's sense boils down via a height function on integers $m \geq 2$ to the finite quotient group $A/mA$ of rational points of an Abelian group $A$, which is then said to be finitely generated: this is the starting point of Galois cohomology inaugurated by Weil. I would designate this form of infinite descent as a general Euclidean algorithm for divisor theory and Kronecker [1882] has used it accordingly in his theory of modular systems where we have an extensive treatment of the elimination or decomposition of forms into irreducible factors. For model theory, Tarski's theory of quantifier elimination has its source in Kronecker's elimination theory following van den Dries [1988]. For algebraic geometry (and Galois cohomology), it should be noted that Kronecker had defined a generalized Galois principle which consists essentially to move from the group of substitutions for algebraic equations to the permutation group of a higher invariant theory, that is for entire functions with integer coefficients and $n$ indeterminates (see Kronecker [1882] : chap. 11, 284-285) : these are the forms for which Kronecker claims that it is the most complete arithmetical theory of algebraic quantities (*ibid*, 377). His principal result quoted above, as Kronecker emphasizes, is the analogue of the fundamental theorem of arithmetic for algebraic forms :

> Any entire algebraic form is representable as a product of irreducible (prime) forms in a canonical way.
>
> (my translation of [1882] : 352)

This amounts in contemporary elementary algebra to the fact that the domain $F[\mathrm{x}]$ of polynomials is a principal ideal domain, a major ingredient of divisor theory. The Noetherian chain condition on ideals is ascending and descending on the field of $F(\mathrm{x})$ of rational functions. The direct method of calculating the g.c.d. here is the Euclidean algorithm or in the more general algebraic-geometric situations, the descent method *à la* Fermat described by Weil above and which I have characterized as a generalized Euclidean algorithm. This is one of the junction points between number theory and algebraic geometry that Weil [1979b] has stressed and it is here that Weil and Grothendieck would agree, despite Weil's reluctance to admit the category-theoretical language. Even the recalcitrant Dieudonné who didn't have much taste for constructive mathematics had recognized that

Grothendieck's notion of scheme was genetically linked to the *Modulsysteme* or modular systems of Kronecker.

If classical infinite descent relied on the well-ordering of the natural numbers to prove impossibility results by *reductio ad absurdum* for example, it consisted also in reduction procedures for Diophantine equations of finite degree with possibly infinite solutions, a kind of *positive* descent. It is also a kind of positive descent that Weil defines and Serre as well as Grothendieck have named that descent "descending induction" or "recurrent induction"– Poincaré preferred the term "*récurrence*" to "induction". It is Weil again who has introduced the practice of infinite descent for field extensions in the theory of Galois cohomology. Cohomology as the computational dual of homology harbours various devices and descent appears under many disguises such as exact sequences, descending sequences, recurrence hypothesis, finiteness conditions, etc. Noetherian rings and spaces have an intrinsic descent (chain) condition. Grothendieck has even invented the term "*dévissage*" to express the unscrewing of the sequence of integers inherent to the descent method. But in his categorical idiom, descent consists in representing the algebraic structures – on a Noetherian frame most of the time – in the geometric universe or universes by arrows pointing to the objects of the ground level (the discrete topology), thus collecting the descent data from above and glueing them below. This is more or less a pictorial or intuitive approach, as Giraud [1964] admits in his treatise on Grothendieck's theory of descent. For a recent example of the use of descending induction in a somewhat more constructive style, one should consult Serre's paper [2009] "How to use finite fields for problems concerning infinite fields", a most elegant illustration of a simple descent technique. For an another illustration, Faltings had built his proof of the Mordell conjecture – on the finite number of rational points of a rational algebraic curve (of genus greater than one) – on moduli spaces which are geometric spaces endowed with a Noetherian algebraic structure and finite coverings (see Faltings [1984]). The method of descent still works by climbing down the ladder of natural numbers from a given $n$ or the sparser rungs of the prime numbers from a given $p$ or $l$ (for $l$-cohomology). The same is true for the ring of integers and the ring of polynomials which are Noetherian, as are finite fields. Geometric descent with functors and morphisms must count on the algebraic-arithmetic descent to recover the ground field or the polynomial ring which is the fundamental arena of algebraic geometry.

A nice illustration of this fact can be found in Jacob Lurie's voluminous *Higher Topos Theory.*

Infinite categories or infinity-categories ($\infty$-categories or $\omega$-categories) along with $n$-categories are new objects of higher category theory and topos theory. For brevity's sake, let us say that $n$-categories for $n$ finite become infinite when $n = \infty$. The same holds for topoi and Grothendieck descent applies to $n$-topoi, where it is the usual descent *à la* Fermat as introduced by Weil. Recall that Serre defines "descending induction" as acting on two ( positive ) integers $m$ and $N$ with $m > N$ descending to $m = N$ (see [2009] :10). What happens in the case of infinite topoi above and beyond cohomology and cobordism for infinite categories? The situation becomes more complicated with hyperdescent, as Lurie admits ([2009] : 67) and one has to introduce the set-theoretic machinery of transfinite iteration (induction) on limit ordinals that reside in the universe of a regular uncountable cardinal; descent consists then in a transfinite sequence of downward-closed subsets (*ibid.*, p. 800) in order to decompose them into pieces before glueing them in a suitable topological space. In the same Grothendieck's lineage or line of thought, V. Voevodsky [2010] is proposing geometric (non set-theoretic) univalent foundations for homotopy types – with equivalence classes of continuous maps between topological spaces – for an axiomatization in a dependent type system *à la* Martin-Löf (intuitionnistic type theory). But he needs, as he says "at least one unreacheable cardinal $\alpha$" (*ibid.*, p. 5), which means that one has to climb the cumulative hierarchy of axiomatic set theory ZFC up to an inaccessible cardinal before redescending to a topological space or to its fundamental groupoid – groupoids are a generalization of the notion of group and they construct all morphisms as isomorphisms in category theory by having a partial function instead of a binary operation between group elements or objects. Here univalent logico-geometrical foundations might be seen as multivalent transfinite arithmetical foundations! It is true that Grothendieck didn't care much about the cardinality of the universes of his *U-topoi*, the totality of which could be called *Utopia* from a finitist point of view! But here we are a far cry from the Kroneckerian point of departure of *SGA 1*, to say the least. The set-theoretic background (the category of sets) – as a matter of fact the category of sets is *reducible* to a point in topoi theory and *contractible* to a point in homotopy type theory – is the starting point of category theory and topos theory, but the algebraic side of algebraic geometry finds its basic objects in simplicial sets, that is finite series of ordinals. Here, I would put Quillen's original work in homotopical algebra with the Quillen-Sullivan rational homotopy theory and Joyal's work on quasi-categories in the algebraic trend, while the geometric side exploits the ground territory of (homogeneous) topological spaces with the full resources of higher set theory. Of course, one could accept the logical, classical equivalence of arithmetic descent with transfinite induction, but from a constructivist point of view, it can be shown that the equivalence does not hold since it involves the excluded middle principle via a double negation over an infinite set of natural numbers (see Appendix 1 "La descente infinie, l'induction transfinie et le tiers exclu" of Gauthier [2010] : 133-151, and also Gauthier [2002]: 51). Let us note that higher topoi theory, not unlike ordinary (lower) topoi theory, makes room for Heyting topoi where a second-order intuitionistic logic leaves no place for the classical excluded middle principle. The arithmetic scope of arithmetic-algebraic geometry appears to be more faithful to its Kroneckerian inspiration and I want to look briefly at Langlands programme in that perspective.

## 4. Langlands' Programme

In his paper on contemporary problems with origins in Kronecker's *Jugendtraum* Langlands [1976] evokes Hibert's $12^{th}$ problem in the 1900 list which reads "Extension of Kronecker's proposition on Abelian fields over an arbitrary algebraic domain of rationality". Hilbert declares that it is one of the deepest and far-reaching problems of number theory and function theory to generalize Kronecker's proposition on the generation of every commutative (Abelian) rational field through the decomposition of fields for the roots of unity; the idea here is to extend the rational field to any algebraic number field – what is called today the Kronecker-Weber theorem asserts that any Abelian extension of $\mathbb{Q}$ belongs to the cyclotomic field $\mathbb{Q}(\zeta_m)$. Hilbert holds the problem to be at the internal junction of number theory, algebra and the theory of functions (analysis). Such a language recalls Kronecker's statement in his inaugural speech at the Berlin Academy of Science in the year 1861 (see Kronecker [1968], vol. V : 388) :

> [. . . ] the study of complex multiplication of elliptic functions leading to works the object of which can be characterized as

> being drawn from analysis, motivated by algebra and driven by number theory.
>
> (my translation)

Kronecker was perfectly aware of the centrality of his programme which he sees in the continuity of Gauss and Dirichlet and there is no doubt that he hoped for a full arithmetization of analysis. The dream of his youth (*Jugendtraum*) was that vision of an arithmetical theory of elliptic functions, an arithmetic of *ellipotomy* or divison of the ellipse, as I venture to say in analogy with the notion of cyclotomy. In a letter to Dedekind, Kronecker goes even as far as to say that the fundamental relation he has found between arithmetic and analysis originates in a philosophical intuition (see Kronecker [1899], vol. V : 453). Kronecker's foundational insight is given its fullest expression in his main paper on the arithmetical theory of algebraic quantities of 1882 where he gives the final formulation of his *Allgemeine Arithmetik* or General Arithmetic. It contains, in Kronecker's words, the complete development of the theory of entire (rational and algebraic) functions of a variable together with the systems of divisors. In such a complete theory, the association of forms allows for the conservation of the laws of factorization, so that the passage from natural and rational domains to the more general algebraic domains (of algebraic integers) is perfectly uniform. The conservative extension of arithmetic up to the highest reaches of algebra – the theory of entire rational and algebraic functions – is the ultimate goal of general arithmetic defined as the theory of all forms, homogeneous polynomials with integer coefficients and an arbitrary number of indeterminates.

Langlands insists also on number theory in connection with algebraic geometry, as Weil has taught. Here Langlands points to the continuation of Kronecker's work on Abelian extensions by the generalization to Abelian varieties in the hands of a long offspring of number theorists from Hilbert to Shimura and Deligne. For example, Shimura varieties embody some ideas of Kronecker who sought arithmetic objects within the analytic core of elliptic functions. Abelian varieties are nowadays spread out over algebraic number fields, finite fields, local fields and extend to contemporary arithmetic algebraic geometry in the work of Faltings and Wiles – by the way, descent is still present in the proof of Fermat's last theorem, if it is only by the Noetherian ring on the sequence of primes, modular and automorphic forms being on their side generalizations of Kronecker's modular sytems.

Langlands' programme or Langlands' philosophy, as it has come to be known, could be seen as a contemporary revival of Kronecker's idea of the deep analogy or correspondence between number fields and function fields. Evidently, as I mentioned above, the contemporay mathematician will not refrain from transcendental methods, but when he comes down to the arithmetic level, he sticks as the typical arithmetician would say, to the motto "denumerable at infinity", which means that denumerable infinity is seen as a limit or that the non-denumerable is unknown territory (*terra incognita*). Langlands' philosophy has met with success in two recent instances, Lafforgue's and Ngô's contributions to Langlands' programme. I want to deal briefly with Laurent Lafforgue's result.

Lafforgue [2002] has succeeded in showing the exact Langlands' correspondence between pieces of the modular space (or algebraic variety) and its (denumerable) rational points with the help of an iteration technique on Drinfel'd chtoukas – chtoukas comes from the Russian *штука* and is drawn from the German *Stücke*, meaning pieces. The ground field of Langlands' correspondence is a finite field $F$ with a Galois group $G$ and we come back to the priviledged arena of applications for Fermat's descent on Weil's conception. I'll not pursue that theme, but only recall that Drinfel'd himself has drawn on some motives from Kronecker : his chtoukas are elliptic modules and have an ancestral relationship to Kronecker's search for arithmetic objects or "discrete pieces" in the complex multiplication of elliptic functions.

Kronecker's dream and programme of a general arithmetic have provided a fertile soil for large-scale foundational projects, if only as deep-seated motivations or inspirational ideas. The immediate and long-term posterity of Kronecker's programme includes a vast number of people from Hurwitz and Hensel to Weil and Langlands – see the overview of the secondary literature by Marion [1995]. One should include in the list Brouwer, Poincaré and the French semi-intuitionists like Borel and Lebesgue to a certain extent and even Hadamard, who did borrow from Kronecker's arithmetical theory of functions for the particular purposes of topology *e.g.* the winding number (*Windungszahl*) which is an integer or index giving the number of times a closed curve $c$ passes around a designated point $P$ in the plane or in contemporary idiom the topological degree for a continuous function to itself on the closed unit ball $D^n$. Russian constructivists like Markov, Shanin, Kolmogorov up to Essenine-Volpin have also some share of Kronecker's finitism. But it is certainly in algebraic geometry that Kronecker's heritage is most strongly felt. Weil [1976] considers Kronecker as the originator of modern algebraic (arithmetic) geometry in the sense that Kronecker has initiated the work on the arithmetic of elliptic functions – they have become the elliptic curves or the modular forms of the contemporary scene. Elliptic curves even play a role in recent cryptography, for they have an arithmetical content hidden under their surface of intersection!

## 5. Kronecker's and Hilbert's programmes in contemporary mathematical logic

I see Hilbert's metamathematical programme as the continuation of Kronecker's arithmetical programme with other means, that is the means of logic. In turn, I consider Hilbert's theory of formal systems and axiomatization as the initiation of the arithmetization of logic after Kronecker's arithmetization programme. Such an arithmetization of logic is manifest in contemporary theoretical computer science and in applied proof theory. I want to emphasize the new developments in Hilbert's proof-theoretical programme.

It is common knowledge that metamathematics or proof theory is concerned with finitary methods, as in Hilbert's conception of the theory of formal systems. I contend (see Gauthier [1989], [1991], [1994], [2002]) that the consistency question is the crux of the matter and that it requires a finitist approach in the sense of Kronecker, as some Hilbert's early manuscripts seem to attest – see Hallett [1995] and also Sieg [1999] for Hilbert's later papers. The rather sketchy attempt on the simultaneous foundation of logic and arithmetic (Hilbert [1905]) puts forward the concept of homogeneous equations in a manner reminiscent of Kronecker's combinatorial theory of homogeneous polynomials. Consistency, following Hilbert boils down to the homogeneous equation $a = a$ or inequation $a \neq a$. In his report on Hilbert's research on the foundations of arithmetic, Bernays says that Hilbert, in spite of his durable opposition to Kronecker whom he accused of dogmatism, has wanted a reconciliation with Kronecker's finitist stand :

> Kronecker has elaborated a clear conception which he has put to use in many cases and his conception accords essentially with our finitist position
>
> (see Hilbert[1935] : vol. III : 203, my translation).

The quotation refers to Hilbert [1930] on the foundations of elementary number theory and is a sequel to Hilbert's paper "*Über das Unendliche*" [1926]. The finitist position in question is the metamathematical idea of proof theory which I would formulate in the following way : To use finite logical rules with transfinite axioms in order to extend the finite into the infinite. The metamathematical conception is modeled after Kronecker's extension of elementary arithmetic into general arithmetic, an extension which should preserve the conceptual determinations of elementary arithmetic, following Kronecker; for Hilbert, the objective was to preserve the laws of finite logic with the excluded middle principle in the transfinite (set-theoretic) realm of ideal elements (*ideale Elemente*) for which classical (Aristotelian) logic could not make place because it did not distinguish between the finite and the transfinite — Kronecker could respond to Hilbert here by saying that there was no actual infinite in Aristotelian logic either, but only a potential infinite for Aristotle and Euclid. Logic will provide the passage from the finite to the infinite, since there is no place for the infinite (Hilbert [1930] : 487) as it was also proclaimed in the lecture "On the Infinite". From my point of view, Hilbert's metamathematical programme is but the continuation or the consequence of Kronecker's arithmetization programme.

A further proof of Kronecker's inspiration, if not direct influence, on Hilbert's proof theory is the introduction by Hilbert of the finite-type functionals in his (unsuccessful) attempt to prove Cantor's continuum hypothesis in his 1926 paper "On the Infinite" (see Hilbert [1926]). As Kohlenbach notes [2008] those finite-type functionals were used by Gödel [1958] in his *Dialectica* interpretation for the consistency of intuitionistic arithmetic and it is an essential tool (with Herbrand's theorem) for applied proof theory (see Gauthier [2009]); it must be added here that Gödel had already referred to Hilbert's construction in his 1931 paper on completeness and consistency (cf. *Über Vollständigkeit und Widerspruchsfreiheit. Ergebnisse eines mathematischen Kolloquiums* 3 (1932): 13) where he imagines a transfinite sequence of formal systems of higher type. Let us remark though that in his proof for the consistency of intuitionnistic arithmetic (the *Dialectica* interpretation), Gödel limits his (impredicative) construction to all finite types up to $\omega$. Hilbert's idea was to introduce number-theoretic functions "as those functions of an integral argument whose values are also integers" and then add up functions of functions, functions of functions of functions in a finite procedure for the iteration of types of functional variables over primitive integral types. Those higher-type functionals according to their height are associated with propositions that are supposed to come into a one-to-one correspondence with the transfinite ordinals up to the $\epsilon_0$ of Cantor's second number class. Hilbert needed transfinite induction here, but since only a finite iteration was necessary for the build-up of the functional hierarchy, the methods of substitution and recursion would suffice to produce a finitary proof, because subtitution and recursion are counted as finitary procedures according to Hilbert. Hilbert's course, I forcibly suggest, follows up or reproduces Kronecker's construction of higher-level (*Stufe*) or (*Rang*) forms or homogeneous polynomials in his 1883 paper (Kronecker [1883]) where substitution and decomposition – or descent for recursion – were used in a radical finitist setting, as I have shown above. Kronecker's idea was to build a finite hierarchy of (polynomial) functions to encompass the content of general arithmetic, what he calls the multilevel extension domain of arithmetic (*stufenweise Gebietserweiterung der Arithmetik*) in the footsteps of Gauss (Kronecker [1882]: 356). Beyond and above this – and despite his repeated finitist commitment (*finite Einstellung*) – Hilbert wanted to include Cantor's transfinite arithmetic (up to $\epsilon_0$).

$$\lim_{n\to\omega} \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\}n = \epsilon_0.$$

One step further and the ordinal rank structure of von Neumann or the cumulative rank structure of Zermelo-Fraenkel set theory would look like transcendental extensions of Kronecker's finite arithmetical rank construction! Indeed, Hilbert followed faithfully the Kroneckerian construction I have outlined above in using the two fundamental procedures of substitution (for new variables) and recursion (*Rekursion*) where the values of a function of height *n+1* are derived from the values of a function of height $n$ (Hilbert [1926] : 184). Kronecker had instead polynomials of order $n$ generating a system of order $n+1$ by the product operation (Kronecker [1883] : 419).

Hilbert's attempt failed, but the construction was pursued by Hilbert's followers like Gentzen, Ackermann, Kalmár – with the infinite descent idea – and Gödel in allegedly extended finitist ways – see my critique in Gauthier [2002 : 57-58] where I insist that tranfinite induction with an (excluding third) double negation on the infinite set of natural numbers cannot be identified with infinite descent from a constructive finitist point of view. Gödel did not use transfinite induction in his *Dialectica* interpretation, but induction on all finite types and I contend that the functional interpretation has a natural translation in the polynomial arithmetic of Kronecker's theory of higher-level forms (see Gauthier 2002: 74-76).

There is no doubt that Hilbert followed Kronecker's steps in mathematics, for instance in algebraic invariant theory (Hilbert [1890] and Hilbert [1893], see also Gauthier [1995]). From my point of view, Hilbert was deeply influenced by Kronecker's mathematical practice and in spite of his reaction to Kronecker's prohibition of transfinite methods (Hilbert [1926]), he could not depart entirely from a finitist pragmatic and philosophical point of view as far as mathematics is concerned (and logic for that matter). It is only in 1917 that Hilbert resumed his foundational research and returned to finitism, not without polemizing with Kronecker (posthumously!), Brouwer and Weyl whom he considers as Kronecker's direct heirs – for the variety of Hilbert's programmatic ideas, see Sieg [1999]. The simultaneous foundation of logic and arithmetic still dominates his preoccupations and the recourse to the notion of formal system is meant as a mechanism (a finite algorithm) for the introduction of ideal elements. My hypothesis is that this process mimicks Kronecker's association of forms in his general arithmetic and the consistency which is required for the association of ideal elements can only be achieved by a formalism which is the exact counterpart of an arithmetic (polynomial) algorithm, *e.g.* the method of descent as a generalized Euclidean algorithm.

The propositions of general arithmetic that are found in Kronecker's 1882 paper on the arithmetical theory of algebraic quantities can be considered as so many axioms from which Kronecker derived his results with arithmetical means alone. In his paper on the axiomatic method ([1935], vol. III : 146-156), Hilbert pinpoints the properties of independence and consistency as the main features of the axiomatic method. Relative consistency of geometry and other scientific disciplines, Hilbert suggests, is based on the consistency of arithmetic, but there is no further foundation for arithmetic and, Hilbert adds, set theory. Logic is the ultimate foundation and it must also be axiomatized; in the final analysis there only remains for the axiomatic method the question of decidability which must be settled "in a finite number of operations" (see [Hilbert 1935], vol. III : 155). Here Hilbert gives the example of the theory of algebraic invariants for which he had provided a finiteness proof inspired by the very method he had used in his major result : Hilbert's finite basis theorem depends heavily on Kronecker's own methods in general arithmetic and becomes the paradigm case for the decidability property of a logical system! But there is no logic involved in Hilbert's result and his paradigmatic case is drawn from polynomial arithmetic (Kronecker's general arithmetic of forms). Decidability implies, of course, that we have an algorithm or a finite procedure to decide of a given question in a "finite number of steps". We then come back to our point of departure and it is not surprising to see that most decidable theories are elementary (first-order) algebraic theories and have ended up as the subject matter of model theory, not proof theory. The method of quantifier elimination, for instance, is a test for decidability and has been employed by Tarski in his well-known model-theoretic results; van den Dries [1988] has stressed the influence of Kronecker's methods in that context. But then what is the logical point of the decision method? A decidable theory, if consistent, is finitely so. In the specific case elementary theories, logic does not play any special role since the equational calculus of polynomials does not need other operations than the purely arithmetical (combinatorial) laws.

The case for logic rests solely on the alleged conservative extensions of arithmetic into the transfinite domain of ideal elements. I have discussed extensively elsewhere (Gauthier [2002]) the relevance of Hilbert's proposal for such a "transfinite logic ". It remains though that even if Hilbert had hoped for a logical introduction of ideal elements, he has constantly stressed that a finite process (or procedure) is the inference engine of internal consistency (*inhaltliches Schliessen*).

Internal consistency is obtained by internal means in the case of general arithmetic as in the case mentioned above of the theory of algebraic invariants. Hilbert was not mistaken there and he saw consistency as internal to the polynomial equation calculus when he defined consistency as the equation $a = a$ and inconsistency as $a \neq a$. We have observed that one of the essential tools of consistency for Kronecker's general arithmetic is the convolution product which generates linear polynomial expressions from higher-level polynomial expressions as in Kronecker's result, Dedekind's Prague theorem or Hilbert's work in invariant theory. The convolution or Cauchy product can be called Cauchy diagonal. A serious blow to Hilbert's programme was administered from the outside, the "external " Cantor diagonal in Gödel's results. Of course, if Gödel's first incompleteness result assumed $\omega$-consistency, the second incompleteness result resting on external consistency could only be obtained from a transcendent point of view, as he says, but Gödel didn't exclude the possibility of an internal (*innere*) consistency proof (see Gauthier [2007] and [2011]). It is not only set-theoretic arithmetic, as Hilbert himself has named it, but also set theory (including analysis) that he wanted to secure. It is a paradoxical situation for the logician Hilbert to see his full-blown programme for consistency of set theory and analysis put in jeopardy by a set-theoretic device! In any case, Hilbert's programme can still be saved to a large extent and to a larger extent than expected if we rethink it in the framework of Kronecker's programme. Herbrand (see [1968] : 152), a follower of Hilbert, wanted also a consistency proof for arithmetic and he had formulated what I call Herbrand conjecture (see Gauthier [1983]) :

> Transcendental methods cannot demonstrate theorems in arithmetic that could not be demonstrated by arithmetical means alone.
>
> (my tranlation)

Herbrand stated his conjecture for a suitable formal system which he does not describe. Herbrand was also a practitioner of (algebraic) number theory and he expressed himself in Kroneckerian terms when he used what he called "intuitionistic" arguments where one supposes that an object, logical or mathematical, does not exist without the means to construct it. In the same line of thought, he defends the potential infinite for his notion of infinite domain (*champ infini*) by saying that it is built iteratively (*pas-à-pas*) or (*Schritt zu Schritt*) in Kroneckerian terms, an expression also used by Skolem. Herbrand worked for instance on finite extensions of infinite fields in the tradition of Hilbert and Kronecker, foreshadowing to some extent the contemporary work of Weil and Serre.

## 6. Conclusion. Finitism and Arithmetism.

≪ Arithmetism ≫ is the name I give to a foundational option in radical opposition to logicism and to Frege's question ([1893] : X) : "How far can one go in arithmetic solely by deductive (logical) means ?" (*Wie weit man in der Arithmetik durch Schlüsse allein gelangen könnte?*), the arithmetician Kronecker would respond : "How far can one go in mathematics with arithmetic alone?" and Hilbert following suit as a logician would ask : "How far can we go into the tranfinite using only finite logical means?". One must admit that after the demise of the logicist programme (Frege and Russell) and despite the efforts of philosophers and logicians to recover Frege's logicist foundations of arithmetic with the second-order Hume principle, it is Kronecker's arithmetist programme which is still alive in the farthest reaches of contemporary mathematics, arithmetic-algebraic geometry. That does not mean however that mathematicians inspired by Kronecker from Hilbert to Weil adhere wholly to the Kroneckerian doctrine of finitism. On the contrary, most would allow for methods that leave Kronecker's arithmetical safe haven and venture into transarithmetical (set-theoretic), geometrical or analytical (trancendental) extended universes. A good pilot here is certainly Hilbert himself.

Hilbert introduced ideal elements (*ideale Elemente*) in order to have a clear-cut divide between the finite and the non-finite, a divide that Aristotelian logic did oversee, because it could not survey – (*Unübersichtlichkeit*) in Hilbert's text [1926] – the extent of its applications. The idea of the epsilon-calculus for the $\epsilon$-symbol was to enable the extension of the simple laws of Aristotelian logic, excluded middle and universal instantiation with existential import, to the transfinite universe of ideal statements. Once this is achieved, one could redescend in the finite by elimination of the ideal elements or the epsilon formulas by a finite process in polynomial arithmetic, that is Hilbert's use of infinite descent (*die Methode der Zurückführung*) (see Gauthier [2011]) which reduces transfinite expressions to arithmetical statements.

Intuitionistic logic, after the work of Brouwer, Kolmogorov, Heyting and Gödel, fares better in discriminating between the finite and the infinite, simply by rejecting the entension of classical logical laws beyond the finite domain and by exploring the potential infinite. This explains why it is the starting point of the functional interpretation privileged by applied proof-theorists; in their hands, intuitionistic logic is extended by various non-constructive principles or one-way translations from intuitionistic logic to classical logic. There is the foundational shift from Hilbert's programme and it has proven successful in recent proof-theoretic research with Kohlenbach [2008] and others.

If applied proof theory and the proof mining enterprise represent a shift of emphasis in original (pure!) proof theory as Kohlenbach repeats after Kreisel, it remains that the idea of extracting more constructive information (with an enrichment of data) from a given classical proof concurs with the idea of certainty (*Sicherheit*) or of certification (*Sicherung*) that Hilbert defined as the ideal goal of his proof theory and the motto of applied proof theory could very well be "More information, more certainty". Detracting from that ideal would mean fruitless prospection for proof-theorists, either in the abstract realm of constructivist principles or in the mining field of promising applications. Of course, the motto has to be substantiated by further foundational research into the historical, logico-mathematical and philosophical motives of proof theory. Hilbert was certainly the first mathematician to think of mathematical proofs in terms of a systematic study of the internal logic of deductive reasoning "*das inhaltliche logische Schliessen*" in line with Kronecker's constructive stance in his general arithmetic "*allgemeine Arithmetik*" for which he claimed "*innere Wahrheit und Folgerichtigkeit*", that is internal truth and consistency ; these objectives could very well be shared by applied proof theory in the search for effective proofs in classical analysis where proofs were made available by the (constructive and non-constructive) means at hand. Proof theory puts the emphasis on proofs with the aim of making manifest their constructive hidden content and I would count such an enterprise as a revival of the Kroneckerian spirit with the logical means that Hilbert introduced in the programme of the arithmetization of logic after the arithmetization of analysis by Cauchy and Weirstrass along the arithmetization of algebra by Kronecker. It is maybe in contemporary theoretical computer science, for example in computational algebraic geometry with the Gröbner basis technique as well as in a variety of computational disciplines, that arithmetization can be pursued with a finite number of procedures as I would translate Hensel's phrase (*eine endliche Anzahl von Versuchen*) in the Preface of Kronecker [1901] to characterize Kronecker's finitist arithmetism.

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