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# ON SEMILATTICE-BASED LOGICS WITH AN ALGEBRAIZABLE ASSERTIONAL COMPANION

A b s t r a c t. This paper studies some properties of the so-called semilattice-based logics (which are defined in a standard way using only the order relation from a variety of algebras that have a semilattice reduct with maximum) under the assumption that its companion assertional logic (defined from the same variety of algebras using the top element as representing truth) is algebraizable. This describes a very common situation, and the conclusion of the paper is that these semilattice-based logics exhibit some of the good behaviour of protoalgebraic logics, without being necessarily so. The main result is that all these logics have enough Leibniz filters, a fact previously known in the literature to occur only for protoalgebraic logics. Another significant result is that the two companion logics coincide if and only if one of them enjoys the characteristic property of the other, that is, if and only if the semilattice-based logic is algebraizable, and if and only if its assertional companion is selfextensional. When these conditions are met, then the (unique) logic is finitely, regularly and

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> strongly algebraizable and fully Fregean; this places it at some of the highest ranks in both the Leibniz hierarchy and the Frege hierarchy.

Let $\mathsf{K}$ be a class of algebras of the same (arbitrary) similarity type. The class $\mathsf{K}$ is said to be ***semilattice-based*** when each algebra in $\mathsf{K}$ has a semilattice reduct uniformly defined by some primitive or term-defined operation. All such algebras have a natural order relation $\leqslant$ such that the semilattice operation is its *meet* or *infimum*; accordingly, the semilattice operation will be denoted by $\wedge$. In this paper $\mathsf{K}$ will always be a *variety of semilattice-based algebras with maximum* 1, which is the interpretation of a *constant term* denoted by 1 as well. In the present paper, phrases like "for all algebras" or "an arbitrary algebra" refer to all algebras of the similarity type under consideration.

Two (finitary) logics can be naturally associated with each such variety $\mathsf{K}$:

**Definition 1.** The logic $\vdash^1_{\mathsf{K}}$, called the **assertional logic** associated with $\mathsf{K}$, is the finitary logic defined by the conditions

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash^1_{\mathsf{K}} \varphi \quad &\text{iff} \quad \mathsf{K} \models \varphi_1 \approx 1 \ \& \ \cdots \ \& \ \varphi_n \approx 1 \Rightarrow \varphi \approx 1\,, \\ \emptyset \vdash^1_{\mathsf{K}} \varphi \quad &\text{iff} \quad \mathsf{K} \models \varphi \approx 1\,. \end{aligned} \tag{1}$$

Its filters on an arbitrary algebra $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_1\boldsymbol{A}$.

The logic $\vdash^{\leqslant}_{\mathsf{K}}$, called the **semilattice-based logic** associated with $\mathsf{K}$, is the finitary logic defined by the conditions

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash^{\leqslant}_{\mathsf{K}} \varphi \quad &\text{iff} \quad \mathsf{K} \models \varphi_1 \wedge \ \dots \ \wedge \varphi_n \preccurlyeq \varphi\,, \\ \emptyset \vdash^{\leqslant}_{\mathsf{K}} \varphi \quad &\text{iff} \quad \mathsf{K} \models \varphi \approx 1\,. \end{aligned} \tag{2}$$

Its filters on an arbitrary algebra $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_{\leqslant}\boldsymbol{A}$.

These two logics are said to be **companions** of each other; more precisely, $\vdash^{\leqslant}_{\mathsf{K}}$ is the **semilattice-based companion** of $\vdash^1_{\mathsf{K}}$, and this one is the **assertional companion** of the former.

In this definition, the symbol $\models$ denotes the usual first-order (or quasi-equational) satisfaction; the symbols $\&$ and $\Rightarrow$ denote first-order conjunction and implication, respectively; the symbol $\approx$ stands for a formal equation, to be interpreted as real equality; and the symbol $\preccurlyeq$ stands for a formal ordering relation, to be interpreted by the natural order relation of the algebras in $\mathsf{K}$. Because of the semilattice structure, the formal order relation $\varphi \preccurlyeq \psi$ is equivalent to the equation $\varphi \wedge \psi \approx \varphi$ over the algebras in $\mathsf{K}$. The meaning of Definition 1 is thus clear.

In the literature these logics have also been called "the logic preserving truth" (for $\vdash^1_{\mathsf{K}}$) and "the logic preserving degrees of truth" (for $\vdash^{\leqslant}_{\mathsf{K}}$) with respect to $\mathsf{K}$; see [4, 9, 11, 26, 27, 29]. These denominations arise in situations where the points in the algebras of $\mathsf{K}$ are interpreted as truth values, and specifically where 1 represents some kind of "absolute" truth in each $\boldsymbol{A}$. This is clearer in the case of $\vdash^1_{\mathsf{K}}$, since (1) is equivalent to

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash^1_{\mathsf{K}} \varphi \quad &\text{iff} \quad \text{if } v(\varphi_1) = \dots = v(\varphi_n) = 1 \text{ then } v(\varphi) = 1 \\ &\qquad \text{for all } v : \boldsymbol{Fm} \to \boldsymbol{A} \text{ and all } \boldsymbol{A} \in \mathsf{K}, \\ \emptyset \vdash^1_{\mathsf{K}} \varphi \quad &\text{iff} \quad v(\varphi) = 1 \text{ for all } v : \boldsymbol{Fm} \to \boldsymbol{A} \text{ and all } \boldsymbol{A} \in \mathsf{K}. \end{aligned} \tag{3}$$

The connection between the given definition of $\vdash^{\leqslant}_{\mathsf{K}}$ and the idea of a logic preserving degrees of truth lies in the fact that, due to the semilattice structure of the algebras in $\mathsf{K}$, (2) is equivalent to

$$\begin{aligned} \varphi_1, \dots, \varphi_n \vdash^{\leqslant}_{\mathsf{K}} \varphi \quad &\text{iff} \quad \text{if } v(\varphi_i) \geqslant a \text{ for all } i = 1, \dots, n, \text{ then } v(\varphi) \geqslant a \\ &\qquad \text{for all } v : \boldsymbol{Fm} \to \boldsymbol{A}, \text{ all } \boldsymbol{A} \in \mathsf{K} \text{ \underline{and all } } \underline{a \in A}, \\ \emptyset \vdash^{\leqslant}_{\mathsf{K}} \varphi \quad &\text{iff} \quad v(\varphi) = 1 \text{ for all } v : \boldsymbol{Fm} \to \boldsymbol{A} \text{ and all } \boldsymbol{A} \in \mathsf{K}. \end{aligned} \tag{4}$$

Incidentally, this equivalence shows that the presence of conjunction $\wedge$ in (2) is not the result of any *a priori* metalogical choice of conjunction as the formal counterpart of the grammatical comma, but rather a consequence of the order having a meet-semilattice structure.

From the discussion in [10, 11] let me highlight the suggestion of interpreting the points $a$ in the models not exactly as representing degrees of truth themselves, but as *determining* the degree of truth "to have a truth-value greater than or equal to $a$". Then, (4) says that a formula follows in $\vdash^{\leqslant}_{\mathsf{K}}$ from a certain set of premises if and only if each valuation giving all premises truth values with a certain degree of truth also gives that formula a truth value with that degree of truth. It is in this sense that the consequence $\vdash^{\leqslant}_{\mathsf{K}}$ can be understood as preserving these degrees of truth.

It can be argued that such interpretations make sense truly only in situations where it is natural to interpret the elements in the models as truth values. For instance, this happens when using definitions (1) and (2) not for a whole class of algebras $\mathsf{K}$ but for a single algebra which is supposed to represent the structure of truth-values of a certain semantics. From (2) it is however clear that $\vdash^{\leqslant}_{\mathsf{K}}$ depends only on the equations satisfied by the chosen algebra, and hence one can equally consider the generated variety, even if it may contain algebras where such truth-value interpretation makes less sense.

Semilattice-based logics were introduced and studied in the context of abstract algebraic logic in [14] under the name of "selfextensional logics with conjunction", and were christened and more thoroughly studied in [24]; specifically Section 3.1 of [24] is devoted to study when can $\vdash^{\leqslant}_{\mathsf{K}}$ coincide with $\vdash^{1}_{\mathsf{K}}$, under the assumption that $\vdash^{\leqslant}_{\mathsf{K}}$ is protoalgebraic. Here I will study some relations between the two logics under a different assumption. The key will be the study of Leibniz filters of $\vdash^{\leqslant}_{\mathsf{K}}$.

If $\mathcal{L}$ is a logic, an $\mathcal{L}$-filter $F$ on an algebra $\boldsymbol{A}$ is a ***Leibniz filter*** (of $\mathcal{L}$) when it is the smallest $\mathcal{L}$-filter on $\boldsymbol{A}$ having the same Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}F$. This notion was introduced and studied in [15, 23] in the context of protoalgebraic logics. The starting result was that if a logic is protoalgebraic, then each of its filters has an associated Leibniz filter with the same Leibniz congruence. The logic defined by the Leibniz filters of a given (protoalgebraic) logic was also studied. In the present paper I will show that some of the results of [15] can also be obtained for (not necessarily protoalgebraic) semilattice-based logics $\vdash^{\leqslant}_{\mathsf{K}}$, provided their assertional companion $\vdash^{1}_{\mathsf{K}}$ is algebraizable. Some proofs in this case are more direct than those in [15].

The reason for considering the alternative assumption that $\vdash^{1}_{\mathsf{K}}$ is algebraizable is that it seems to describe a more common situation than that of assuming that $\vdash^{\leqslant}_{\mathsf{K}}$ is protoalgebraic; although in general this assumption does not imply the alternative one, all the examples analyzed in [15] actually fall also under the scope of the alternative assumption. Moreover, several examples falling under this assumption but not under the original one have been separately studied in the literature, always coming to similar results but in each case obtained by working with the particular logics and classes of algebras. In [4] the logics $\vdash^{\leqslant}_{\mathsf{K}}$ for varieties $\mathsf{K}$ of commutative and integral residuated lattices have been studied, and it has been proved that for many of these varieties the logic is not protoalgebraic (later I will mention examples), while the associated assertional logic $\vdash^1_{\mathsf{K}}$, which is the logic usually associated with $\mathsf{K}$ in the context of substructural logics [21], is always algebraizable. In this case, for each $\boldsymbol{A} \in \mathsf{K}$, $\mathcal{F}i_{\leqslant}\boldsymbol{A}$ equals the family of all lattice filters of $\boldsymbol{A}$ while $\mathcal{F}i_1\boldsymbol{A}$ equals the family of all implicative filters of $\boldsymbol{A}$. Something similar had already been found in [9, 12] when $\mathsf{K}$ is the variety of MV-algebras; in this case $\vdash^1_{\mathsf{K}}$ is the usual (finitary) infinite-valued Łukasiewicz logic $Ł_\infty$, while $\vdash^{\leqslant}_{\mathsf{K}}$ is the so-called "Łukasiewicz logic preserving degrees of truth" $Ł^{\leqslant}_\infty$, which was shown to be non-protoalgebraic.

The main group of (protoalgebraic) examples analyzed in [15], and the one which actually provided most of the starting intuitions for that research, corresponds to the field of modal logic. In this case, $\mathsf{K}$ is a variety of modal algebras, and the logic $\vdash^1_{\mathsf{K}}$ is the so-called "global consequence" of the normal modal logic associated with $\mathsf{K}$ (i.e., the one having the Necessitation Rule in the strong sense: $\varphi \vdash^1_{\mathsf{K}} \Box\varphi$) while $\vdash^{\leqslant}_{\mathsf{K}}$ is the corresponding "local consequence" (the one with the Necessitation Rule in the weak sense: $\vdash^{\leqslant}_{\mathsf{K}} \varphi$ implies $\vdash^{\leqslant}_{\mathsf{K}} \Box\varphi$); the terms "local" and "global" come from the semantical characterizations of these logics in terms of classes of Kripke frames. Both logics are protoalgebraic, and for each $\boldsymbol{A} \in \mathsf{K}$, $\mathcal{F}i_{\leqslant}\boldsymbol{A}$ equals the family of all filters of $\boldsymbol{A}$ (since $\boldsymbol{A}$ is a Boolean algebra lattice filters and implicative filters coincide) and $\mathcal{F}i_1\boldsymbol{A}$ equals the family of all open filters. The case where $\mathsf{K}$ is a variety of orthomodular lattices is also analyzed in [15], resulting in a similar situation (but in this case the filters of the assertional companion are not determined as those of the semilattice-based logic that are also closed under certain rule).

The present paper treats this situation in general, that is, it deals with the common features of these cases that do not depend on logical connectives other than conjunction (such as join, implication, fusion, necessity). I assume from now on the following:

> **Basic assumption**: $\mathsf{K}$ is a variety of semilattice-based algebras with maximum, $\vdash^{\leqslant}_{\mathsf{K}}$ is the semilattice-based logic determined by $\mathsf{K}$, and its assertional companion $\vdash^1_{\mathsf{K}}$ is algebraizable with $\mathsf{K}$ as its equivalent algebraic semantics and with $x \approx 1$ as defining equation.

At the beginning of the paper I assumed the constant 1 is already in the similarity type; this is just a simple way of ensuring that the maximum of the order in the algebras is term-definable, which implies that both logics have theorems; otherwise these facts would have to be postulated.

If $\boldsymbol{A} \in \mathsf{K}$ and $F \subseteq A, F \neq \emptyset$, one says that $F$ is a ***semilattice filter*** when $a, b \in F$ iff $a \wedge b \in F$ (if $\boldsymbol{A}$ is a lattice this is just the usual notion of lattice filter). Note that this implies that $1 \in F$, and that if $a \in F$ and $a \leqslant b$ then $b \in F$. The next result summarizes the main basic properties of the two logics needed here; when the symbol $\vdash$ appears in a property it means that the two logics $\vdash^1_{\mathsf{K}}$ and $\vdash^{\leqslant}_{\mathsf{K}}$ have the stated property. For a logic $\mathcal{L}$, the notations $\mathsf{Alg}(\mathcal{L})$ and $\mathsf{Alg}^*(\mathcal{L})$ refer respectively to the classes of algebra reducts of all reduced generalized models of $\mathcal{L}$ and of the algebra reducts of the reduced models of $\mathcal{L}$. The class $\mathsf{Alg}(\mathcal{L})$ is arguably *the algebraic counterpart* of $\mathcal{L}$. See [6, 14, 17, 29] for more details and discussion, and for background on abstract algebraic logic.

**Lemma 2.**

1. $\vdash^1_{\mathsf{K}}$ *and* $\vdash^{\leqslant}_{\mathsf{K}}$ *have the same theorems.*
2. $\vdash^1_{\mathsf{K}}$ *is an extension of* $\vdash^{\leqslant}_{\mathsf{K}}$ *; in symbols,* $\vdash^{\leqslant}_{\mathsf{K}} \subseteq \vdash^1_{\mathsf{K}}$. *Therefore, in any algebra* $\boldsymbol{A}$, $\mathcal{F}i_1\boldsymbol{A} \subseteq \mathcal{F}i_{\leqslant}\boldsymbol{A}$.
3. *Both logics are **conjunctive**, that is, they satisfy:*
$$\varphi, \psi \vdash \xi \ \text{ iff } \ \varphi \wedge \psi \vdash \xi\,. \tag{5}$$
4. *As a consequence, each of them satisfies, for all* $n \geqslant 1$,
$$\varphi_1, \ldots, \varphi_n \vdash \psi \ \text{ iff } \ \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \psi \dashv\vdash \varphi_1 \wedge \cdots \wedge \varphi_n\,, \tag{6}$$
*where* $\dashv\vdash$ *is the respective relation of interderivability. Thus, the consequences of finite non-empty assumptions are determined by the interderivability relation.*
5. *The logic* $\vdash^{\leqslant}_{\mathsf{K}}$ *is **selfextensional**, that is, its relation of interderivability* $\dashv\vdash^{\leqslant}_{\mathsf{K}}$ *is a congruence of the formula algebra* $\boldsymbol{Fm}$. *Actually,*
$$\varphi \dashv\vdash^{\leqslant}_{\mathsf{K}} \psi \ \text{ iff } \ \mathsf{K} \models \varphi \approx \psi\,. \tag{7}$$
6. *If* $\boldsymbol{A} \in \mathsf{K}$ *then* $\mathcal{F}i_{\leqslant}\boldsymbol{A}$ *is the family of all semilattice filters of* $\boldsymbol{A}$. *The smallest one is* $\{1\}$, *which is also a filter of* $\vdash^1_{\mathsf{K}}$, *and the smallest one as well.*

**7.** $\mathsf{Alg}(\vdash_{\mathsf{K}}^{\leqslant}) = \mathsf{Alg}(\vdash_{\mathsf{K}}^{1}) = \mathsf{K}$.

**8.** $\mathsf{Alg}^*(\vdash_{\mathsf{K}}^{1}) = \mathsf{K}$*, and a matrix* $\langle \boldsymbol{A}, F\rangle$ *is a reduced model of* $\vdash_{\mathsf{K}}^{1}$ *if and only if* $\boldsymbol{A} \in \mathsf{K}$ *and* $F = \{1\}$.

**Proofs.** 1. Obvious from the definitions.
2. Same, because 1 is the maximum of the order in the algebras in $\mathsf{K}$.
3. Being conjunctive is a property inherited by extensions, because it is equivalent to the three following Hilbert-style rules:

$$\varphi \wedge \psi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi \qquad \varphi, \psi \vdash \varphi \wedge \psi \tag{8}$$

So it is only necessary to check it for $\vdash_{\mathsf{K}}^{\leqslant}$. But for this logic, (5) is trivially true.
4. Just use the rules in (8).
5. By definition, $\varphi \dashv\vdash_{\mathsf{K}} \psi$ iff $\varphi \vdash_{\mathsf{K}}^{\leqslant} \psi$ and $\psi \vdash_{\mathsf{K}}^{\leqslant} \varphi$. Therefore, $\varphi \dashv\vdash_{\mathsf{K}} \psi$ iff $\mathsf{K} \models \varphi \preccurlyeq \psi$ and $\mathsf{K} \models \psi \preccurlyeq \varphi$ iff $\mathsf{K} \models \varphi \approx \psi$. This relation is obviously a congruence, hence the logic is selfextensional.
6. The first statement is proved in Lemma 3.8 of [24], modulo the observation that here the empty set has to be excluded, as all the logics have theorems by assumption. Then clearly $\{1\}$ is the smallest semilattice filter and hence the smallest filter of the two logics.
7. That $\mathsf{Alg}(\vdash_{\mathsf{K}}^{\leqslant}) = \mathsf{K}$ is proved in Theorem 3.12 of [24], and by algebraizability, using Proposition 3.2 of [14], it follows that $\mathsf{Alg}(\vdash_{\mathsf{K}}^{1}) = \mathsf{K}$.
8. The first equality is again a consequence of the just mentioned result, and the characterization of reduced models is a general property of algebraizable logics, given that their filter is uniquely determined by the defined equations, which in this case reduce to $x \approx 1$, so that the filter reduces to the set $\{1\}$. □

Note that neither the general results of abstract algebraic logic nor the more specific ones in [24] yield a workable description of reduced models of $\vdash_{\mathsf{K}}^{\leqslant}$ nor of their algebraic reducts, the class $\mathsf{Alg}^*(\vdash_{\mathsf{K}}^{\leqslant})$. Later on I will give one.

The general theory of abstract algebraic logic can be used to obtain an alternative picture of the situation under scrutiny, this time in terms of properties of the class $\mathsf{K}$ of algebras:

**Lemma 3.** *Let* $\mathsf{K}$ *be a semilattice-based variety of algebras with maximum. Then the following conditions are equivalent:*

**(i)** *The assertional logic* $\vdash^1_{\mathsf{K}}$ *is algebraizable, with* $x \approx 1$ *as defining equation and* $\mathsf{K}$ *as its equivalent algebraic semantics.*

**(ii)** *There is a binary term* $\leftrightarrow$ *such that*

$$\mathsf{K} \models x \leftrightarrow x \approx 1, \tag{9}$$

$$\mathsf{K} \models x \leftrightarrow y \approx 1 \;\Rightarrow\; x \approx y. \tag{10}$$

*If these conditions hold, then* $\vdash^1_{\mathsf{K}}$ *is finitely, regularly and strongly algebraizable.*

**Proof.** Notice that, since $\mathsf{K}$ is a variety, the relative equational consequence $\models_{\mathsf{K}}$ associated with $\mathsf{K}$ is finitary, and hence the assertional logic $\vdash^1_{\mathsf{K}}$ defined from $\mathsf{K}$ for all $\Gamma \cup \{\varphi\} \subseteq Fm$, as

$$\Gamma \vdash^1_{\mathsf{K}} \varphi \ \text{ iff } \ \{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathsf{K}} \varphi \approx 1 \tag{11}$$

coincides with the finitary one defined in (1). By definition, this logic has $\mathsf{K}$ as its equivalent algebraic semantics.

(i)⇒(ii) If $\vdash^1_{\mathsf{K}}$ is algebraizable, there is a set of binary terms $\{\delta_i(x,y) : i \in I\}$ such that

$$x \approx y \mathrel{=\!\mid\models_{\mathsf{K}}} \bigl\{\delta_i(x,y) \approx 1 : i \in I\bigr\}.$$

By finitarity of the consequence $\models_{\mathsf{K}}$, there is a finite family of terms $\delta_1, \dots, \delta_k$ such that actually

$$x \approx y \mathrel{=\!\mid\models_{\mathsf{K}}} \bigl\{\delta_1(x,y) \approx 1, \dots, \delta_k(x,y) \approx 1\bigr\}.$$

Then, define $x \leftrightarrow y = \delta_1(x,y) \wedge \cdots \wedge \delta_k(x,y)$. The fact that $\wedge$ is the semilattice operation and 1 is the maximum of the associated order in the algebras in $\mathsf{K}$ implies that

$$\bigl\{\delta_1(x,y) \approx 1, \dots, \delta_k(x,y) \approx 1\bigr\} \mathrel{=\!\mid\models_{\mathsf{K}}} x \leftrightarrow y \approx 1,$$

therefore

$$x \approx y \mathrel{=\!\mid\models_{\mathsf{K}}} x \leftrightarrow y \approx 1, \tag{12}$$

which is trivially equivalent to having both (9) and (10).

(ii)⇒(i) Now, assume that there is a binary term $\leftrightarrow$ satisfying conditions (9) and (10). As observed before, these amount to the condition (12), which is equivalent to the property that $\vdash^1_{\mathsf{K}}$ is algebraizable with the defining equation $x \approx 1$ and with $\mathsf{K}$ as its equivalent algebraic semantics (given that it is already its algebraic semantics).

Finally, the fact that algebraizability is witnessed by a single equivalence formula implies that the logic is finitely algebraizable; that the defining equation is of the form $x \approx 1$ means it is regularly algebraizable; and that its equivalent algebraic semantics is a variety is described by saying that it is *strongly* algebraizable (see [17] for details). □

It is interesting to notice that the assumption that $\mathsf{K}$ is semilattice-based with maximum can be weakened in Lemma 3 to require that the binary term $\wedge$ and the constant 1 just satisfy

$$\{x \approx 1\,,\, y \approx 1\} \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} x \wedge y \approx 1\,.$$

This assumption is strictly weaker than that of $\mathsf{K}$ being semilattice-based with maximum; as an example, in residuated lattices this property is satisfied both by the ordinary (additive) conjunction $\wedge$ and by the multiplicative conjunction or fusion operation $\star$, while they are semilattice-based only with respect to $\wedge$ and not with respect to $\star$ (unless they are Heyting algebras, where the two operations coincide).

Putting the semilattice properties in algebraic form quickly gives:

**Corollary 4.** *Let $\mathsf{K}$ be a variety of algebras. Then the basic assumptions of this paper hold (that is, $\mathsf{K}$ is semilattice-based with maximum and its assertional logic $\vdash^1_{\mathsf{K}}$ is algebraizable with defining equation $x \approx 1$ and equivalent algebraic semantics $\mathsf{K}$) if and only if the following equations and quasi-equations are satisfied in $\mathsf{K}$,*

$$x \wedge x \approx x \tag{13}$$

$$x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \tag{14}$$

$$x \wedge y \approx y \wedge y \tag{15}$$

$$x \wedge 1 \approx x \tag{16}$$

$$x \leftrightarrow x \approx 1 \tag{9}$$

$$x \leftrightarrow y \approx 1 \;\Rightarrow\; x \approx y \tag{10}$$

*for certain binary terms $\wedge, \leftrightarrow$ and some constant term 1.* □

As a particular case of (10), the following apparently weaker property is obtained

$$\mathsf{K} \models 1 \leftrightarrow y \approx 1 \;\Rightarrow\; 1 \approx y\,. \tag{17}$$

The fact that properties (9) and (17) are satisfied means that the class $\mathsf{K}$ is *1-protoregular* in the sense of [3]; the classes of algebras of this kind are very interesting from the point of view of abstract algebraic logic, and in [7, Theorem 6.8] it is shown that their associated assertional logic is always weakly algebraizable. But, by Theorem 3.10 of [14], if a weakly algebraizable logic has a variety as its algebraic counterpart, then the logic is actually algebraizable, which is the present situation, so here (17) implies (10) and can thus replace it in Corollary 4. Notice that this result concerns a variety and that all properties appearing in it are equations, save for (10) or (17) which are quasi-equations[1].

Leaving the previous results aside, no explicit use of $\leftrightarrow$ will be made in the paper; the key properties of $\vdash^1_{\mathsf{K}}$ will be

- that $\mathsf{Alg}(\vdash^1_{\mathsf{K}}) = \mathsf{K}$, and
- that in each algebra $\boldsymbol{A}$ the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an order-isomorphism between $\mathcal{F}i_1\boldsymbol{A}$ and $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$.

Both are consequences of the algebraizability of $\vdash^1_{\mathsf{K}}$ with respect to $\mathsf{K}$. That they are indeed necessary is discussed at the end of the paper, where it is shown by a counterexample that regular algebraizability can be replaced neither by regular weak algebraizability nor by ordinary algebraizability.

It is also worth recalling that just monotonicity of this operator on the filters of some logic on every algebra characterizes that logic as protoalgebraic; see [2, 6] for more details.

The central result in the present paper is the following one:

**Theorem 5.** *For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, there exists a unique Leibniz filter $F^+ \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, namely*

$$F^+ = \min\{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F\} \subseteq F.$$

*Moreover $F^+ \in \mathcal{F}i_1\boldsymbol{A}$ and it is determined from $\boldsymbol{\Omega}^{\boldsymbol{A}}F$ by the expression*

$$F^+ = \big\{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F\big\}\,.$$

[1] An anonymous referee has pointed out the intrinsic interest of investigating whether either of them can be replaced by an equation, or a finite set of equations. This issue is not dealt with in the present paper.

**Proof.** The proof of the second statement is actually part of the construction that proves the first statement. For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, the matrix $\langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F, F/\boldsymbol{\Omega}^{\boldsymbol{A}}F\rangle$ is always reduced, hence $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{Alg}^*(\vdash^{\leqslant}_{\mathsf{K}})$. By Corollary 2.24 of [14] and Lemma 2.7, $\mathsf{Alg}^*(\vdash^{\leqslant}_{\mathsf{K}}) \subseteq \mathsf{Alg}(\vdash^{\leqslant}_{\mathsf{K}}) = \mathsf{K}$, therefore actually $\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$. By the algebraizability of $\vdash^{1}_{\mathsf{K}}$, there is a unique $F^+ \in \mathcal{F}i_1\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. By Lemma 2.2, $F^+ \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$. Thus $F^+ \in \{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F\}$. Now let $a \in F^+$. Since $x \approx 1$ is the defining equation of $\vdash^{1}_{\mathsf{K}}$, $a \in F^+$ if and only if $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. Since by assumption 1 is a theorem of $\vdash^{1}_{\mathsf{K}}$, by Lemma 2.1 it is also a theorem of $\vdash^{\leqslant}_{\mathsf{K}}$, and hence $1 \in F$. Then, by compatibility it follows that $a \in F$. This shows that $F^+ \subseteq F$. Now it is possible to show that $F^+ = \min\{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F\}$: let $G \in \{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F\}$ and consider $G^+ \in \mathcal{F}i_1\boldsymbol{A}$ constructed in the same way as $F^+$ but with respect to $G$; then $\boldsymbol{\Omega}^{\boldsymbol{A}}F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}}F = \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}G^+$, and since both are filters of $\vdash^{1}_{\mathsf{K}}$, it follows that $F^+ = G^+$ because $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is one-to-one on $\mathcal{F}i_1\boldsymbol{A}$. But $G^+ \subseteq G$, so $F^+ \subseteq G$. This proves that $F^+ = \min\{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F\}$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}}F$, this fact can also be expressed as $F^+ = \min\{G \in \mathcal{F}i_{\leqslant}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}F^+\}$, which means that $F^+$ is a Leibniz filter of $\vdash^{\leqslant}_{\mathsf{K}}$. □

Obviously, a filter $F$ is Leibniz if and only if $F = F^+$. This yields several characterizations of the notion; the one contained in part (iv) of the next result was proved in [14, 15] in the context of protoalgebraic logics, and with a more involved proof:

**Proposition 6.** *For any $\boldsymbol{A}$, if $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, then the following conditions are equivalent:*

**(i)** *$F$ is a Leibniz filter of $\vdash^{\leqslant}_{\mathsf{K}}$.*

**(ii)** $F \in \mathcal{F}i_1\boldsymbol{A}$.

**(iii)** $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = \{1\}$.

**(iv)** *$F/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the smallest filter of $\vdash^{\leqslant}_{\mathsf{K}}$ on $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$.*

**Proof.** If $F$ is Leibniz then $F = F^+$, so $F \in \mathcal{F}i_1\boldsymbol{A}$. Conversely, if $F \in \mathcal{F}i_1\boldsymbol{A}$, then $F^+$ is a filter of $\vdash^{1}_{\mathsf{K}}$ with the same Leibniz congruence as $F$, and since $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is one-to-one on $\mathcal{F}i_1\boldsymbol{A}$, $F = F^+$, hence $F$ is Leibniz. This shows that (i) is equivalent to (ii). Due to the way how $\vdash^{1}_{\mathsf{K}}$ is algebraized (i.e., that the defining equation is $x \approx 1$), the only filter in a reduced model of $\vdash^1_{\mathsf{K}}$ is of the form $\{1\}$. Hence, if $F$ is a filter of $\vdash^1_{\mathsf{K}}$ then $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = \{1\}$. Conversely, if $F/\boldsymbol{\Omega}^{\boldsymbol{A}}F = \{1\}$, then $F = \pi^{-1}\big[\{1\}\big]$, where $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F$ is the canonical projection. Since by Lemma 2.6 $\{1\}$ is a filter of $\vdash^1_{\mathsf{K}}$ on the quotient algebra, by Proposition 1.19 of [14] it follows that $F \in \mathcal{F}i_1\boldsymbol{A}$. This shows that (ii) is equivalent to (iii). Finally, (iii) is equivalent to (iv) because if $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ then $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}F \in \mathsf{K}$ and by Lemma 2.6 the smallest filter of $\vdash^{\leqslant}_{\mathsf{K}}$ coincides with the smallest filter of $\vdash^1_{\mathsf{K}}$, which is $\{1\}$ by the same property. □

The situation described in the two previous results can be summarized in the terminology of [16] by saying that the logic $\vdash^{\leqslant}_{\mathsf{K}}$ *has enough Leibniz filters* and that the logic $\vdash^1_{\mathsf{K}}$ is its *strong version*[2]. As a matter of fact, here the following three senses of the notion of a strong version of the logic $\vdash^{\leqslant}_{\mathsf{K}}$ coincide:

- The assertional companion $\vdash^1_{\mathsf{K}}$ of $\vdash^{\leqslant}_{\mathsf{K}}$, as introduced in Definition 1. Starting from a semilattice-based variety $\mathsf{K}$ this construction always yields a logic, which is stronger that $\vdash^{\leqslant}_{\mathsf{K}}$.
- The logic defined by the class of matrices $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an arbitrary algebra and $F$ is a Leibniz filter of $\vdash^{\leqslant}_{\mathsf{K}}$, as introduced and studied in [15] for protoalgebraic logics. The notion of Leibniz filter can be considered for an arbitrary logic, and since they are a special family of the filters of the logic, this construction always produces a logic stronger than the original one. The phrase *strong version* was introduced in [15], as a technical term, to refer to the logic defined in this way.
- The logic defined by the class of matrices $\langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an arbitrary algebra and $F$ is the smallest filter of $\vdash^{\leqslant}_{\mathsf{K}}$ on $\boldsymbol{A}$. This is a construction that, again, is always possible starting from an arbitrary logic, since the smallest filter on a given algebra always exists; it always produces a logic stronger than that produced by the previous procedure (because the smallest filter is always Leibniz), and hence stronger than the original logic.

Each of the logics resulting from the three procedures can claim to be a "strong version" of the initial logic, and by Proposition 6, in the present sit-

[2]These expressions however do not cover the equivalent conditions (iii) and (iv) of Proposition 6; these appear here because the logic $\vdash^1_{\mathsf{K}}$ is algebraizable, something that is not assumed in the more general framework of [16].

uation they coincide. Up to now this was only known to happen when the initial logic is protoalgebraic [15]. This adds to the idea that semilattice-based logics with an algebraizable assertional companion behave particularly well and share some of the properties of protoalgebraic logics without being so.

One property than can be obtained here, in contrast with the general theory [14, 24] of semilattice-based logics, is the determination of the reduced models of $\vdash_{\mathsf{K}}^{\leqslant}$. The first step is to characterize their algebra reducts, that is, the class $\mathsf{Alg}^*(\mathcal{L})$. As already recalled, if a logic $\mathcal{L}$ is protoalgebraic then $\mathsf{Alg}^*(\mathcal{L}) = \mathsf{Alg}(\mathcal{L})$, but since $\vdash_{\mathsf{K}}^{\leqslant}$ is not assumed to be protoalgebraic, one has to find $\mathsf{Alg}^*(\vdash_{\mathsf{K}}^{\leqslant})$ with a direct, *ad hoc* proof. As shown at the end of the paper, for arbitrary non-protoalgebraic logics this class can be rather weird and unexpected. In the present case the result is indeed the expected one:

**Proposition 7.** $\mathsf{Alg}^*(\vdash_{\mathsf{K}}^{\leqslant}) = \mathsf{K}$.

**Proof.** In general $\mathsf{Alg}^*(\vdash_{\mathsf{K}}^{\leqslant}) \subseteq \mathsf{Alg}(\vdash_{\mathsf{K}}^{\leqslant}) = \mathsf{K}$. If $\boldsymbol{A} \in \mathsf{K}$ then by Lemma 2.8 the matrix $\langle \boldsymbol{A}, \{1\} \rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{1}$. Lemma 2.2 implies that it is also a model of $\vdash_{\mathsf{K}}^{\leqslant}$. Since being reduced is an intrinsic property of a matrix and does not depend on the logic envisaged to be modelled, $\langle \boldsymbol{A}, \{1\} \rangle$ is also a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$. Therefore, one concludes that $\boldsymbol{A} \in \mathsf{Alg}^*(\vdash_{\mathsf{K}}^{\leqslant})$. ☐

The characterization of the class of algebra reducts of the reduced models of $\vdash_{\mathsf{K}}^{\leqslant}$ yields one of the reduced models themselves, thus complementing Lemma 2.8:

**Proposition 8.** *A matrix $\langle \boldsymbol{A}, F \rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$ if and only if $\boldsymbol{A} \in \mathsf{K}$ and $F$ is a semilattice filter of $\boldsymbol{A}$ such that $F^+ = \{1\}$.*

**Proof.** If $\langle \boldsymbol{A}, F \rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$, then by Proposition 7, $\boldsymbol{A} \in \mathsf{K}$, and by Lemma 2.6 this implies that $F$ is a semilattice filter of $\boldsymbol{A}$. By Theorem 5, $F^+ \in \mathcal{F}i_1 \boldsymbol{A}$, therefore the matrix $\langle \boldsymbol{A}, F^+ \rangle$ is a model of $\vdash_{\mathsf{K}}^{1}$, and moreover $\boldsymbol{\Omega}^{\boldsymbol{A}} F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}} F = Id$, so that $\langle \boldsymbol{A}, F^+ \rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{1}$. By Lemma 2.8 this implies that $F^+ = \{1\}$. The converse follows also from $\boldsymbol{\Omega}^{\boldsymbol{A}} F^+ = \boldsymbol{\Omega}^{\boldsymbol{A}} F$. ☐

Using the characterization of $F^+$ in Theorem 5, the characterization in Proposition 8 can be rewritten in the following, less cryptic way:

**Corollary 9.** *A matrix* $\langle \boldsymbol{A}, F\rangle$ *is a reduced model of* $\vdash_{\mathsf{K}}^{\leqslant}$ *if and only if* $\boldsymbol{A} \in \mathsf{K}$ *and* $F$ *is a semilattice filter of* $\boldsymbol{A}$ *such that for all* $a \in A$*, if* $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ *then* $a = 1$. □

Notice how the stated condition is a restricted form of the property of being reduced, which would be "if $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}} F$ then $a = b$". It is perhaps easier to understand it as saying that $\{1\}$ constitutes an equivalence class of $\boldsymbol{\Omega}^{\boldsymbol{A}} F$. Proposition 8 also implies:

**Corollary 10.** *If* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{A}, G\rangle$ *are reduced models of* $\vdash_{\mathsf{K}}^{\leqslant}$ *on the same algebra* $\boldsymbol{A}$*, then* $F^{+} = G^{+}$. □

Another, more interesting consequence is:

**Proposition 11.** *If* $\langle \boldsymbol{A}, F\rangle$ *and* $\langle \boldsymbol{B}, G\rangle$ *are models of* $\vdash_{\mathsf{K}}^{\leqslant}$ *and* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a strict surjective homomorphism between them, then* $h^{-1}[G^{+}] = F^{+}$*, and* $h$ *is also a strict surjective homomorphism between* $\langle \boldsymbol{A}, F^{+}\rangle$ *and* $\langle \boldsymbol{B}, G^{+}\rangle$.

**Proof.** Since $h$ is strict and surjective, $F = h^{-1}[G]$. Moreover, $h$ is also a strict and surjective homomorphism between $\left\langle \boldsymbol{A}, h^{-1}[G^{+}]\right\rangle$ and $\langle \boldsymbol{B}, G^{+}\rangle$. By Proposition 0.5.5 of [6], the Leibniz operator commutes with strict and surjective homomorphisms between matrices, therefore

$$\boldsymbol{\Omega}^{\boldsymbol{A}} h^{-1}[G^{+}] = h^{-1}\big[\boldsymbol{\Omega}^{\boldsymbol{B}} G^{+}\big] = h^{-1}\big[\boldsymbol{\Omega}^{\boldsymbol{B}} G\big] = \boldsymbol{\Omega}^{\boldsymbol{A}} h^{-1}[G] = \boldsymbol{\Omega}^{\boldsymbol{A}} F = \boldsymbol{\Omega}^{\boldsymbol{A}} F^{+}.$$

Since $G^{+} \in \mathcal{F}i_1 \boldsymbol{B}$, also $h^{-1}[G^{+}] \in \mathcal{F}i_1 \boldsymbol{A}$, and since $F^{+} \in \mathcal{F}i_1 \boldsymbol{A}$ as well and $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is one-to-one on $\mathcal{F}i_1 \boldsymbol{A}$, it follows that $h^{-1}[G^{+}] = F^{+}$. □

The same result is shown in Theorem 8 of [15], assuming protoalgebraicity of what would be here $\vdash_{\mathsf{K}}^{\leqslant}$, but in a much more complicated manner; here it is rather the assumption on $\vdash_{\mathsf{K}}^{1}$ that provides more powerful tools. Straightforward consequences of this proposition are that, under the same assumptions, $F$ is Leibniz if and only if $G$ is Lebniz, and that in general a filter of $\vdash_{\mathsf{K}}^{\leqslant}$ is Leibniz if and only if its reduction is Leibniz as well.

In [15] it is shown that for protoalgebraic logics the Leibniz filter $F^{+}$ associated with an arbitrary filter is also the largest of all Leibniz filters included in it. This property no longer holds in the present, more general case; on the contrary, the said property precisely characterizes the protoalgebraic logics among the semilattice-based logics that have an algebraizable assertional companion:

**Proposition 12.** *The following conditions are equivalent:*

**(i)** *The logic* $\vdash_{\mathsf{K}}^{\leqslant}$ *is protoalgebraic.*

**(ii)** *For each* $\boldsymbol{A}$ *and each* $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, $F^{+}$ *is the largest Leibniz filter of* $\vdash_{\mathsf{K}}^{\leqslant}$ *contained in* $F$*.*

**(iii)** *For each* $\boldsymbol{A}$ *and each* $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, $F^{+}$ *is the largest filter of* $\vdash_{\mathsf{K}}^{1}$ *contained in* $F$*.*

**Proof.** (i)⇒(ii) is proved in Corollary 4 of [15].

(ii)⇒(i) Let $\boldsymbol{A}$ be any algebra, and let $F, G \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ with $G \subseteq F$. Then $G^{+} \subseteq G \subseteq F$ and $G^{+}$ is a Leibniz filter of $\vdash_{\mathsf{K}}^{\leqslant}$. Since by the assumption $F^{+}$ is the largest Leibniz filter of $\vdash_{\mathsf{K}}^{\leqslant}$ contained in $F$, it follows that $G^{+} \subseteq F^{+}$. Now the algebraizability of $\vdash_{\mathsf{K}}^{1}$ implies in particular that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotonic on $\mathcal{F}i_{1}\boldsymbol{A}$, so that $\boldsymbol{\Omega}^{\boldsymbol{A}}G = \boldsymbol{\Omega}^{\boldsymbol{A}}G^{+} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F^{+} = \boldsymbol{\Omega}^{\boldsymbol{A}}F$. Thus $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotonic on $\mathcal{F}i_{\leqslant}\boldsymbol{A}$ for any $\boldsymbol{A}$, and this amounts to $\vdash_{\mathsf{K}}^{\leqslant}$ being protoalgebraic.

(ii)⇔(iii) because by Proposition 6 the Leibniz filters of $\vdash_{\mathsf{K}}^{\leqslant}$ coincide with the filters of $\vdash_{\mathsf{K}}^{1}$. ☐

Notice that in [15], where all logics are assumed to be protoalgebraic, only property (ii) holds, but not (iii); the reason is that not all items in Proposition 6 are equivalent in the setting of [15]. Here we also have condition (iii), which is probably more useful when looking at particular logics.

Protoalgebraicity of semilattice-based logics in the case where $\mathsf{K}$ is a variety of residuated lattices[3] is analyzed in Section 4 of [4]. This (large) group of examples happens to have stronger properties, concerning the topic of the present paper, than the general case analyzed here. For instance, surprisingly, the logic $\vdash_{\mathsf{K}}^{\leqslant}$ is protoalgebraic if and only if it is finitely equivalential, and also if and only if the variety $\mathsf{K}$ satisfies the equation $x \wedge \big((x \to y)^{n} \star (y \to x)^{n}\big) \preccurlyeq y$ for some $n < \omega$ (where $\star$ is the fusion connective, and $\to$ is its residuum). These and other, purely algebraic

[3] As in other works in the literature, all residuated lattices studied in the cited paper are assumed to be commutative and integral. This last condition amounts to saying that 1, the unit of the fusion operation, is also the maximum of the order; this condition is thus unavoidable in order to obtain the cases that fall under the framework of the present paper.

characterizations allow to identify many cases where $\vdash_{\mathsf{K}}^{\leqslant}$ is *not protoalgebraic*; for instance, this happens when $\mathsf{K}$ is any of the following varieties: all residuated lattices, MV-algebras, product algebras, basic algebras, MTL algebras, $\mathrm{FL_{ew}}$-algebras, and any variety generated by a family of continuous t-norms over the real unit interval which is not the variety of Gödel algebras; general references for these classes of algebras and their associated logics are [5, 21, 22]. This is interesting because it provides a host[4] of natural and well-behaved examples of non-protoalgebraic logics; these examples were lacking in the early stages of the development of abstract algebraic logic, to the point that they had been considered rather *pathological*[5].

The next result settles the precise conditions under which the description of reduced models of $\vdash_{\mathsf{K}}^{\leqslant}$ in Proposition 8 can be improved in a definite way that avoids referring to the filter $F^{+}$, thus making it more practical. This improved characterization generalizes several ones found in the field of modal logic, as explained below. Besides, the result is technically interesting in itself:

**Lemma 13.** *The following conditions are equivalent:*

**(i)** *A matrix $\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$ if and only if $\boldsymbol{A} \in \mathsf{K}$, $F$ is a semilattice filter of $\boldsymbol{A}$ and $\{1\}$ is the only filter of $\vdash_{\mathsf{K}}^{1}$ contained in $F$.*

**(ii)** *For every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, if $G \subseteq F$ and $\langle \boldsymbol{A}, F\rangle$ is reduced, then also $\langle \boldsymbol{A}, G\rangle$ is reduced.*

**Proof.** (i)⇒(ii) If the matrix $\langle \boldsymbol{A}, F\rangle$ is reduced and $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, then $\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$ and thus it satisfies the equivalent condition in (i). Now let $G \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ be such that $G \subseteq F$. Then any $\vdash_{\mathsf{K}}^{1}$-filter contained in $G$ will also be contained in $F$, therefore $\{1\}$ will also be the only $\vdash_{\mathsf{K}}^{1}$-filter contained in $G$. Moreover, $\boldsymbol{A} \in \mathsf{K}$ by assumption, and by Lemma 2.6 $G$ is also a semilattice filter of $\boldsymbol{A}$. Thus, the matrix $\langle \boldsymbol{A}, G\rangle$ satisfies the equivalent condition in (i), and therefore it is reduced.

(ii)⇒(i) Assume that $\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\vdash_{\mathsf{K}}^{\leqslant}$. By Proposition 8, $\boldsymbol{A} \in \mathsf{K}$, $F$ is a semilattice filter of $\boldsymbol{A}$, and $F^{+} = \{1\}$. Let $G \in \mathcal{F}i_{1}\boldsymbol{A}$

[4]Actually, there is at least a denumerable number (perhaps a continuum) of such examples.

[5]Blok and Pigozzi, in their seminal paper [2, p. 355], wrote that "all the logics we are aware of in the literature, except for pathological cases, are protoalgebraic".

be such that $G \subseteq F$. Then $G \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ and by (ii) $\boldsymbol{\Omega}^{\boldsymbol{A}}G = Id$ and since $\boldsymbol{\Omega}^{\boldsymbol{A}}\{1\} = Id$, and both $\{1\}$ and $G$ are $\vdash^{1}_{\mathsf{K}}$-filters, it follows that $G = \{1\}$. Conversely, assume that the conditions are satisfied for some $\langle \boldsymbol{A}, F\rangle$. Then clearly $\langle \boldsymbol{A}, F\rangle$ is a model of $\vdash^{\leqslant}_{\mathsf{K}}$. Since $F^{+}$ is a $\vdash^{1}_{\mathsf{K}}$-filter contained in $F$, by the assumptions it follows that $F^{+} = \{1\}$, and by Proposition 8 this means that the matrix is reduced. □

Notice how condition 13(ii) can be rephrased as a restricted version of the monotonicity condition of the Leibniz operator:

If $F, G \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$ with $G \subseteq F$ then $\boldsymbol{\Omega}^{\boldsymbol{A}}G \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}F$, provided that $\boldsymbol{\Omega}^{\boldsymbol{A}}F = Id$.

Since the Leibniz operator is monotonic, without restrictions, over all filters of protoalgebraic logics, the result corresponding to Proposition 6 of [15] follows immediately:

**Corollary 14.** *If* $\vdash^{\leqslant}_{\mathsf{K}}$ *is protoalgebraic, then a matrix* $\langle \boldsymbol{A}, F\rangle$ *is a reduced model of* $\vdash^{\leqslant}_{\mathsf{K}}$ *if and only if* $\boldsymbol{A} \in \mathsf{K}$, $F$ *is a semilattice filter of* $\boldsymbol{A}$ *and* $\{1\}$ *is the only filter of* $\vdash^{1}_{\mathsf{K}}$ *contained in* $F$. □

Theorem 4.4 of [4] shows that the converse implication holds when $\mathsf{K}$ is a variety of residuated lattices, so that the two equivalent conditions in Lemma 13 can be added to those in Proposition 12. However, the proof uses several properties of residuated lattices in an essential way, and it is not likely that this converse holds in general.

The standard examples of this situation analyzed in [15] are that of the local and the global consequences associated with a normal system of modal logic and that of the weak and strong quantum logics associated with a variety of orthomodular lattices, as already explained in the introductory discussion. A close example, still not analyzed in the literature from this point of view, is that of the weakest *classical* system of modal logic $\mathcal{E}$. The associated variety $\mathsf{K}$ is here that of all Boolean algebras expanded with an arbitrary unary operator $\Box$, and it originates two logics, that is, two consequence relations, which can be seen to conform to the framework of this paper, that is, they have the forms $\vdash^{\leqslant}_{\mathsf{K}}$ and $\vdash^{1}_{\mathsf{K}}$ for the said variety $\mathsf{K}$. Syntactically, the two are separated only by the Extensionality Rule: the logic $\vdash^{\leqslant}_{\mathsf{K}}$ has it in the weak form (if $\vdash^{\leqslant}_{\mathsf{K}} \alpha \leftrightarrow \beta$ then $\vdash^{\leqslant}_{\mathsf{K}} \Box\alpha \leftrightarrow \Box\beta$) while $\vdash^{1}_{\mathsf{K}}$ is the extension of $\vdash^{\leqslant}_{\mathsf{K}}$ with the strong form of the rule ($\alpha \leftrightarrow \beta \vdash^{1}_{\mathsf{K}} \Box\alpha \leftrightarrow \Box\beta$). The logic $\vdash^{1}_{\mathsf{K}}$ is algebraizable, hence this example falls under the scope of the present paper. In contrast with what happens in the case of the *normal* modal logics, here $\vdash_{\mathsf{K}}^{\leqslant}$ is not equivalential, but it is still protoalgebraic. On a Boolean algebra the filters of $\vdash_{\mathsf{K}}^{1}$ are the filters of $\vdash_{\mathsf{K}}^{\leqslant}$ that are closed under the Extensionality Rule. Therefore, Corollary 14 effectively generalizes Malinowski's characterization [25, Theorem II.1] of the reduced models of $\vdash_{\mathsf{K}}^{\leqslant}$ as the matrices $\langle \boldsymbol{A}, F\rangle$ such that $\boldsymbol{A}$ is a Boolean algebra with a unary operator $\Box$ and $F$ is a filter of $\boldsymbol{A}$ such that $\{1\}$ is the only filter $G \subseteq F$ that is closed under the rule "if $a \leftrightarrow b \in G$ then $\Box a \leftrightarrow \Box b \in G$". The same result holds for other classical systems of modal logic extending $\mathcal{E}$; in each case it suffices to restrict the class of algebras to those Boolean algebras with a unary $\Box$ satisfying the particular axioms or rules of the logic. When the logic is also normal (i.e., it is stronger than the so-called Kripke's logic $\mathcal{K}$) then the condition of being closed under the Extensionality Rule may be replaced by that of being open, which amounts to being closed under the Necessitation Rule. For more details see [6, § 3.4] or [25].

The last result of the paper tells us that the key abstract properties that distinguish the logics $\vdash_{\mathsf{K}}^{\leqslant}$ and $\vdash_{\mathsf{K}}^{1}$ (that $\vdash_{\mathsf{K}}^{\leqslant}$ is selfextensional and $\vdash_{\mathsf{K}}^{1}$ is algebraizable) effectively separate them in the sense that each cannot enjoy the characteristic property of the other unless the two are actually the same logic. Moreover, two further equivalent properties can be added, one concerning the location of the logic $\vdash_{\mathsf{K}}^{1}$ in the Frege hierarchy and the other concerning the location of the logic $\vdash_{\mathsf{K}}^{\leqslant}$ in the Leibniz hierarchy; the proof uses some notions and results of abstract algebraic logic not described here.

**Theorem 15.** *The following conditions are equivalent:*

**(i)** $\vdash_{\mathsf{K}}^{1}$ *is selfextensional.*

**(ii)** $\vdash_{\mathsf{K}}^{1}$ *is fully Fregean.*

**(iii)** $\vdash_{\mathsf{K}}^{\leqslant}$ *is algebraizable.*

**(iv)** $\vdash_{\mathsf{K}}^{\leqslant}$ *is weakly algebraizable.*

**(v)** $\vdash_{\mathsf{K}}^{\leqslant} = \vdash_{\mathsf{K}}^{1}$.

**Proof.** (i)⇒(ii) By Lemma 2.3 the logic $\vdash_{\mathsf{K}}^{1}$ is conjunctive. Now the assumption is that it is in addition selfextensional. Therefore Theorem 4.28 of [14] applies, and as a consequence it is also fully selfextensional. Since $\vdash_{\mathsf{K}}^{1}$ is by assumption algebraizable, it is *a fortiori* weakly algebraizable, and for these logics Corollary 3.21 of the same work shows that being fully selfextensional is equivalent to being Fregean. Finally, being algebraizable, it is protoalgebraic, and Corollary 80 of [8] shows that for a protoalgebraic logic, to be Fregean implies to be fully Fregean. (ii)⇒(v) All fully Fregean logics are fully selfextensional. This means that the natural equivalence or Frege relation associated with the closure system of all filters of the logic on an arbitrary algebra is a congruence, and hence it coincides with the Tarski relation, which is the identity in algebras of $\mathsf{Alg}(\vdash^1_{\mathsf{K}}) = \mathsf{K}$. Thus, if $\boldsymbol{A} \in \mathsf{K}$ and $a, b \in A$ are such that $Fi_1^{\boldsymbol{A}}(a) = Fi_1^{\boldsymbol{A}}(b)$, then $a = b$, where $Fi_1^{\boldsymbol{A}}$ is the operator of $\vdash^1_{\mathsf{K}}$-filter generation. Now, let $\varphi, \psi \in Fm$ be such that $\varphi \dashv\vdash^1_{\mathsf{K}} \psi$, let $\boldsymbol{A} \in \mathsf{K}$ and $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. The interderivability of $\varphi$ and $\psi$ implies that $Fi_1^{\boldsymbol{A}}\big(v(\varphi)\big) = Fi_1^{\boldsymbol{A}}\big(v(\psi)\big)$, and by the previous observation this implies that $v(\varphi) = v(\psi)$. Therefore $\mathsf{K} \models \varphi \approx \psi$. Conversely, using that $\vdash^1_{\mathsf{K}}$ is complete with respect to the class all all its matrix models over the algebras in $\mathsf{K}$, the same argument works backwards and shows that $\varphi \dashv\vdash^1_{\mathsf{K}} \psi$ if and only if $\mathsf{K} \models \varphi \approx \psi$. Now, using this fact and (6),

$$\varphi \vdash^1_{\mathsf{K}} \psi \;\; \textsf{iff} \;\; \varphi \wedge \psi \dashv\vdash^1_{\mathsf{K}} \varphi \;\; \textsf{iff} \;\; \mathsf{K} \models \varphi \wedge \psi \approx \varphi \;\; \textsf{iff} \;\; \mathsf{K} \models \varphi \preccurlyeq \psi \;\; \textsf{iff} \;\; \varphi \vdash^{\leqslant}_{\mathsf{K}} \psi\,.$$

That is, the two logics coincide on single-premise consequences. Since both logics are finitary, have the same theorems (Lemma 2.1) and are conjunctive (Lemma 2.3), this implies that $\Gamma \vdash^1_{\mathsf{K}} \psi$ if and only if $\Gamma \vdash^{\leqslant}_{\mathsf{K}} \psi$ for all $\Gamma$ and $\psi$, that is, that $\vdash^1_{\mathsf{K}} = \vdash^{\leqslant}_{\mathsf{K}}$.

(iv)⇒(v) Since $\mathsf{Alg}(\vdash^{\leqslant}_{\mathsf{K}}) = \mathsf{Alg}(\vdash^1_{\mathsf{K}}) = \mathsf{K}$, in order to show that $\vdash^{\leqslant}_{\mathsf{K}} = \vdash^1_{\mathsf{K}}$ it is enough to show that in any $\boldsymbol{A} \in \mathsf{K}$, $\mathcal{F}i_{\leqslant}\boldsymbol{A} = \mathcal{F}i_1\boldsymbol{A}$. By Lemma 2.2 one needs only prove that $\mathcal{F}i_{\leqslant}\boldsymbol{A} \subseteq \mathcal{F}i_1\boldsymbol{A}$. The assumption that the logic $\vdash^{\leqslant}_{\mathsf{K}}$ is weakly algebraizable implies that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is one-to-one on $\mathcal{F}i_{\leqslant}\boldsymbol{A}$, and this implies that for every $F \in \mathcal{F}i_{\leqslant}\boldsymbol{A}$, $F = F^{+}$. Therefore, by Theorem 5, $F \in \mathcal{F}i_1\boldsymbol{A}$.

The remaining implications are trivial. □

Thus, if any of the conditions in this theorem holds, the two logics coincide, and this unique logic is actually finitely, regularly and strongly algebraizable (Lemma 3) and moreover it is fully Fregean; these properties together place it very high in both the Leibniz and the Frege hierarchies of abstract algebraic logic.

Classical and intuitionistic logic are examples of the coincidence of the logics $\vdash^{\leqslant}_{\mathsf{K}}$ and $\vdash^1_{\mathsf{K}}$, with $\mathsf{K}$ being the variety of Boolean and of Heyting algebras respectively. Other examples are contained in the paper [4] in the context of substructural logics determined by varieties of residuated lattices; in this case, the filters of the logic $\vdash^{1}_{\mathsf{K}}$, which are the implicative filters of the algebras, can be characterized as the filters of $\vdash^{\leqslant}_{\mathsf{K}}$ closed under the fusion operation. This fact provides stronger algebraic tools which allow the authors to show, among other properties, that the varieties for which the conditions in Theorem 15 hold are exactly those contained in the variety of Heyting algebras (more precisely, of generalized Heyting algebras, because in the setting of [4] no minimum element is required to exist); see Theorems 4.3 and 4.12 of [4].

Some of the equivalences in Theorem 15 correspond to those in Theorem 4.10 of [24], which assumes that the logic under consideration is protoalgebraic; thus, this is another result showing that these logics share some of the properties of protoalgebraic logics without being so. It is also possible to see that the implication (iii)⇒(v) is very close to one of the two contained in Theorem 3.16 of [24]; however, this one assumes and uses in an essential way the *regular* algebraizability of $\vdash^{\leqslant}_{\mathsf{K}}$, which is not assumed in (iii) but appears only as a consequence of its coincidence with $\vdash^{1}_{\mathsf{K}}$.

To end the paper let's discuss the possibilities of extending its scope by relaxing its basic assumptions.

Algebraizability of the assertional companion seems to be the key property for all these enhanced properties to hold in a non-protoalgebraic logic. Since this algebraizable companion is the assertional logic of the variety $\mathsf{K}$, it is in fact regularly algebraizable. Note that a semilattice-based logic may have an algebraizable companion that is not regularly algebraizable (which indicates it is not an *assertional* companion), and it need not satisfy the properties found in this paper. An example showcasing this is the weak relevance logic $\mathcal{WR}$ suggested by Wójcicki in [28, 29] and studied with abstract algebraic logic tools in [19]: It is a non-protoalgebraic and semilattice-based logic, relatively to the variety of algebras called *R-algebras* identified in [18]. The logic $\mathcal{WR}$ has an algebraizable companion, the (usual) relevance logic $\mathcal{R}$, whose equivalent algebraic semantics is the same variety of R-algebras, but since these need not have a maximum, it is not regularly algebraizable, the defining equation being $x \to x \preccurlyeq x$ (that is, the equation $x \wedge (x \to x) \approx x \to x$). It is possible to show that the R-algebra described in pages 380–381 of [19] provides an example witnessing the failure of Theorem 5; the details are dealt with in [16]. It is reasonable to expect that similar failures will be found in the logics associated with other varieties of non-integral residuated lattices.

Another situation where the results do not apply is that where an assertional companion does exit, but is not algebraizable. An extreme and extremely simple example is the logic associated with the variety $\mathsf{D}_1$ of distributive lattices with maximum. Let **2** denote the 2-element distributive lattice with maximum, i.e., the $\{\wedge, \vee, 1\}$-reduct of the 2-element Boolean algebra, which generates $\mathsf{D}_1$. The $\{\wedge, \vee, 1\}$-fragment of classical logic is the logic defined as in (1) from **2** alone. It is not difficult to see that this logic is equal to the assertional logic $\vdash^1_{\mathsf{D}_1}$ defined as in (1) from the whole variety $\mathsf{D}_1$. It is then natural to consider also its semilattice-based companion $\vdash^{\leqslant}_{\mathsf{D}_1}$ defined as in (2) from the variety $\mathsf{D}_1$. Again, it is possible to see that the two logics coincide: $\vdash^1_{\mathsf{D}_1} = \vdash^{\leqslant}_{\mathsf{D}_1}$. Thus, the assertional companion logic is here selfextensional. A very close situation, where 1 does not appear in the similarity type, was studied in [13, 20], but its results carry over without difficulty to this expanded case. Thus, one can show that this logic is not protoalgebraic (actually, the same counterexample in Proposition 2.8 of [20] also works here, because it is a distributive lattice with maximum), and therefore it is not algebraizable either. This shows that Theorem 15 fails. One can also show that $\mathsf{Alg}^*(\vdash^1_{\mathsf{D}_1})$ is the class of distributive lattices with maximum 1 satisfying the condition that for all $a, b \in A$, if $a < b$ then there is some $c \in A$ with $a \vee c \neq 1$ and $b \vee c = 1$. This condition is dual to the so-called "Wallman disjunctive property" [1], and the class of all distributive lattices satisfying it is not even a quasi-variety, therefore in particular $\mathsf{Alg}^*(\vdash^1_{\mathsf{D}_1}) \neq \mathsf{D}_1$. These facts show that Propositions 7 and 8 fail as well[6].

Finally, another possible weakening of the basic assumptions of the paper, suggested by the observed role of the conditions highlighted in page 118, would be to require that the assertional companion is just regularly *weakly* algebraizable. In this case the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ will still be an order-isomorphism between $\mathcal{F}i_1\boldsymbol{A}$ and $\mathrm{Co}_{\mathsf{K}}\boldsymbol{A}$, which is one of the key points in several proofs. However, the other condition (that $\mathsf{Alg}(\vdash^1_{\mathsf{K}}) = \mathsf{K}$) may

[6] There are no metalogical reasons to consider this class as the real algebraic counterpart of the logic. Moreover, since the logic is selfextensional and conjunctive, Theorem 4.27 of [14] applies, and as a consequence $\mathsf{Alg}(\vdash^1_{\mathsf{D}_1}) = \mathsf{Alg}(\vdash^{\leqslant}_{\mathsf{D}_1}) = \mathsf{D}_1$; thus, under the more general framework of abstract algebraic logic based on generalized matrices developed in [14], one may conclude that the class $\mathsf{D}_1$ is indeed the right algebraic counterpart of the logic.

fail, so that the proofs of the main results are blocked. Orthologics constitute examples of this situation. Let $\mathsf{K}$ be any variety of ortholattices that includes at least a non-orthomodular one; for instance, the variety of all ortholattices. The logic $\vdash^1_{\mathsf{K}}$ is known to be regularly weakly algebraizable but non-algebraizable (see [6], Corollaries 4.7.5 and 5.6.7). Since we are dealing with bounded semilattices, it makes sense to consider the semilattice-based companion $\vdash^{\leqslant}_{\mathsf{K}}$ of $\vdash^1_{\mathsf{K}}$, but its particular properties have not been studied. Anyway, we know from the general theory that $\mathsf{Alg}(\vdash^{\leqslant}_{\mathsf{K}}) = \mathsf{K}$, while we also know that $\mathsf{Alg}^*(\vdash^1_{\mathsf{K}}) = \mathsf{Alg}(\vdash^1_{\mathsf{K}})$ is strictly smaller than $\mathsf{K}$; actually, it is not a quasivariety, because it is not closed under subalgebras (this follows from Theorem 4.7.3 of [6]). As in the previous example, Propositions 7 and 8 do not hold in this example.

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## References

[1] G. Birkhoff, Lattice Theory, 3rd. ed., vol. XXV of Colloquium Publications. American Mathematical Society, Providence, 1973. (1st. ed. 1940).

[2] W. Blok and D. Pigozzi, *Protoalgebraic logics.* Studia Logica **45** (1986), pp. 337–369.

[3] W. Blok and J. G. Raftery, *Ideals in quasivarieties of algebras.* In Models, algebras and proofs, X. Caicedo and C. H. Montenegro, Eds., vol. 203 of Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, New York, 1999, pp. 167–186.

[4] F. Bou, F. Esteva, J. M. Font, A. Gil, L. Godo, A. Torrens, A., and V. Verdú, *Logics preserving degrees of truth from varieties of residuated lattices.* Journal of Logic and Computation **19** (2009), pp. 1031–1069. DOI: `10.1093/logcom/exp030`.

[5] R. Cignoli, D. Mundici, and I. D'Ottaviano, Algebraic foundations of many-valued reasoning, vol. 7 of Trends in Logic - Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 2000.

[6] J. Czelakowski, Protoalgebraic logics, vol. 10 of Trends in Logic - Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 2001.

[7] J. Czelakowski and R. Jansana, *Weakly algebraizable logics.* The Journal of Symbolic Logic **65** (2000), pp. 641–668.

[8] J. Czelakowski and D. Pigozzi, *Fregean logics.* Annals of Pure and Applied Logic **127** (2004), pp. 17–76.

[9] J. M. Font, *An abstract algebraic logic view of some multiple-valued logics.* In Beyond two: Theory and applications of multiple-valued logic, M. Fitting and E. Orlowska, Eds., vol. 114 of Studies in Fuzziness and Soft Computing. Physica-Verlag, Heidelberg, 2003, pp. 25–58.

[10] J. M. Font, *Generalized matrices in abstract algebraic logic.* In Trends in Logic. 50 years of Studia Logica, V. F. Hendriks and J. Malinowski, Eds., vol. 21 of Trends in Logic - Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 2003, pp. 57–86.

[11] J. M. Font, *Taking degrees of truth seriously.* Studia Logica (Special issue on Truth Values, Part I) **91** (2009), pp. 383–406.

[12] J. M. Font, A. Gil, A. Torrens, and V. Verdú, *On the infinite-valued Łukasiewicz logic that preserves degrees of truth.* Archive for Mathematical Logic **45** (2006), pp. 839–868.

[13] J. M. Font, F. Guzmán, and V. Verdú, *Characterization of the reduced matrices for the* $\{\wedge, \vee\}$*-fragment of classical logic.* Bulletin of the Section of Logic **20** (1991), pp. 124–128.

[14] J. M. Font and R. Jansana, A general algebraic semantics for sentential logics, vol. 7 of Lecture Notes in Logic. Springer-Verlag, 1996. Out of print. *Second revised edition* published in 2009 by the Association for Symbolic Logic, freely available through Project Euclid at `http://projecteuclid.org/euclid.lnl/1235416965`.

[15] J. M. Font and R. Jansana, *Leibniz filters and the strong version of a protoalgebraic logic.* Archive for Mathematical Logic **40** (2001), pp. 437–465.

[16] Font, J. M., and Jansana, R. Leibniz-linked pairs of deductive systems. Studia Logica, to appear.

[17] J. M. Font, R. Jansana, and D. Pigozzi, *A survey of abstract algebraic logic.* Studia Logica (Special issue on Abstract Algebraic Logic, Part II) **74** (2003), 13–97. With an update in vol. 91 (2009), 125–130.

[18] J. M. Font and G. Rodríguez, *Note on algebraic models for relevance logic.* Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **36** (1990), pp. 535–540.

[19] J. M. Font and G. Rodríguez, *Algebraic study of two deductive systems of relevance logic.* Notre Dame Journal of Formal Logic **35** (1994), 369–397.

[20] J. M. Font and V. Verdú, *Algebraic logic for classical conjunction and disjunction.* Studia Logica (Special issue on Algebraic Logic) **50** (1991), 391–419.

[21] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, Residuated lattices: an algebraic glimpse at substructural logics, vol. 151 of Studies in Logic and the Foundations of Mathematics. Elsevier, Amsterdam, 2007.

[22] P. Hájek, Metamathematics of fuzzy logic, vol. 4 of Trends in Logic - Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 1998.

[23] R. Jansana, *Leibniz filters revisited.* Studia Logica **75** (2003), pp. 305–317.

[24] R. Jansana, *Selfextensional logics with a conjunction.* Studia Logica **84** (2006), pp. 63–104.

[25] J. Malinowski, *Modal equivalential logics.* Journal of Non-Classical Logic **3** (1986), pp. 13–35.

[26] M. Nowak, *A characterization of consequence operations preserving degrees of truth.* Bulletin of the Section of Logic **16** (1987), pp. 159–166.

[27] M. Nowak, *Logics preserving degrees of truth.* Studia Logica **49** (1990), pp. 483–499.

[28] R. Wójcicki, Lectures on propositional calculi. Ossolineum, Wrocław, 1984.

[29] R. Wójcicki, Theory of logical calculi. Basic theory of consequence operations, vol. 199 of Synthèse Library. Reidel, Dordrecht, 1988.

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