REPORTS ON MATHEMATICAL LOGIC
45 (2010), 37–95

Juan CLIMENT VIDAL and Juan SOLIVERES TUR

A 2-CATEGORICAL FRAMEWORK FOR THE SYNTAX AND SEMANTICS OF MANY-SORTED EQUATIONAL LOGIC

A b s t r a c t. For, not necessarily similar, single-sorted algebras Fujiwara defined, through the concept of family of basic mapping-formulas between single-sorted signatures, a notion of morphism which generalizes the ordinary notion of homomorphism between algebras. Subsequently he also defined an equivalence relation, the relation of conjugation, on the families of basic mapping-formulas. In this article we extend the theory of Fujiwara to the, not necessarily similar, many-sorted algebras, by defining the concept of polyderivor between many-sorted signatures under which are subsumed the standard signature morphisms, the derivors of Goguen-Thatcher-Wagner, and the basic mapping-formulas of Fujiwara.

Received 30 January 2008; revised 7 July 2008

In memory of Professor Tsuyoshi Fujiwara (1923–2006)

Primary: 03C05, 03F25, 08A68, 08A40, 18A23, 18C10, 18D05; Secondary: 03C95, 68Q65.

Keywords: Many-sorted set, many-sorted algebra, Ehresmann-Grothendieck construction, Kleisli category for a monad, many-sorted term, clone, many-sorted algebraic theory, Hall algebra, Bénabou algebra, polyderivor, transformation of polyderivors.

After this, by means of the homomorphisms between Bénabou algebras, we define the composition of polyderivors from which we get the category $\mathbf{Sig}_{\mathfrak{pd}}$, of many-sorted signatures and polyderivors. Next, by defining the notion of transformation between polyderivors, which generalizes the relation of conjugation of Fujiwara, we endow the category $\mathbf{Sig}_{\mathfrak{pd}}$ with a 2-category structure. From this we obtain a 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications, in which we prove that, for every set of sorts S, the specifications B_S, of Bénabou for S, and H_S, of Hall for S, are *equivalent*, and, after defining a pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ from $\mathbf{Spf}_{\mathfrak{pd}}$ to $\mathbf{Cat}$, we prove that, for every set of sorts S, the categories $\mathbf{Alg}(\mathrm{H}_S)$, of Hall algebras for S, and $\mathbf{Alg}(\mathrm{B}_S)$, of Bénabou algebras for S, are *equivalent*. These last equivalences were used in an earlier article to give an alternative proof of the Completeness Theorem of many-sorted Equational Logic based on the categories $\mathbf{Alg}(\mathrm{B}_S)$, which are *isomorphic* to the categories $\mathbf{BTh}_{\mathrm{f}}(S)$, of finitary many-sorted algebraic theories for S. Therefore, in this case, $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ provides a justification for the existence of such alternative proofs.

1. Introduction.

The closed sets of operations, or clones, on a set A were initially defined and investigated by P. Hall, as pointed out by Cohn in [6], pp. 127 and 132 (who attended the lectures by Professor P. Hall from 1944 to 1951), to show that the crucial mathematical properties of a Σ-algebra $\mathbf{A} = (A, (F_\sigma)_{\sigma\in\Sigma})$ do not depend on the family of primitive operations $(F_\sigma)_{\sigma\in\Sigma}$ on A, but on the system of all operations on A obtainable from $(F_\sigma)_{\sigma\in\Sigma}$ by means of the operations of composition.

The concept of an ordinary clone, axiomatized by P. Hall as a single-sorted *partial* algebra subject to satisfy some laws (see [6], p. 132) and, independently but subsequently, by M. Lazard as a compositor (see [20], p. 327), was generalized to that of a many-sorted clone by Goguen and Meseguer in [12], and axiomatically defined by them (in [12], pp. 318–319) as any many-sorted algebra (of the appropriate signature) that satisfies a definite system of many-sorted equational laws. Given its origin in P. Hall, we agree to refer to the many-sorted algebras that are models of the just mentioned laws as Hall algebras.

Hall algebras, as reflected by the defining axioms, are a species of algebraic construct in which the essential properties of the fundamental procedures of *substitution*, for the many-sorted terms in the free many-sorted algebras, and of *composition*, for the many-sorted-operations on sorted sets are embodied. And this is precisely one of the reasons why Hall algebras are a powerful and fundamental instrument to investigate many-sorted algebras. To this we add that Hall algebras are not only worth of study because of its source in the above mentioned procedures. Besides that, Hall algebras are interesting in themselves since they furnish important examples of equationally defined many-sorted algebras, and also because they have been used by Goguen and Meseguer in [12] to prove the Completeness Theorem of finitary many-sorted equational logic (that generalizes the classical Completeness Theorem of finitary equational logic of Birkhoff), providing in this way, a full algebraization of many-sorted equational deduction.

Another approximation to the study of many-sorted algebras has been proposed by Bénabou in [1], by making use of the finitary many-sorted algebraic theories, that are the generalization to the many-sorted case of the finitary single-sorted algebraic theories of Lawvere in [19]. The equational presentation of the finitary many-sorted algebraic theories of Bénabou gives rise to what we have called Bénabou algebras. And the Bénabou algebras, even having a many-sorted specification different from that of the Hall algebras, are also models of the essential properties of the clones for the many-sorted operations.

For an arbitrary, but fixed, set of sorts S, the many-sorted specifications H_S, for Hall algebras, and B_S, for Bénabou algebras, are *not isomorphic* in the category **Spf**, of many-sorted specifications and many-sorted specification morphisms, because between the corresponding categories of models: $\mathbf{Alg}(\mathrm{H}_S)$, of Hall algebras, and $\mathbf{Alg}(\mathrm{B}_S)$, of Bénabou algebras, there is not any isomorphism. However, the many-sorted specifications H_S and B_S can be considered, in some definite way, as being *equivalent*, as a consequence of the proof, in the fourth section about Hall and Bénabou algebras, of the categorical equivalence between the categories $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$.

But, the semantical equivalence of the many-sorted specifications H_S and B_S, or, for that matter, of any two many-sorted specifications, understood, by convention, as meaning the categorical equivalence of the canonically associated categories of models, can not be properly reflected at the purely syntactical level of the many-sorted specifications and many-sorted specification morphisms, i.e., can not be mathematically defined in the category **Spf**. And this is so, essentially, as a consequence of the fact of not having actually endowed **Spf** with a (non trivial) 2-category structure. Thus, if one remains anchored in the tradition of viewing **Spf** as being, simply, a category, then the only reasonable way of classifying many-sorted specifications from within the category **Spf** is through the categorical concept of isomorphism, and not, by structural impossibility, by means of some other notion of equivalence between many-sorted specifications, itself being strictly weaker than that of isomorphism (as it would be the case if instead of having a category, we had a 2-category).

Therefore, what is really needed to settle the problem of the equivalence between many-sorted specifications (i.e., the problem of determining whether or not two many-sorted specifications determine equivalent categories) is to have at one's disposal some way of comparing many-sorted specifications that goes, strictly, beyond the mere isomorphisms, in the same way as equivalences go beyond the isomorphisms when comparing categories among them. We suggest in this article that an adequate way of providing a solution to the just mentioned problem is by constructing suitable 2-categories of many-sorted signatures and many-sorted specifications, through the appropriate definitions of the 2-cells between the 1-cells. This bidimensionality, by supplying one additional degree of freedom, generates a richer world, that opens the possibility to deal not only with isomorphic but also with adjoint and equivalent many-sorted specifications. Thus carrying further the previous development which was not sufficiently complete because of its restriction to categories. The methodology we have followed in order to find a solution of the equivalence problem will now be considered.

It consists in generalizing the theory of Fujiwara in [8] and [9] into several directions. Firstly, by defining the concept of *morphism of Fujiwara*, henceforth abbreviated to *polyderivor*, from a many-sorted signature into another, which assigns to basic sorts, words and to formal operations, families of derived terms, and this in such a way that under the concept of polyderivor falls the concept of derivor, defined in [13], and that of morphism between many-sorted algebraic theories. Secondly, by endowing the category of many-sorted signatures and polyderivors with a 2-category structure, by defining the appropriate transformations between the polyderivors, that generalize the equivalences defined by Fujiwara in [9], and allow richer comparisons between many-sorted signatures than the usually considered. Lastly, by introducing the corresponding 2-category of many-sorted specifications, polyderivors between many-sorted specifications, and transformations from such a polyderivor into another.

By using the machinery introduced we prove, as a notable example, the equivalence between the many-sorted specifications of Hall and Bénabou. And from this we get, as an immediate consequence of the existence of a certain pseudo-functor from the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of many-sorted specifications, to the 2-category **Cat**, the equivalence between the categories of Hall and Bénabou algebras. This, we believe, helps to understand, from a purely categorical standpoint, how some equivalences between categories, e.g., that between clones (represented by Hall algebras) and finitary many-sorted algebraic theories (represented by Bénabou algebras), arise from more primitive syntactical equivalences between some many-sorted specifications associated to them.

Every set we consider, unless otherwise stated, will be a $\boldsymbol{\mathcal{U}}$-small set or a $\boldsymbol{\mathcal{U}}$-large set, i.e., an element or a subset, respectively, of a Grothendieck universe $\boldsymbol{\mathcal{U}}$ (as defined, e.g., in [21], p. 22), fixed once and for all. Besides, we write **Set** for the category canonically associated to $\boldsymbol{\mathcal{U}}$, and, depending on the context, we agree that **Cat** denotes either, the category of the $\boldsymbol{\mathcal{U}}$-categories (i.e., categories $\mathbf{C}$ such that the set of objects of $\mathbf{C}$ is a subset of $\boldsymbol{\mathcal{U}}$, and the hom-sets of $\mathbf{C}$ elements of $\boldsymbol{\mathcal{U}}$), and functors between $\boldsymbol{\mathcal{U}}$-categories, or the 2-category of the $\boldsymbol{\mathcal{U}}$-categories, functors between $\boldsymbol{\mathcal{U}}$-categories, and natural transformations between functors.

In all that follows we use standard concepts and constructions from category theory, see e.g., [3], [7], [15], [18], and [21]; classical universal algebra, see e.g., [6] and [14]; categorical universal algebra, see e.g., [1] and [19]; and many-sorted algebra, see e.g., [1], [2], [12], [16], and [22]. Nevertheless, we have generically adopted the following notational and terminological conventions. For a set S we write $\mathbf{T}_\star(S) = (S^\star, \curlywedge, \lambda)$ for the *free monoid on* S, where $S^\star$, the underlying set of $\mathbf{T}_\star(S)$, is $\bigcup_{n\in\mathbb{N}} S^n$, the set of all *words on* S, $\curlywedge$ the *concatenation* of words on S, and λ the *empty word* on S. For a word w on S, $|w|$ is the length of w. Moreover, $\mathbb{T}_\star = (\mathrm{T}_\star, \between, \curlywedge)$ is the standard monad in **Set** for the monoid specification, where $\mathrm{T}_\star$ is the composition of the free monoid functor $\mathbf{T}_\star\colon \mathbf{Set} \longrightarrow \mathbf{Mon}$ and the forgetful functor $\mathrm{G}_{\mathbf{Mon}}\colon \mathbf{Mon} \longrightarrow \mathbf{Set}$, and, for every set S, $\between_S\colon S \longrightarrow S^\star$ is the inclusion of S into $S^\star$, and $\curlywedge_S\colon S^{\star\star} \longrightarrow S^\star$ is the merging of strings of words to words. To simplify notation, we let (s) stand for $\between_S(s)$. Furthermore, if $\varphi\colon S \longrightarrow T$ and $\psi\colon S \longrightarrow T^\star$ are mappings, then $\varphi^\star$ is the unique homomorphism from $\mathbf{T}_\star(S)$ to $\mathbf{T}_\star(T)$ such that $\varphi^\star \circ \between_S = \between_T \circ \varphi$, $\psi^\sharp\colon S^\star \longrightarrow T^\star$ the underlying mapping of the canonical extension of ψ to the free monoid $\mathbf{T}_\star(S)$ on S and $\psi^\star$ the unique monoid homomorphism from $\mathbf{T}_\star(S)$ to $\mathbf{T}_\star(T^\star)$ such that $\psi^\star \circ \between_S = \between_{T^\star} \circ \psi$. More specific notational conventions will be included and explained in the successive sections.

2. Many-sorted sets, signatures, algebras, and generalized terms.

In this section we begin by defining the category **MSet**, of many-sorted sets, by applying the Ehresmann-Grothendieck construction (we write it EG-construction for short) (see [7], pp. 89–91, and [15], pp. (sub.) 175–177) to a contravariant functor MSet from **Set** to **Cat**. Following this we define the categories **Sig**, of many-sorted signatures, and **Alg**, of many-sorted algebras, by applying also the EG-construction to suitable contravariant functors Sig from **Set** to **Cat**, and Alg from **Sig** to **Cat**, respectively. Besides, we prove that there exists a left adjoint **T** to a "forgetful" functor G from **Alg** to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. On the basis of the functor **T** we assign to every many-sorted signature $\boldsymbol{\Sigma}$, by applying the construction of Kleisli (we write it Kl-construction for short), the category $\mathbf{Ter}(\boldsymbol{\Sigma})$, of generalized terms for $\boldsymbol{\Sigma}$, as the dual of the Kleisli category for $\mathbb{T}_{\boldsymbol{\Sigma}}$ (the standard monad derived from the adjunction between the category $\mathbf{Alg}(\boldsymbol{\Sigma})$, of $\boldsymbol{\Sigma}$-algebras, and the category $\mathbf{Set}^S$, of S-sorted set), and to every signature morphism $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ the functor $\mathbf{d}_\diamond\colon \mathbf{Ter}(\boldsymbol{\Sigma}) \longrightarrow \mathbf{Ter}(\boldsymbol{\Lambda})$. In this way we obtain a pseudo-functor Ter from **Sig** to the 2-category **Cat** (where the concept of pseudo-functor from a category to a 2-category has to be understood as in [3], pp. 289–290) which formalizes the procedure of translation for many-sorted terms.

Before stating the first proposition of this section, we agree upon calling, henceforth, for a set (of sorts) $S \in \boldsymbol{\mathcal{U}}$, the objects of the category $\mathbf{Set}^S$ (i.e., the functions $A = (A_s)_{s\in S}$ from S to $\boldsymbol{\mathcal{U}}$) *S-sorted sets*; and the morphisms of $\mathbf{Set}^S$ from an S-sorted set A to another B (i.e., the ordered triples (A, f, B), abbreviated to $f\colon A \longrightarrow B$, where $f \in \prod_{s\in S} \mathrm{Hom}(A_s, B_s)$) *$S$-sorted mappings from A to B*.

In the following proposition, that is basic for a great deal of what follows, for a mapping φ from S to T, we prove the existence of an adjunction $\coprod_\varphi \dashv \Delta_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$, as well as the existence of a contravariant functor MSet and of a pseudo-functor $\mathrm{MSet}^{\amalg}$ (related, respectively, to the right and left components of the adjunction) from **Set** to **Cat**.

Proposition 2.1. *Let $\varphi\colon S \longrightarrow T$ be a mapping. Then there are functors Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ and $\coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ such that $\coprod_\varphi \dashv \Delta_\varphi$. We write θ^φ, η^φ, and ε^φ, respectively, for the natural isomorphism, the unit, and the counit of the adjunction. Moreover, there exists a contravariant functor* MSet *from* **Set** *to* **Cat** *which sends a set S to the category $\mathrm{MSet}(S) = \mathbf{Set}^S$, and a mapping φ from S to T to the functor Δ_φ from $\mathbf{Set}^T$ to $\mathbf{Set}^S$; and a pseudo-functor* $\mathrm{MSet}^{\amalg}$ *from* **Set** *to the* 2*-category* **Cat** *given by the following data: its object mapping sends each set S to the category $\mathrm{MSet}^{\amalg}(S) = \mathbf{Set}^S$; its arrow mapping sends each mapping φ from S to T to the functor $\mathrm{MSet}^{\amalg}(\varphi) = \coprod_\varphi$ from $\mathbf{Set}^S$ to $\mathbf{Set}^T$; for every $\varphi\colon S \longrightarrow T$ and $\psi\colon T \longrightarrow U$, the natural isomorphism $\gamma^{\varphi,\psi}$ from $\coprod_\psi \circ \coprod_\varphi$ to $\coprod_{\psi\circ\varphi}$ is that which is defined, for every S-sorted set A, as the U-sorted mapping that in the u-th coordinate is $((a, s), \varphi(s)) \mapsto (a, s)$, if there exists an $s \in S$ such that $u = \psi(\varphi(s))$, and is the identity at $\varnothing$, otherwise; for every set S, the natural isomorphism ν^S from $\mathrm{Id}_{\mathbf{Set}^S}$ to $\coprod_{\mathrm{id}_S}$ is that which is defined, for every S-sorted set A and $s \in S$, as the canonical isomorphism from A_s to $A_s \times \{s\}$.*

Proof. Let Δ_φ be the functor from $\mathbf{Set}^T$ to $\mathbf{Set}^S$ defined as follows: its object mapping sends each T-sorted set A to the S-sorted set $A_\varphi = (A_{\varphi(s)})_{s\in S}$, i.e., the composite mapping $A \circ \varphi$; its arrow mapping sends each T-sorted mapping $f\colon A \longrightarrow B$ to the S-sorted mapping $f_\varphi = (f_{\varphi(s)})_{s\in S}\colon A_\varphi \longrightarrow B_\varphi$. Let $\coprod_\varphi$ be the functor from $\mathbf{Set}^S$ to $\mathbf{Set}^T$ defined as follows: its object mapping sends each S-sorted set A to the T-sorted set $\coprod_\varphi A = (\coprod_{s\in\varphi^{-1}[t]} A_s)_{t\in T}$; its arrow mapping sends each S-sorted mapping $f\colon A \longrightarrow B$ to the T-sorted mapping

$$\textstyle\coprod_\varphi f = (\coprod_{s\in\varphi^{-1}[t]} f_s)_{t\in T}\colon \coprod_\varphi A \longrightarrow \coprod_\varphi B.$$

Then $\coprod_\varphi$ is a left adjoint for Δ_φ since, for every S-sorted set A, $\coprod_\varphi A$ is $\mathrm{Lan}_\varphi A$, the left Kan extension of A along φ.

The proof that $\mathrm{MSet}^{\amalg}$ is a pseudo-functor follows easily from its definition and can therefore be left to the reader. □

By applying the EG-construction to MSet we get the category of many-sorted sets as stated in the following definition.

Definition 2.2. The category **MSet**, of *many-sorted sets* and *many-sorted mappings*, is given by $\mathbf{MSet} = \int^{\mathbf{Set}} \mathrm{MSet}$. Therefore **MSet** has as objects the pairs (S, A), where S is a set and A an S-sorted set, and as morphisms from (S, A) to (T, B) the pairs (φ, f), where $\varphi\colon S \longrightarrow T$ and $f\colon A \longrightarrow B_\varphi$.

From the definition of **MSet** it follows, immediately, that the projection functor π_{MSet} for **MSet** is a split bifibration, i.e., a split fibration and a split opfibration. Moreover, from Theorem 1, pp. 247–248, and Theorem 2, pp. 250–251, in [25], it follows that **MSet** is bicomplete.

Our next goal is to define the category **Sig**. But before doing that we agree that, for a set of sorts S in $\boldsymbol{\mathcal{U}}$, $\mathbf{Sig}(S)$ denotes the category of S-sorted signatures and S-sorted signature morphisms, i.e., the category $\mathbf{Set}^{S^\star \times S}$. Therefore an *$S$-sorted signature* is a mapping Σ from $S^\star \times S$ to $\boldsymbol{\mathcal{U}}$ which sends a pair (w, s) in $S^\star \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w, *sort* (or *coarity*) s, and *biarity* (w, s); and an *S-sorted signature morphism* from Σ to Σ' is an ordered triple (Σ, d, Σ'), written as $d\colon \Sigma \longrightarrow \Sigma'$, where $d = (d_{w,s})_{(w,s)\in S^\star\times S}$ is an element of $\prod_{(w,s)\in S^\star\times S} \mathrm{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for every $(w, s) \in S^\star \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$ ($d(\sigma)$ for short) in $\Sigma'_{w,s}$. Sometimes we will write $\sigma\colon w \longrightarrow s$ to indicate that the formal operation σ belongs to $\Sigma_{w,s}$.

Proposition 2.3. *There exists a contravariant functor* Sig *from* **Set** *to* **Cat**. *Its object mapping sends each set of sorts S to* $\mathrm{Sig}(S) = \mathbf{Sig}(S)$*; its arrow mapping sends each mapping φ from S to T to the functor* $\mathrm{Sig}(\varphi) = \Delta_{\varphi^\star\times\varphi}$ *from* $\mathbf{Sig}(T)$ *to* $\mathbf{Sig}(S)$ *which relabels T-sorted signatures into S-sorted signatures, i.e.,* $\mathrm{Sig}(\varphi)$ *assigns to a T-sorted signature Λ the S-sorted signature* $\mathrm{Sig}(\varphi)(\Lambda) = \Lambda_{\varphi^\star\times\varphi}$*, and assigns to a morphism of T-sorted signatures d from Λ to Λ' the morphism of S-sorted signatures* $\mathrm{Sig}(\varphi)(d) = d_{\varphi^\star\times\varphi}$ *from* $\Lambda_{\varphi^\star\times\varphi}$ *to* $\Lambda'_{\varphi^\star\times\varphi}$.

By applying the EG-construction to Sig we get the category of many-sorted signatures as stated in the following definition.

Definition 2.4. The category **Sig**, of *many-sorted signatures* and *many-sorted signature morphisms*, is given by $\mathbf{Sig} = \int^{\mathbf{Set}} \mathrm{Sig}$. Therefore **Sig** has as objects the pairs (S, Σ), where S is a set of sorts and Σ an S-sorted signature and as many-sorted signature morphisms from (S, Σ) to (T, Λ) the pairs (φ, d), where $\varphi\colon S \longrightarrow T$ is a morphism in **Set** while $d\colon \Sigma \longrightarrow \Lambda_{\varphi^\star \times \varphi}$ is a morphism in $\mathbf{Sig}(S)$. The composition of $(\varphi, d)\colon (S, \Sigma) \longrightarrow (T, \Lambda)$ and $(\psi, e)\colon (T, \Lambda) \longrightarrow (U, \Omega)$, denoted by $(\psi, e) \circ (\varphi, d)$, is $(\psi \circ \varphi, e_{\varphi^\star \times \varphi} \circ d)$, where

$$e_{\varphi^\star \times \varphi}\colon \Lambda_{\varphi^\star \times \varphi} \longrightarrow (\Omega_{\psi^\star \times \psi})_{\varphi^\star \times \varphi} (= \Omega_{(\psi \circ \varphi)^\star \times (\psi \circ \varphi)}).$$

Henceforth, unless otherwise stated, we will write $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, $\mathbf{\Omega}$, and $\mathbf{\Xi}$ instead of (S, Σ), (T, Λ), (U, Ω), and (X, Ξ), respectively, and $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{h}$, instead of (φ, d), (ψ, e), and (γ, h), respectively. Furthermore, to shorten terminology, we will say *signature* and *signature morphism* instead of *many-sorted signature* and *many-sorted signature morphism*, respectively.

Remark. In [16], P.J. Higgins allows the variation of S but holds Σ fixed, while, in [1], J. Bénabou follows precisely the inverse criterium.

Because **Sig** can be identified to a subcategory of the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, defined in the fifth section, we refer to that section for examples of signature morphisms.

From the definition of **Sig** it follows that the projection functor π_{Sig} from **Sig** to **Set** is a split bifibration. Moreover, from Theorem 1, pp. 247–248, and Theorem 2, pp. 250–251, in [25], it follows that **Sig** is bicomplete.

Since it will be used afterwards we introduce, for a signature $\mathbf{\Sigma}$, an S-sorted set A, an S-sorted mapping f from A to B, and a word w on S, the following notation and terminology. We write A_w for $\prod_{i \in |w|} A_{w_i}$, and f_w for the mapping $\prod_{i \in |w|} f_{w_i}$ from A_w to B_w which sends $(a_i)_{i \in |w|}$ in A_w to $(f_{w_i}(a_i))_{i \in |w|}$ in B_w. Moreover, we let $\mathrm{HO}_S(A)$ stand for the $S^\star \times S$-sorted set $(\mathrm{Hom}(A_w, A_s))_{(w,s) \in S^\star \times S}$ and we call it the $S^\star \times S$-sorted set of the *finitary operations on* A.

We proceed next to define the category **Alg** of many-sorted algebras. But before doing that we agree that, for an arbitrary but fixed signature $\mathbf{\Sigma}$, $\mathbf{Alg}(\mathbf{\Sigma})$ denotes the category of $\mathbf{\Sigma}$-algebras (and $\mathbf{\Sigma}$-homomorphisms). By a $\mathbf{\Sigma}$-*algebra* is meant a pair $\mathbf{A} = (A, F)$, where A is an S-sorted set and F a $\mathbf{\Sigma}$-*algebra structure on* A, i.e., a morphism $F = (F_{w,s})_{(w,s)\in S^{\star}\times S}$ in $\mathbf{Sig}(S)$ from Σ to $\mathrm{HO}_S(A)$ (for a pair $(w,s) \in S^{\star} \times S$ and a $\sigma \in \Sigma_{w,s}$, to simplify notation we let F_σ stand for $F_{w,s}(\sigma)$). A $\mathbf{\Sigma}$-*homomorphism* from a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ to another $\mathbf{B} = (B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, written as $f\colon \mathbf{A} \longrightarrow \mathbf{B}$, where f is an S-sorted mapping from A to B that preserves the structure in the sense that, for every (w,s) in $S^{\star} \times S$, every σ in $\Sigma_{w,s}$, and every $(a_i)_{i\in|w|}$ in A_w, it happens that $f_s(F_\sigma((a_i)_{i\in|w|})) = G_\sigma(f_w((a_i)_{i\in|w|}))$.

Proposition 2.5. *There exists a contravariant functor* Alg *from* $\mathbf{Sig}$ *to* $\mathbf{Cat}$. *Its object mapping sends each signature* $\mathbf{\Sigma}$ *to* $\mathrm{Alg}(\mathbf{\Sigma}) = \mathbf{Alg}(\mathbf{\Sigma})$, *the category of* $\mathbf{\Sigma}$-*algebras; its arrow mapping sends each signature morphism* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *to the functor* $\mathrm{Alg}(\mathbf{d}) = \mathbf{d}^{*}\colon \mathbf{Alg}(\mathbf{\Lambda}) \longrightarrow \mathbf{Alg}(\mathbf{\Sigma})$ *defined as follows: its object mapping sends each* $\mathbf{\Lambda}$-*algebra* $\mathbf{B} = (B, G)$ *to the* $\mathbf{\Sigma}$-*algebra* $\mathbf{d}^{*}(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$, *where* $G^{\mathbf{d}}$ *is the composition of the* $S^{\star} \times S$-*sorted mappings* d *from* Σ *to* $\Lambda_{\varphi^{\star}\times\varphi}$ *and* $G_{\varphi^{\star}\times\varphi}$ *from* $\Lambda_{\varphi^{\star}\times\varphi}$ *to* $\mathrm{HO}_T(B)_{\varphi^{\star}\times\varphi}$ *(for* $\sigma \in \Sigma_{w,s}$, *to shorten notation, we let* $G_{d(\sigma)}$ *stand for the value of* $G^{\mathbf{d}}$ *at* σ*); its arrow mapping sends each* $\mathbf{\Lambda}$-*homomorphism* f *from* $\mathbf{B}$ *to* $\mathbf{B}'$ *to the* $\mathbf{\Sigma}$-*homomorphism* $\mathbf{d}^{*}(f) = f_\varphi$ *from* $\mathbf{d}^{*}(\mathbf{B})$ *to* $\mathbf{d}^{*}(\mathbf{B}')$.

Proof. For every $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ it is the case that G is a morphism from Λ to $\mathrm{HO}_T(B)$. Then, by composing d and $G_{\varphi^{\star}\times\varphi}$, and taking into account that $\mathrm{HO}_T(B)_{\varphi^{\star}\times\varphi} = \mathrm{HO}_S(B_\varphi)$, we infer that $G^{\mathbf{d}} = G_{\varphi^{\star}\times\varphi} \circ d$ is a $\mathbf{\Sigma}$-algebra structure on the S-sorted set B_φ. On the other hand, for every (w,s) in $S^{\star} \times S$ and every $\sigma \in \Sigma_{w,s}$, it happens that $d(\sigma) \in \Lambda_{\varphi^{\star}(w),\varphi(s)}$. Thus, f being a $\mathbf{\Lambda}$-homomorphism from (B, G) to (B', G'), we infer that $f_{\varphi(s)} \circ G_{d(\sigma)} = G'_{d(\sigma)} \circ f_{\varphi^{\star}(w)}$. Hence, since $G^{\mathbf{d}}_\sigma = G_{d(\sigma)}$ and $G'^{\mathbf{d}}_\sigma = G'_{d(\sigma)}$, we have that $(f_\varphi)_s \circ G^{\mathbf{d}}_\sigma = G'^{\mathbf{d}}_\sigma \circ (f_\varphi)_w$. Therefore f_φ is a $\mathbf{\Sigma}$-homomorphism from $(B_\varphi, G^{\mathbf{d}})$ to $(B'_\varphi, G'^{\mathbf{d}})$.

Just because identities and composites are, obviously, preserved by $\mathbf{d}^{*}$, it follows that $\mathbf{d}^{*}$ is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$. $\square$

By applying the EG-construction to Alg we get the category of many-sorted algebras as stated in the following definition.

Definition 2.6. The category $\mathbf{Alg}$, of *many-sorted algebras* and *many-sorted algebra homomorphisms*, is given by $\mathbf{Alg} = \int^{\mathbf{Sig}} \mathrm{Alg}$. Therefore the category $\mathbf{Alg}$ has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*(\mathbf{B})$. Henceforth, to shorten terminology, we will say *algebra* and *algebra homomorphism*, or, simply, *homomorphism*, instead of *many-sorted algebra* and *many-sorted algebra homomorphism*, respectively.

Because **Alg** can be identified to a subcategory of the category $\mathbf{Alg}_{\mathbf{p}\mathfrak{d}}$, defined in the fifth section, we refer to that section for examples of homomorphisms between algebras.

From the definition of **Alg** it follows that the projection functor π_{Alg} from **Alg** to **Sig** is a fibration.

Proposition 2.7. *The category* **Alg** *is a concrete and uniquely transportable category.*

Proof. It is enough to specify a functor from **Alg** to a convenient category of sorted sets labelled by signatures. Let $\mathrm{G}_{\mathbf{MSet}}$ be the forgetful functor from **Alg** to **MSet** (that is not a fibration), and $(\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}, (\mathrm{P}_0, \mathrm{P}_1))$ the pullback of the projection functors $\pi_{\mathrm{MSet}} \colon \mathbf{MSet} \longrightarrow \mathbf{Set}$ and $\pi_{\mathrm{Sig}} \colon \mathbf{Sig} \longrightarrow \mathbf{Set}$. Then we have that the structural functors P_0 and P_1 are fibrations, and that the unique functor G from **Alg** to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ such that $\mathrm{P}_0 \circ \mathrm{G} = \mathrm{G}_{\mathbf{MSet}}$ and $\mathrm{P}_1 \circ \mathrm{G} = \pi_{\mathrm{Alg}}$ makes the category **Alg** a concrete and uniquely transportable category on the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. □

Before we prove the existence of a left adjoint $\mathbf{T}$ to the functor G from **Alg** to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, we agree on the following notation and terminology. For a signature $\mathbf{\Sigma}$ in **Sig**, the functor $\mathbf{T}_{\mathbf{\Sigma}}$ from $\mathbf{Set}^S$ to $\mathbf{Alg}(\mathbf{\Sigma})$ is the left adjoint to the forgetful functor $\mathrm{G}_{\mathbf{\Sigma}}$ from $\mathbf{Alg}(\mathbf{\Sigma})$ to $\mathbf{Set}^S$. For a signature $\mathbf{\Sigma}$ and an S-sorted set of variables X, $\mathbf{T}_{\mathbf{\Sigma}}(X)$ is the *free* (also called the *term* or *word*) $\mathbf{\Sigma}$-*algebra on* X, and η_X is the *insertion (of the generators)* X *into* $\mathrm{T}_{\mathbf{\Sigma}}(X)$, the underlying S-sorted set of $\mathbf{T}_{\mathbf{\Sigma}}(X)$. For a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and a *valuation* f *of the* S*-sorted set of variables* X *in* A, i.e., an S-sorted mapping f from X to A, we will denote by $f^\sharp$ the *canonical extension of* f *to* $\mathbf{T}_{\mathbf{\Sigma}}(X)$, i.e., the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{A}$ such that $f^\sharp \circ \eta_X = f$. For an S-sorted mapping f from X to Y, we will denote by $f^@$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Sigma}}(Y)$ such that $f^@ \circ \eta_X = \eta_Y \circ f$, i.e., the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at f. Therefore $f^@$ is also $(\eta_Y \circ f)^\sharp$. Moreover, transposing to the many-sorted case the terminology coined for the single-sorted case, we call, for $s \in S$, the elements of $\mathrm{T}_{\mathbf{\Sigma}}(X)_s$, *many-sorted terms for* $\mathbf{\Sigma}$ *of type* (X, s), henceforth abbreviated to *terms for* $\mathbf{\Sigma}$ *of type* (X, s), or, simply, to *terms of type* (X, s).

Proposition 2.8. *There exists a functor* $\mathbf{T}\colon \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ *left adjoint to the functor* $\mathrm{G}\colon \mathbf{Alg} \longrightarrow \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$.

Proof. The functor $\mathbf{T}$ from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to $\mathbf{Alg}$ given on objects (S, Σ, X) by $\mathbf{T}(S, \Sigma, X) = (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ and on arrows

$$(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$$

as

$$\mathbf{T}(\varphi, d, f) = (\mathbf{d}, f^{\mathbf{d}})\colon (\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X)) \longrightarrow (\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(Y)),$$

where $f^{\mathbf{d}} = ((\eta_Y)_\varphi \circ f)^\sharp$ is the canonical extension of the S-sorted mapping $(\eta_Y)_\varphi \circ f$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(Y)_\varphi$ to the free $\mathbf{\Sigma}$-algebra on X, is a left adjoint for G. □

For a morphism $(\varphi, d, f)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, the functor $\mathbf{T}$ acting on (φ, d, f) allows us to get the $\mathbf{\Sigma}$-homomorphism $f^{\mathbf{d}}$ from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(Y)_\varphi$. Hence, for each $s \in S$, $f^{\mathbf{d}}_s$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(Y, \varphi(s))$. In particular, the unit η^φ of $\coprod_\varphi \dashv \Delta_\varphi$ provides, for every S-sorted set X, the S-sorted mapping $\eta^\varphi_X\colon X \longrightarrow (\coprod_\varphi X)_\varphi$ and if $\mathbf{d}$ is a morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, then $(\varphi, d, \eta^\varphi_X)\colon (S, \Sigma, X) \longrightarrow (T, \Lambda, \coprod_\varphi X)$ is a morphism in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Hence the functor $\mathbf{T}$ acting on $(\varphi, d, \eta^\varphi_X)$ determines the morphism $(\mathbf{d}, \eta^{\mathbf{d}}_X)$ from $(\mathbf{\Sigma}, \mathbf{T}_{\mathbf{\Sigma}}(X))$ to $(\mathbf{\Lambda}, \mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X))$, where $\eta^{\mathbf{d}}_X = ((\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X)^\sharp$ is the $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$ that extends the S-sorted mapping $(\eta_{\coprod_\varphi X})_\varphi \circ \eta^\varphi_X$ from X to $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$. Therefore, for each $s \in S$, $\eta^{\mathbf{d}}_{X,s}$ translates terms for $\mathbf{\Sigma}$ of type (X, s) into terms for $\mathbf{\Lambda}$ of type $(\coprod_\varphi X, \varphi(s))$. The $\mathbf{\Sigma}$-homomorphisms $\eta^{\mathbf{d}}_X$, as stated in the following proposition, are in fact the components of a natural transformation, and this contributes to explain their relevance as *translators*.

Proposition 2.9. *Let* $\mathbf{d}$ *be a morphism of signatures from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$. *Then the family* $\eta^{\mathbf{d}} = (\eta^{\mathbf{d}}_X)_{X \in \boldsymbol{\mathcal{U}}}$, *which to an* S*-sorted set* X *assigns the* $\mathbf{\Sigma}$*-homomorphism* $\eta^{\mathbf{d}}_X$ *from* $\mathbf{T}_{\mathbf{\Sigma}}(X)$ *to* $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi X)_\varphi$, *is a natural transformation from* $\mathbf{T}_{\mathbf{\Sigma}}$ *to* $\mathbf{d}^* \circ \mathbf{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi$. *Therefore, for the forgetful functor* $\mathrm{G}_{\mathbf{\Sigma}}$

from $\mathbf{Alg}(\mathbf{\Sigma})$ *to* $\mathbf{Set}^S$, $\mathrm{G}_{\mathbf{\Sigma}} * \eta^{\mathbf{d}}$, *i.e., the horizontal composition of the natural transformations* $\eta^{\mathbf{d}}$ *and* $\mathrm{id}_{\mathrm{G}_{\mathbf{\Sigma}}}$, *also denoted, for simplicity, by* $\eta^{\mathbf{d}}$, *is a natural transformation from* $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$ *to* $\Delta_\varphi \circ \mathrm{T}_{\mathbf{\Lambda}} \circ \coprod_\varphi$, *taking into account that* $\mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{d}^* = \Delta_\varphi \circ \mathrm{G}_{\mathbf{\Lambda}}$ *and* $\mathrm{T}_{\mathbf{\Lambda}} = \mathrm{G}_{\mathbf{\Lambda}} \circ \mathbf{T}_{\mathbf{\Lambda}}$.

The category **Alg** is bicomplete. This follows from Theorem 1, pp. 247–248, and Theorem 2, pp. 250–251, in [25], taking into account that, for every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the functor $\mathbf{d}^*$, defined in Proposition 2.5 has a left adjoint $\mathbf{d}_*$.

Since it will be used afterwards we recall next, for a $\mathbf{\Sigma}$-algebra $\mathbf{A}$ and an S-sorted set X, the concepts of many-sorted X-ary operation on $\mathbf{A}$ and of many-sorted X-ary term operation on $\mathbf{A}$, and the procedure of realization of terms P of type (X, s) as term operations $P^{\mathbf{A}}$ on $\mathbf{A}$.

Definition 2.10. Let X be an S-sorted set, $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, s a sort in S and $P \in \mathrm{T}_{\mathbf{\Sigma}}(X)_s$ a term for $\mathbf{\Sigma}$ of type (X, s). Then the $\mathbf{\Sigma}$-algebra of the *many-sorted* X*-ary operations on* $\mathbf{A}$, $\mathbf{O}_X(\mathbf{A})$, is $\mathbf{A}^{A_X}$, i.e., the direct A_X-power of $\mathbf{A}$, where A_X is $\mathrm{Hom}(X, A)$, the (ordinary) set of the S-sorted mappings from X to A. For abbreviation we let X*-ary operations on* $\mathbf{A}$ stand for *many-sorted* X*-ary operations on* $\mathbf{A}$. The $\mathbf{\Sigma}$-algebra of the *many-sorted* X*-ary term operations on* $\mathbf{A}$, $\mathbf{Ter}_X(\mathbf{A})$, is the subalgebra of $\mathbf{O}_X(\mathbf{A})$ generated by $\mathcal{P}^A_X = (\mathcal{P}^A_{X,s})_{s\in S} = (\{\, \mathrm{pr}^A_{X,s,x} \mid x \in X_s \,\})_{s\in S}$, the subfamily of $\mathrm{O}_X(A) = A^{A_X}$, where, for every $s \in S$ and every $x \in X_s$, $\mathrm{pr}^A_{X,s,x}$ is the mapping from A_X to A_s which sends $a \in A_X$ to $a_s(x)$. For abbreviation we let X*-ary term operations on* $\mathbf{A}$ stand for *many-sorted* X*-ary term operations on* $\mathbf{A}$. We denote by $\mathrm{Tr}^{X,\mathbf{A}}$ the unique $\mathbf{\Sigma}$-homomorphism from $\mathbf{T}_{\mathbf{\Sigma}}(X)$ to $\mathbf{O}_X(\mathbf{A})$ such that $\mathrm{pr}^A_X = \mathrm{Tr}^{X,\mathbf{A}} \circ \eta_X$, where pr^A_X is the S-sorted mapping $(\mathrm{pr}^A_{X,s})_{s\in S}$ from X to $\mathrm{O}_X(A)$ whose s-th coordinate, for each $s \in S$, is $\mathrm{pr}^A_{X,s} = (\mathrm{pr}^A_{X,s,x})_{x\in X_s}$. For abbreviation, we let $P^{\mathbf{A}}$ stand for the image of P under $\mathrm{Tr}^{X,\mathbf{A}}_s$, and we call the mapping $P^{\mathbf{A}}$ from A_X to A_s, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$. For simplicity of notation, we continue to write $\mathrm{Tr}^{X,\mathbf{A}}$ for the co-restriction of the $\mathbf{\Sigma}$-homomorphism $\mathrm{Tr}^{X,\mathbf{A}}\colon \mathbf{T}_{\mathbf{\Sigma}}(X) \longrightarrow \mathbf{O}_X(\mathbf{A})$ to the subalgebra $\mathbf{Ter}_X(\mathbf{A})$ of $\mathbf{O}_X(\mathbf{A})$.

As it is well-known, for a signature $\mathbf{\Sigma}$, the conglomerate of terms for $\mathbf{\Sigma}$ is precisely the set $\bigcup_{X\in\boldsymbol{\mathcal{U}}} \bigcup_{s\in S} \mathrm{T}_{\mathbf{\Sigma}}(X)_s$, but such an amorphous set is not adequate, because of its lack of structure, for some tasks, as e.g., to explain the invariant character of the realization of terms as term operations on algebras, under change of signature (or to state a Completeness Theorem for finitary many-sorted equational logic).

However, by conveniently generalizing the concept of term for a signature $\mathbf{\Sigma}$ (as explained immediately below), it is possible to endow, in a natural way, to the corresponding generalized terms for $\mathbf{\Sigma}$, taken as *morphisms*, with a category structure, that enables us to give a categorical explanation of the existing relation between terms and algebras. To this we add that the use of the generalized terms and related notions, such as, e.g., that of generalized equation (to be defined in the following section), has allowed us, in [4], to provide a purely categorical proof of the Completeness Theorem for monads in categories of sorted sets.

Actually, we associate to every signature $\mathbf{\Sigma}$ the category $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})^{\mathrm{op}}$, of generalized terms for $\mathbf{\Sigma}$, that we denote, to shorten notation, by $\mathbf{Ter}(\mathbf{\Sigma})$, i.e., the dual of the Kleisli category for $\mathbb{T}_{\mathbf{\Sigma}} = (\mathrm{T}_{\mathbf{\Sigma}}, \eta, \mu)$, the standard monad derived from the adjunction $\mathbf{T}_{\mathbf{\Sigma}} \dashv \mathrm{G}_{\mathbf{\Sigma}}$ between $\mathbf{Alg}(\mathbf{\Sigma})$ and $\mathbf{Set}^{S}$, with $\mathrm{T}_{\mathbf{\Sigma}} = \mathrm{G}_{\mathbf{\Sigma}} \circ \mathbf{T}_{\mathbf{\Sigma}}$.

The construction of the category $\mathbf{Ter}(\mathbf{\Sigma})$ is a natural one. This is so, essentially, because it has been obtained by applying a category-theoretic construction, specifically the Kl-construction (defined by Kleisli in [18]). However, to understand more plainly how the category $\mathbf{Ter}(\mathbf{\Sigma})$ is obtained, or, more precisely, from where the morphisms of $\mathbf{Ter}(\mathbf{\Sigma})$ arise, the following observation could be helpful. For a signature $\mathbf{\Sigma}$, an S-sorted set X, and a sort $s \in S$, an ordinary term $P \in \mathrm{T}_{\Sigma}(X)_s$ for $\mathbf{\Sigma}$ of type (X, s) is, essentially, an S-sorted mapping $P\colon \delta^s \longrightarrow \mathrm{T}_{\Sigma}(X)$ where, for $s \in S$, $\delta^s = (\delta^s_t)_{t \in S}$, the Kronecker delta at s, is the S-sorted set such that $\delta^s_t = \varnothing$ if $s \neq t$ and $\delta^s_s = 1$. But the just mentioned S-sorted mappings do not constitute the morphisms of a category. Therefore, in order to get a category, it seems natural to replace the special S-sorted sets that are the Kronecker deltas, as domains of morphisms, by arbitrary S-sorted sets, thus obtaining the generalized terms, that are the categorical rendering of the ordinary terms, since they are now S-sorted mappings from an S-sorted set to the free $\mathbf{\Sigma}$-algebra on another S-sorted set, i.e., morphisms in a category $\mathbf{Ter}(\mathbf{\Sigma})$.

This category-theoretic perspective about terms, in its turn, will allow us to get a functor $\mathrm{Tr}^{\mathbf{\Sigma}}$, of realization of terms as term operations, from $\mathbf{Alg}(\mathbf{\Sigma}) \times \mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Set}$, and therefore to define (in the next section) the validation of equations, understood as ordered pairs of coterminal terms in the corresponding generalized sense, in an algebra.

Since it will be used afterwards we provide, for a signature $\mathbf{\Sigma}$, the full definition of $\mathbf{Ter}(\mathbf{\Sigma})$ and also the explicit definition of the procedure of realization of the terms for $\mathbf{\Sigma}$ as term operations on a given $\mathbf{\Sigma}$-algebra. Observe that we depart, in the definition of the category $\mathbf{Ter}(\mathbf{\Sigma})$, but only for this type of category, from the (non-Ehresmannian) tradition, in calling a category by the name of its morphisms.

Definition 2.11. Let $\mathbf{\Sigma}$ be a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then $\mathbf{Ter}(\mathbf{\Sigma})$, the category of *generalized terms for* $\mathbf{\Sigma}$, is the dual of $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$: the objects are the elements of $\boldsymbol{\mathcal{U}}^S$; the morphisms from an S-sorted set X to another Y, which we call *generalized terms for* $\mathbf{\Sigma}$ *of type* (X,Y), or, simply, *terms of type* (X,Y), are the S-sorted mappings P from Y to $\mathrm{T}_{\mathbf{\Sigma}}(X)$; the composition, denoted in $\mathbf{Ter}(\mathbf{\Sigma})$ and $\mathbf{Kl}(\mathbb{T}_{\mathbf{\Sigma}})$ by $\diamond$, is the operation which sends $P\colon X \longrightarrow Y$ and $Q\colon Y \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$ to $Q\diamond P\colon X \longrightarrow Z$ in $\mathbf{Ter}(\mathbf{\Sigma})$, where $Q \diamond P$ is $\mu_X \circ P^{@} \circ Q$, with μ_X the value at X of the multiplication μ of the monad $\mathbb{T}_{\mathbf{\Sigma}}$ and $P^{@}$ the value of the functor $\mathbf{T}_{\mathbf{\Sigma}}$ at the S-sorted mapping $P\colon Y \longrightarrow \mathrm{T}_{\mathbf{\Sigma}}(X)$; and the identities are the values of η, the unit of the monad $\mathbb{T}_{\mathbf{\Sigma}}$, at the S-sorted sets.

If $P\colon X \longrightarrow Y$ is a term for $\mathbf{\Sigma}$ of type (X,Y), then $P^{\mathbf{A}}$, the *term operation on* $\mathbf{A}$ *determined by* P, or the *term realization of* P *on* $\mathbf{A}$, is the mapping from A_X to A_Y which assigns to a valuation f of the variables X in A the valuation $f^{\sharp} \circ P$ of the variables Y in A.

We proceed next to assign to every signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ the corresponding functor $\mathbf{d}_{\diamond}$ from $\mathbf{Ter}(\mathbf{\Sigma})$ to $\mathbf{Ter}(\mathbf{\Lambda})$.

Proposition 2.12. *Let* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *be a signature morphism. Then there exists a functor* $\mathbf{d}_{\diamond}$ *from* $\mathbf{Ter}(\mathbf{\Sigma})$ *to* $\mathbf{Ter}(\mathbf{\Lambda})$. *Its object mapping assigns to each* S*-sorted set* X *the* T*-sorted set* $\mathbf{d}_{\diamond}(X) = \coprod_{\varphi} X$*; its morphism mapping assigns to each morphism* P *from* X *to* Y *in* $\mathbf{Ter}(\mathbf{\Sigma})$ *the morphism* $\mathbf{d}_{\diamond}(P) = (\theta^{\varphi})^{-1}(\eta^{\mathbf{d}}_X \circ P)$ *from* $\coprod_{\varphi} X$ *to* $\coprod_{\varphi} Y$ *in* $\mathbf{Ter}(\mathbf{\Lambda})$*, where* $\eta^{\mathbf{d}}_X$ *is the* $\mathbf{\Sigma}$*-homomorphism from* $\mathbf{T}_{\mathbf{\Sigma}}(X)$ *to* $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi}$ *that extends the* S*-sorted mapping* $(\eta_{\coprod_{\varphi} X})_{\varphi} \circ \eta^{\varphi}_X$ *from* X *to* $\mathrm{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} X)_{\varphi}$.

Corollary 2.13. *The mappings that associate, respectively, to a signature* $\mathbf{\Sigma}$ *the category* $\mathbf{Ter}(\mathbf{\Sigma})$*, and to a signature morphism* $\mathbf{d}$ *from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ *the functor* $\mathbf{d}_{\diamond}$ *from* $\mathbf{Ter}(\mathbf{\Sigma})$ *to* $\mathbf{Ter}(\mathbf{\Lambda})$*, are the components of a pseudo-functor* Ter *from* $\mathbf{Sig}$ *to the* 2*-category* $\mathbf{Cat}$.

We state now for the generalized terms the invariant character under signature change of the realization of terms as term operations in arbitrary, but fixed, algebras. We notice that from this fact we will get, in the third section, the invariance of the relation of satisfaction under signature change.

Proposition 2.14. *Let* $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ *be a signature morphism. Then, for each* $\boldsymbol{\Lambda}$*-algebra* $\mathbf{A}$ *and term* P *for* $\boldsymbol{\Sigma}$ *of type* (X,Y)*, the mappings* $P^{\mathbf{d}^*(\mathbf{A})} \circ \theta^{\varphi}_{X,A}$ *and* $\theta^{\varphi}_{Y,A} \circ \mathbf{d}_{\diamond}(P)^{\mathbf{A}}$ *from* $A_{\coprod_{\varphi} X}$ *to* $(A_{\varphi})_Y$ *are identical.*

In the following proposition we assign to a signature $\boldsymbol{\Sigma}$ and a $\boldsymbol{\Sigma}$-algebra $\mathbf{A}$ a functor $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}$ from $\mathbf{Ter}(\boldsymbol{\Sigma})$ to $\mathbf{Set}$. From the definition of the object and morphism mappings of the functors of the type $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}$, we see that they encapsulate the procedure of realization of terms. And, from the fact that they preserve identities and compositions in $\mathbf{Ter}(\boldsymbol{\Sigma})$, we conclude that they formally represent the two basic intuitions about the behavior of the just named procedure, i.e., that the realization of an identity term is an identity term operation, and that the realization of a composite of two terms is the composite of their respective realizations (in the same order).

Proposition 2.15. *Let* $\boldsymbol{\Sigma}$ *be a signature and* $\mathbf{A}$ *a* $\boldsymbol{\Sigma}$*-algebra. Then there exists a functor* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}$ *from* $\mathbf{Ter}(\boldsymbol{\Sigma})$ *to* $\mathbf{Set}$ *which sends an* S*-sorted set* X *to the set* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}(X) = A_X$ *and a term* $P\colon X \longrightarrow Y$ *to* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}(P) = P^{\mathbf{A}}\colon A_X \longrightarrow A_Y$*, the term operation on* $\mathbf{A}$ *determined by* P*.*

Since it will be used afterwards we recall that, for an S-sorted mapping f from an S-sorted set A to another B and an S-sorted set X, f_X is the value at X of the natural transformation $\mathrm{H}(\cdot, f)$ from the contravariant functor $\mathrm{H}(\cdot, A)$ to the contravariant functor $\mathrm{H}(\cdot, B)$, both from $(\mathbf{Set}^S)^{\mathrm{op}}$ to $\mathbf{Set}$.

Proposition 2.16. *Let* $\boldsymbol{\Sigma}$ *be a signature and* f *a* $\boldsymbol{\Sigma}$*-homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$*. Then there exists a natural transformation* $\mathrm{Tr}^{\boldsymbol{\Sigma},f}$ *from the functor* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}$ *to the functor* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{B}}$ *which sends an* S*-sorted set* X *to the mapping* $\mathrm{Tr}^{\boldsymbol{\Sigma},f}_X = f_X$ *from* A_X *to* B_X*. Moreover,* $\mathrm{Tr}^{\boldsymbol{\Sigma},\mathrm{id}_{\mathbf{A}}} = \mathrm{id}_{\mathrm{Tr}^{\boldsymbol{\Sigma},\mathbf{A}}}$*, and, for another* $\boldsymbol{\Sigma}$*-homomorphism* $g\colon \mathbf{B} \longrightarrow \mathbf{C}$*,* $\mathrm{Tr}^{\boldsymbol{\Sigma},g\circ f} = \mathrm{Tr}^{\boldsymbol{\Sigma},g} \circ \mathrm{Tr}^{\boldsymbol{\Sigma},f}$*.*

Therefore, the naturalness of the procedure of realization of terms as term operations on the different algebras is embodied in the natural transformations of the type $\mathrm{Tr}^{\boldsymbol{\Sigma},f}$.

Corollary 2.17. *Let* $\mathbf{\Sigma}$ *be a signature. Then the family of functors* $(\mathrm{Tr}^{\mathbf{\Sigma},\mathbf{A}})_{\mathbf{A}\in\mathbf{Alg}(\mathbf{\Sigma})}$ *together with the family of natural transformations* $(\mathrm{Tr}^{\mathbf{\Sigma},f})_{f\in\mathrm{Mor}(\mathbf{Alg}(\mathbf{\Sigma}))}$ *are the object and morphism mappings, respectively, of a functor* $\mathrm{Tr}^{\mathbf{\Sigma},(\cdot)}$ *from* $\mathbf{Alg}(\mathbf{\Sigma})$ *to* $\mathbf{Set}^{\mathbf{Ter}(\mathbf{\Sigma})}$*, or, what is equivalent, by the Schönfinkel-Curry fundamental transformation law, of a functor* $\mathrm{Tr}^{\mathbf{\Sigma}}$ *from* $\mathbf{Alg}(\mathbf{\Sigma})\times\mathbf{Ter}(\mathbf{\Sigma})$ *to* $\mathbf{Set}$ *(which formalizes the realization of terms as term operations on algebras, but taking into account the variation of the algebras through the homomorphisms between them). Moreover, for a signature morphism* $\mathbf{d}\colon\mathbf{\Sigma}\longrightarrow\mathbf{\Lambda}$*, there exists a natural isomorphism* $\theta^{\mathbf{d}}$ *from* $\mathrm{Tr}^{\mathbf{\Lambda}}\circ(\mathrm{Id}_{\mathbf{Alg}(\mathbf{\Lambda})}\times\mathbf{d}_{\diamond})$ *to* $\mathrm{Tr}^{\mathbf{\Sigma}}\circ(\mathbf{d}^{*}\times\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})})$*, where, for every* $(\mathbf{A},X)\in\mathbf{Alg}(\mathbf{\Lambda})\times\mathbf{Ter}(\mathbf{\Sigma})$*,* $\theta^{\mathbf{d}}_{\mathbf{A},X}$ *is precisely* $\theta^{\varphi}_{X,A}$*, i.e., the natural isomorphism of the adjunction* $\coprod_{\varphi}\dashv\Delta_{\varphi}$*, and this fact shows the invariant character of the procedure of realization of terms under signature change.*

3. Many-sorted specifications and morphisms.

In this section we begin by defining, for a signature $\mathbf{\Sigma}$, the concept of $\mathbf{\Sigma}$-equation, but for the generalized terms defined in the preceding section, the binary relation of satisfaction between $\mathbf{\Sigma}$-algebras and $\mathbf{\Sigma}$-equations, and the semantical consequence operators $\mathrm{Cn}_{\mathbf{\Sigma}}$. Then, after extending the translation of generalized terms to generalized equations, we prove the corresponding satisfaction condition.

After this we define, for the generalized terms, the concepts of many-sorted specification and of many-sorted specification morphism, from which we get the corresponding category, denoted by **Spf**. We notice that, conveniently generalized, the many-sorted specification morphisms will be used, together with some other concepts, in the last section of this article, to prove the equivalence between the many-sorted specifications of Hall and Bénabou.

We now define the equations over a given signature through the morphisms of the category of terms for the signature, what it means for an equation to be valid in an algebra, and the consequence operator on the many-sorted set of the equations.

Definition 3.1. Let $\mathbf{\Sigma}$ be a signature, X, Y two S-sorted sets and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra. Then a $\mathbf{\Sigma}$-*equation of type* (X,Y) is a pair $(P,Q)\colon X\longrightarrow Y$

of parallel morphisms in $\mathbf{Ter}(\boldsymbol{\Sigma})$ (hence $(P,Q)\in \mathrm{Hom}(Y,\mathrm{T}_{\boldsymbol{\Sigma}}(X))^2$), and a $\boldsymbol{\Sigma}$-*equation* is a $\boldsymbol{\Sigma}$-equation of type (X,Y) for some S-sorted sets X, Y. We will denote by $\mathrm{Eq}(\boldsymbol{\Sigma})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of all $\boldsymbol{\Sigma}$-equations. A $\boldsymbol{\Sigma}$-equation $(P,Q)\colon X \longrightarrow Y$ is *valid* in $\mathbf{A}$, denoted by $\mathbf{A} \models^{\boldsymbol{\Sigma}}_{X,Y} (P,Q)$, if and only if, for every $s\in S$ and $y\in Y_s$, $\mathbf{A}\models^{\boldsymbol{\Sigma}}_{X,s} (P_s(y),Q_s(y))$, i.e., $(P_s(y))^{\mathbf{A}} = (Q_s(y))^{\mathbf{A}}$. We extend this satisfaction relation between $\boldsymbol{\Sigma}$-algebras $\mathbf{A}$ and $\boldsymbol{\Sigma}$-equations $(P,Q)\colon X\longrightarrow Y$ to $\boldsymbol{\Sigma}$-algebras $\mathbf{A}$ and families $\mathcal{E}\subseteq \mathrm{Eq}(\boldsymbol{\Sigma})$ by agreeing that $\mathbf{A}\models^{\boldsymbol{\Sigma}}\mathcal{E}$ if and only if, for every $X,Y\in\boldsymbol{\mathcal{U}}^S$ and $(P,Q)\in\mathcal{E}_{X,Y}$, we have that $\mathbf{A}\models^{\boldsymbol{\Sigma}}_{X,Y}(P,Q)$. We will denote by $\mathrm{Cn}_{\boldsymbol{\Sigma}}$ the endomapping of $\mathrm{Sub}(\mathrm{Eq}(\boldsymbol{\Sigma}))$, the set of all sub-$(\boldsymbol{\mathcal{U}}^S)^2$-sorted sets of $\mathrm{Eq}(\boldsymbol{\Sigma})$, which sends $\mathcal{E}\subseteq\mathrm{Eq}(\boldsymbol{\Sigma})$ to $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})$, where, for every $X,Y\in\boldsymbol{\mathcal{U}}^S$ and $(P,Q)\in\mathrm{Eq}(\boldsymbol{\Sigma})_{X,Y}$, $(P,Q)\in\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})_{X,Y}$ if and only if, for every $\boldsymbol{\Sigma}$-algebra $\mathbf{A}$, if $\mathbf{A}\models^{\boldsymbol{\Sigma}}\mathcal{E}$, then $\mathbf{A}\models^{\boldsymbol{\Sigma}}_{X,Y}(P,Q)$. We call $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})$ the $(\boldsymbol{\mathcal{U}}^S)^2$-sorted set of the *semantical consequences* of $\mathcal{E}$.

If we keep in mind that for a term $P\colon X\longrightarrow Y$ for $\boldsymbol{\Sigma}$ of type (X,Y), $P^{\mathbf{A}}$, the term operation on $\mathbf{A}$ determined by P, is the mapping from A_X to A_Y which assigns to an S-sorted mapping $f\colon X\longrightarrow A$ precisely the S-sorted mapping $f^\sharp\circ P\colon Y\longrightarrow A$, then we get the following convenient characterization of the relation $\mathbf{A}\models^{\boldsymbol{\Sigma}}_{X,Y}(P,Q)$:

$$\mathbf{A}\models^{\boldsymbol{\Sigma}}_{X,Y}(P,Q) \text{ if and only if } P^{\mathbf{A}} = Q^{\mathbf{A}}.$$

Besides, by the Completeness Theorem in [4], for $\mathcal{E}\subseteq\mathrm{Eq}(\boldsymbol{\Sigma})$, we have that $\mathrm{Cn}_{\boldsymbol{\Sigma}}(\mathcal{E})$ is precisely $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\boldsymbol{\Sigma})}(\mathcal{E})$, i.e., the smallest Π-compatible congruence on $\mathbf{Ter}(\boldsymbol{\Sigma})$ that contains $\mathcal{E}$, where the superscript Π in the operator $\mathrm{Cg}^{\Pi}_{\mathbf{Ter}(\boldsymbol{\Sigma})}$ abbreviates "product". Therefore the operator $\mathrm{Cn}_{\boldsymbol{\Sigma}}$ on $\mathrm{Eq}(\boldsymbol{\Sigma})$ is a closure operator.

Next we formalize the procedure of translation, by means of a signature morphism, of equations for a signature into equations for another signature in the following definition.

Definition 3.2. Let $\mathbf{d}$ be a signature morphism from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$. Then $\mathbf{d}$ induces a many-sorted mapping

$$((\textstyle\coprod_{\varphi})^2,\mathbf{d}^2_{\diamond})\colon ((\boldsymbol{\mathcal{U}}^S)^2,\mathrm{Eq}(\boldsymbol{\Sigma}))\longrightarrow((\boldsymbol{\mathcal{U}}^T)^2,\mathrm{Eq}(\boldsymbol{\Lambda})),$$

the so-called *translation of equations for* $\boldsymbol{\Sigma}$ *into equations for* $\boldsymbol{\Lambda}$ *relative to* $\mathbf{d}$, where $(\coprod_{\varphi})^2$ is the mapping from $(\boldsymbol{\mathcal{U}}^S)^2$ to $(\boldsymbol{\mathcal{U}}^T)^2$ which sends a pair of

S-sorted sets (X, Y) to the pair $(\coprod_{\varphi} X, \coprod_{\varphi} Y)$ of T-sorted sets, and $\mathbf{d}_{\diamond}^{2}$ the $(\boldsymbol{\mathcal{U}}^{S})^{2}$-sorted mapping which to a $\mathbf{\Sigma}$-equation (P, Q) of type (X, Y) assigns the $\mathbf{\Lambda}$-equation $(\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q))$ of type $(\coprod_{\varphi} X, \coprod_{\varphi} Y)$.

In the following lemma we prove the invariance of the relation of satisfaction under signature change, also known, for those following the terminology coined by Goguen and Burstall in [11], p. 229, as the *satisfaction condition*.

Lemma 3.3. *Let* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *be a signature morphism,* (P, Q) *a* $\mathbf{\Sigma}$-*equation of type* (X, Y) *and* $\mathbf{A}$ *a* $\mathbf{\Lambda}$-*algebra. Then we have that*

$$\mathbf{d}^{*}(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q) \text{ if and only if } \mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_{\varphi} X, \coprod_{\varphi} Y} (\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q)).$$

Proof. The condition $\mathbf{d}^{*}(\mathbf{A}) \models^{\mathbf{\Sigma}}_{X,Y} (P, Q)$ is equivalent to $P^{\mathbf{d}^{*}(\mathbf{A})} = Q^{\mathbf{d}^{*}(\mathbf{A})}$. But this condition is equivalent to $\mathbf{d}_{\diamond}(P)^{\mathbf{A}} = \mathbf{d}_{\diamond}(Q)^{\mathbf{A}}$. Therefore it is also equivalent to the condition $\mathbf{A} \models^{\mathbf{\Lambda}}_{\coprod_{\varphi} X, \coprod_{\varphi} Y} (\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q))$. $\square$

Following this we proceed to define the concept of many-sorted specification (also known as many-sorted pretheory) and that of many-sorted specification morphism.

Definition 3.4. A *many-sorted specification* (or *many-sorted pretheory*) is a pair $(\mathbf{\Sigma}, \mathcal{E})$, where $\mathbf{\Sigma}$ is a signature while $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$. A *many-sorted specification morphism* (or *many-sorted pretheory morphism*) from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$ is a signature morphism $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ such that $\mathbf{d}_{\diamond}^{2}[\mathcal{E}] \subseteq \mathrm{Cn}_{\mathbf{\Lambda}}(\mathcal{H})$. For abbreviation we let *specification* and *specification morphism* stand for *many-sorted specification* and *many-sorted specification morphism*, respectively. Besides, if in a specification $(\mathbf{\Sigma}, \mathcal{E})$ the set $\mathcal{E}$ of equations is closed, i.e., $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E}) = \mathcal{E}$, then we call $(\mathbf{\Sigma}, \mathcal{E})$ a *(many-sorted) theory.* To shorten notation, we write, sometimes, $\overline{\mathcal{E}}$ instead of $\mathrm{Cn}_{\mathbf{\Sigma}}(\mathcal{E})$.

Proposition 3.5. *The specifications and the specification morphisms determine a category denoted as* **Spf**.

What we want now is to lift the contravariant functor Alg from **Sig** to **Cat** to **Spf**, by assigning, in particular, to a specification $(\mathbf{\Sigma}, \mathcal{E})$ the category $\mathbf{Alg}(\mathbf{\Sigma}, \mathcal{E})$ of its models.

Proposition 3.6. *There exists a contravariant functor* $\mathrm{Alg}^{\mathrm{sp}}$ *from* **Spf** *to* **Cat***; its object mapping sends each specification* $(\mathbf{\Sigma}, \mathcal{E})$ *to the category* $\mathrm{Alg}^{\mathrm{sp}}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Alg}(\mathbf{\Sigma}, \mathcal{E})$ *of its models, i.e., the full subcategory of* $\mathbf{Alg}(\mathbf{\Sigma})$ *determined by those* $\mathbf{\Sigma}$*-algebras which satisfy all the equations in* $\mathcal{E}$*; its arrow mapping sends each specification morphism* $\mathbf{d}$ *from* $(\mathbf{\Sigma}, \mathcal{E})$ *to* $(\mathbf{\Lambda}, \mathcal{H})$ *to the functor* $\mathrm{Alg}^{\mathrm{sp}}(\mathbf{d}) = \mathbf{d}^*$ *from* $\mathbf{Alg}(\mathbf{\Lambda}, \mathcal{H})$ *to* $\mathbf{Alg}(\mathbf{\Sigma}, \mathcal{E})$*, obtained from the functor* $\mathbf{d}^*$ *from* $\mathbf{Alg}(\mathbf{\Lambda})$ *to* $\mathbf{Alg}(\mathbf{\Sigma})$ *by bi-restriction.*

Proof. Let $\mathbf{B}$ be a $\mathbf{\Lambda}$-algebra such that $\mathbf{B} \models^{\mathbf{\Lambda}} \mathcal{H}$. Then $\mathbf{B} \models^{\mathbf{\Lambda}} \mathrm{Cn}_{\mathbf{\Lambda}}(\mathcal{H})$, therefore $\mathbf{B} \models^{\mathbf{\Lambda}} \mathbf{d}^2_{\diamond}[\mathcal{E}]$ hence, by Lemma 3.3, $\mathbf{d}^*(\mathbf{B}) \models^{\mathbf{\Sigma}} \mathcal{E}$. □

By applying the EG-construction to $\mathrm{Alg}^{\mathrm{sp}}$ we get the category $\int^{\mathbf{Spf}} \mathrm{Alg}^{\mathrm{sp}}$ denoted by $\mathbf{Alg}^{\mathrm{sp}}$. The category $\mathbf{Alg}^{\mathrm{sp}}$ has as objects the pairs $((\mathbf{\Sigma}, \mathcal{E}), \mathbf{A})$, where $(\mathbf{\Sigma}, \mathcal{E})$ is a specification and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra which is a model of $\mathcal{E}$, and as morphisms from $((\mathbf{\Sigma}, \mathcal{E}), \mathbf{A})$ to $((\mathbf{\Lambda}, \mathcal{H}), \mathbf{B})$, the pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a specification morphism from $(\mathbf{\Sigma}, \mathcal{E})$ to $(\mathbf{\Lambda}, \mathcal{H})$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*(\mathbf{B})$.

On the other hand, taking care of the Completeness Theorem in [4], every family of equations $\mathcal{E} \subseteq \mathrm{Eq}(\mathbf{\Sigma})$ determines a congruence on the category $\mathbf{Ter}(\mathbf{\Sigma})$, hence a quotient category $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$. Besides, this procedure can be completed, as stated in the following proposition, to a pseudo-functor $\mathrm{Ter}^{\mathrm{sp}}$ from **Spf** to **Cat**, and the restriction of $\mathrm{Ter}^{\mathrm{sp}}$ to **Sig** is precisely the pseudo-functor Ter.

Proposition 3.7. *There exists a pseudo-functor* $\mathrm{Ter}^{\mathrm{sp}}$ *from* **Spf** *to* **Cat** *defined as follows*

1. $\mathrm{Ter}^{\mathrm{sp}}$ *sends a specification* $(\mathbf{\Sigma}, \mathcal{E})$ *to the category* $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$*, where* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ *is the quotient category* $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$*.*
2. $\mathrm{Ter}^{\mathrm{sp}}$ *sends a specification morphism* $\mathbf{d}$ *from* $(\mathbf{\Sigma}, \mathcal{E})$ *to* $(\mathbf{\Lambda}, \mathcal{H})$ *to the functor* $\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})$*, also occasionally denoted by* $\mathbf{d}_{\diamond}$*, from* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$ *to* $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H}) = \mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$*, which assigns to a morphism* $[P]_{\overline{\mathcal{E}}} \colon X \longrightarrow Y$ *in* $\mathbf{Ter}(\mathbf{\Sigma}, \mathcal{E})$ *the morphism*

$$\mathrm{Ter}^{\mathrm{sp}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}_{\diamond}(P)]_{\overline{\mathcal{H}}} \colon \coprod_{\varphi} X \longrightarrow \coprod_{\varphi} Y$$

   in $\mathbf{Ter}(\mathbf{\Lambda}, \mathcal{H})$*.*

To finalize this section we notice that, for each specification $(\boldsymbol{\Sigma}, \mathcal{E})$, there exists a functor $\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})}$ from $\mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E}) \times \mathbf{Ter}(\boldsymbol{\Sigma}, \mathcal{E})$ to $\mathbf{Set}$. On the other hand, for each specification morphism $\mathbf{d}\colon (\boldsymbol{\Sigma}, \mathcal{E}) \longrightarrow (\boldsymbol{\Lambda}, \mathcal{H})$, there exists a natural isomorphism $\theta^{\mathrm{sp},\mathbf{d}}$ from $\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Lambda},\mathcal{H})} \circ (\mathrm{Id}_{\mathbf{Alg}(\boldsymbol{\Lambda},\mathcal{H})} \times \mathbf{d}_{\diamond})$ to $\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})} \circ (\mathbf{d}^{*} \times \mathrm{Id}_{\mathbf{Ter}(\boldsymbol{\Sigma},\mathcal{E})})$. Moreover, from the category $\mathbf{Spf}^{\mathrm{op}} \times \mathbf{Spf}$ to the 2-category $\mathbf{Cat}$ there is a pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}(\cdot)$ and the pair $(\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}})$, where $\mathrm{Tr}^{\mathrm{sp}}$ is $(\mathrm{Tr}^{\mathrm{sp},(\boldsymbol{\Sigma},\mathcal{E})})_{(\boldsymbol{\Sigma},\mathcal{E}) \in \mathbf{Spf}}$ and θ^{sp} is $(\theta^{\mathrm{sp},\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Spf})}$ is a pseudo-extranatural transformation from $\mathrm{Alg}^{\mathrm{sp}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}(\cdot)$ to $\mathrm{K}_{\mathbf{Set}}$.

4. Hall and Bénabou algebras.

The concept of many-sorted clone, that generalizes both that of single-sorted clone axiomatized by P. Hall as a single-sorted partial algebra subject to satisfy some laws (see e.g., [6], pp. 127 and 132) and by M. Lazard as a compositor (see [20], p. 327), and that of Boolean clone investigated, among others, by E. Post (see e.g., [23] and [24]), was axiomatically defined by Goguen and Meseguer (in [12], pp. 318–319) as any many-sorted algebra (of the appropriate signature) that satisfies a system of many-sorted equational laws. The corresponding categories of many-sorted algebras, called categories of Hall algebras, are the algebraic rendering of the categories of finitary many-sorted algebraic theories of Bénabou, i.e., both types of categories, as it is well-known, are equivalent.

Our main aim in this section is to define, for each set of sorts, through a system of many-sorted equational laws the, so-called, Bénabou algebras as those many-sorted algebras that satisfy them, and to state that the corresponding category of Bénabou algebras, for a given set of sorts, is *isomorphic* to the category of finitary many-sorted algebraic theories of Bénabou, for the same set of sorts. Besides, we state that the Hall and Bénabou algebras, even having different specification, are models of the essential properties of the clones for the many-sorted operations, i.e., that the respective categories of Hall and Bénabou algebras are *equivalent*.

The homomorphisms between Bénabou algebras, as we will show later on (in the fifth section), are also adequate to define the composition of the morphisms of Fujiwara between signatures, that are a strict generalization of both the standard morphisms and the derivors (defined in the fifth section) between signatures. Informally, we can say that the Bénabou algebras are to the composition of morphisms of Fujiwara between signatures as the Hall algebras are to the composition of derivors between signatures.

Before we define the Hall algebras as the models of a specification, we agree that for a set of sorts U, a word $x \in U^\star$ and a standard U-sorted set of variables $V^U = (\{ v_n^u \mid n \in \mathbb{N} \})_{u \in U}$, $\downarrow x$ is the U-sorted subset of V^U defined, for every $u \in U$ as $(\downarrow x)_u = \{ v_i^u \mid i \in x^{-1}[u] \}$, this will apply, in particular, when $U = S^\star \times S$ or $U = S^\star \times S^\star$.

Definition 4.1. Let S be a set of sorts and V^{H_S} the $S^\star \times S$-sorted set of variables $(V_{w,s})_{(w,s) \in S^\star \times S}$ where $V_{w,s} = \{ v_n^{w,s} \mid n \in \mathbb{N} \}$, for every $(w,s) \in S^\star \times S$. A *Hall algebra for S* is a $\mathrm{H}_S = (S^\star \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S})$-algebra, where Σ^{H_S} is the $S^\star \times S$-sorted signature, i.e., the $(S^\star \times S)^\star \times (S^\star \times S)$-sorted set, defined as follows:

HS_1. For every $w \in S^\star$ and $i \in |w|$, $\pi_i^w \colon \lambda \longrightarrow (w, w_i)$, where $|w|$ is the *length* of the word w and λ the *empty word* in $(S^\star \times S)^\star$.

HS_2. For every u, $w \in S^\star$ and $s \in S$,

$$\xi_{u,w,s} \colon ((w,s),(u,w_0),\dots,(u,w_{|w|-1})) \longrightarrow (u,s);$$

while $\mathcal{E}^{\mathrm{H}_S}$ is the many-sorted subset of $\mathrm{Eq}(\Sigma^{\mathrm{H}_S})$ defined as follows:

H_1. *Projection.* For every u, $w \in S^\star$ and $i \in |w|$, the equation

$$\xi_{u,w,w_i}(\pi_i^w, v_0^{u,w_0}, \dots, v_{|w|-1}^{u,w_{|w|-1}}) = v_i^{u,w_i}$$

of type $(((u,w_0),\dots,(u,w_{|w|-1})),(u,w_i))$.

H_2. *Identity.* For every $u \in S^\star$ and $j \in |u|$, the equation

$$\xi_{u,u,u_j}(v_j^{u,u_j}, \pi_0^u, \dots, \pi_{|u|-1}^u) = v_j^{u,u_j}$$

of type $(((u,u_j)),(u,u_j))$.

H_3. *Associativity.* For every u, v, $w \in S^\star$ and $s \in S$, the equation

$$\begin{aligned}\xi_{u,v,s}(\xi_{v,w,s}(v_0^{w,s}, v_1^{v,w_0}, \dots, v_{|w|}^{v,w_{|w|-1}}), v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|}^{u,v_{|v|-1}}) = \\ \xi_{u,w,s}(v_0^{w,s}, \xi_{u,v,w_0}(v_1^{v,w_0}, v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|}^{u,v_{|v|-1}}), \dots, \\ \xi_{u,v,w_{|w|-1}}(v_{|w|}^{v,w_{|w|-1}}, v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|}^{u,v_{|v|-1}}))\end{aligned}$$

of type $(((w,s),(v,w_0),\dots,(v,w_{|w|-1}),(u,v_0),\dots,(u,v_{|v|-1})),(u,s))$.

We call the formal constants π_i^w *projections*, and the formal operations $\xi_{u,w,s}$ *substitution operators*. Furthermore, we denote by $\mathbf{Alg}(\mathrm{H}_S)$ the category of Hall algebras for S and homomorphisms between Hall algebras. Since $\mathbf{Alg}(\mathrm{H}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{H}_S}$ from $\mathbf{Alg}(\mathrm{H}_S)$ to $\mathbf{Set}^{S^\star \times S}$ has a left adjoint $\mathbf{T}_{\mathrm{H}_S}$ which assigns to an $S^\star \times S$-sorted set Σ the corresponding free Hall algebra $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$.

For every S-sorted set A, $\mathrm{HO}_S(A) = (\mathrm{Hom}(A_w, A_s))_{(w,s)\in S^\star \times S}$, the $S^\star \times S$-sorted set of operation for A, is naturally endowed with a Hall algebra structure if we realize the projections as the true projections and the substitution operators as the generalized composition of mappings (see [5], Proposition 2.8, pp. 134–135, for more details). Let us denote by $\mathbf{HO}_S(A)$ the corresponding Σ^{H_S}-algebra.

Remark. The closed sets of the Hall algebra $\mathbf{HO}_S(A)$ for (S, A) are precisely the clones of (many-sorted) operations on the S-sorted set A.

For every S-sorted signature Σ, $\mathrm{HT}_S(\Sigma) = (\mathrm{T}_\Sigma(\downarrow w)_s)_{(w,s)\in S^\star \times S}$ is also endowed with a Hall algebra structure that formalizes the concept of substitution (see [5], Proposition 2.9, pp. 135–136, for more details). Let us denote by $\mathbf{HT}_S(\Sigma)$ the corresponding Σ^{H_S}-algebra.

Our next goal is to state that, for every $S^\star \times S$-sorted set Σ, $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, the free Hall algebra on Σ, is isomorphic to $\mathbf{HT}_S(\Sigma)$. We remark that the existence of this isomorphism is interesting because it enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$, and, on the other hand, to give, in the fifth section, an alternative, but equivalent, definition of the concept of derivor (defined by Goguen, Thatcher and Wagner in [13], p. 86) between signatures.

Proposition 4.2. *Let Σ be an S-sorted signature, i.e., an $S^\star \times S$-sorted set. Then the Hall algebra $\mathbf{HT}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_S}(\Sigma)$.*

Proof. See [5]. □

For a set of sorts S, the fundamental objects in the approach to the many-sorted completeness theorem in [12], i.e., the Hall algebras for S, have an alternative, but equivalent, description in terms of, what we call, Bénabou algebras for S, that, as we will show below are more strongly linked to the finitary many-sorted theories algebraic theories than are the

Hall algebras. Besides, the Bénabou algebras will be shown to be more adequate in order to work with morphisms between signatures more general than the standard ones. Actually there exists an equivalence between the category $\mathbf{Alg}(\mathrm{H}_S)$, of Hall algebras for S, and the category $\mathbf{Alg}(\mathrm{B}_S)$, of Bénabou algebras for S, i.e., the category defined as follows.

Definition 4.3. Let S be a set of sorts and V^{B_S} the $(S^\star)^2$-sorted set of variables $(V_{u,w})_{(u,w)\in(S^\star)^2}$ where $V_{u,w} = \{\, v_n^{u,w} \mid n \in \mathbb{N} \,\}$, for every $(u,w) \in (S^\star)^2$. A *Bénabou algebra for* S is a $\mathrm{B}_S = ((S^\star)^2, \Sigma^{\mathrm{B}_S}, \mathcal{E}^{\mathrm{B}_S})$-algebra, where Σ^{B_S} is the $(S^\star)^2$-sorted signature defined as follows:

BS_1. For the empty word $\lambda \in S^\star$, every $w \in S^\star$ and $i \in |w|$, where $|w|$ is the domain of the word w, the formal operation of *projection*: $\pi_i^w\colon \lambda \longrightarrow (w, (w_i))$.

BS_2. For every u, $w \in S^\star$, the formal operation of *finite tupling*:

$$\langle\,\rangle_{u,w}\colon ((u,(w_0)),\ldots,(u,(w_{|w|-1}))) \longrightarrow (u,w).$$

BS_3. For every u, x, $w \in S^\star$, the formal operation of *substitution*:

$$\circ_{u,x,w}\colon ((u,x),(x,w)) \longrightarrow (u,w);$$

while $\mathcal{E}^{\mathrm{B}_S}$ is the many-sorted subset of $\mathrm{Eq}(\Sigma^{\mathrm{B}_S})$ defined as follows:

B_1. For every u, $w \in S^\star$ and $i \in |w|$, the equation:

$$\pi_i^w \circ_{u,w,(w_i)} \langle v_0^{u,(w_0)},\ldots,v_{|w|-1}^{u,(w_{|w|-1})}\rangle_{u,w} = v_i^{u,(w_i)},$$

of type $(((u,(w_0)),\ldots,(u,(w_{|w|-1}))),(u,(w_i)))$.

B_2. For every u, $w \in S^\star$, the equation:

$$v_0^{u,w} \circ_{u,u,w} \langle \pi_0^u,\ldots,\pi_{|u|-1}^u\rangle_{u,u} = v_0^{u,w},$$

of type $(((u,w)),(u,w))$.

B_3. For every u, $w \in S^\star$, the equation:

$$\langle \pi_0^w \circ_{u,w,w_0} v_0^{u,w},\ldots,\pi_{|w|-1}^w \circ_{u,w,w_{|w|-1}} v_0^{u,w}\rangle_{u,w} = v_0^{u,w},$$

of type $(((u,w)),(u,w))$.

B_4. For every $w \in S^\star$, the equation:

$$\langle \pi_0^w \rangle_{w,(w_0)} = \pi_0^w,$$

of type $(((w,(w_0))),(w,(w_0)))$.

B_5. For every u, x, w, $y \in S^\star$, the equation:

$$v_0^{w,y} \circ_{u,w,y} (v_1^{x,w} \circ_{u,x,w} v_2^{u,x}) = (v_0^{w,y} \circ_{x,w,y} v_1^{x,w}) \circ_{u,x,y} v_2^{u,x},$$

of type $(((w,y),(x,w),(u,x)),(u,y))$,

where $v_n^{u,w}$ is the n-th variable of type (u,w), $Q \circ_{u,x,w} P$ is $\circ_{u,x,w}(P,Q)$, and $\langle P_0,\dots,P_{|w|-1}\rangle_{u,w}$ is $\langle\,\rangle_{u,w}(P_0,\dots,P_{|w|-1})$.

Since $\mathbf{Alg}(\mathrm{B}_S)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{B}_S}$ from $\mathbf{Alg}(\mathrm{B}_S)$ to $\mathbf{Set}^{S^\star\times S^\star}$ has a left adjoint $\mathbf{T}_{\mathrm{B}_S}$ which assigns to an $S^\star\times S^\star$-sorted set the corresponding free Bénabou algebra.

For every S-sorted set A, $\mathrm{BO}_S(A) = (\mathrm{Hom}(A_w, A_u))_{(w,u)\in S^\star\times S^\star}$ is endowed with a Bénabou algebra structure (see [5], Proposition 3.2, pp. 149–150, for more details). Let us denote by $\mathbf{BO}_S(A)$ the corresponding Σ^{B_S}-algebra.

For every S-sorted signature Σ, $\mathrm{BT}_S(\Sigma) = (\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow w)_u)_{(w,u)\in S^\star\times S^\star}$, that is naturally isomorphic to $(\mathrm{Hom}(\downarrow u, \mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow w))_{(w,u)\in S^\star\times S^\star}$, is endowed with a Bénabou algebra structure (see [5], Proposition 3.3, p. 150, for more details). Let us denote by $\mathbf{BT}_S(\Sigma)$ the corresponding Σ^{B_S}-algebra.

Next, after defining the category $\mathbf{BTh}_{\mathrm{f}}(S)$, of finitary many-sorted algebraic theories of Bénabou (defined for the first time in [1]), which is a strict generalization of the category $\mathbf{LTh}_{\mathrm{f}}(S)$, of finitary single-sorted algebraic theories of Lawvere, we state that there exists an isomorphism between $\mathbf{BTh}_{\mathrm{f}}(S)$ and $\mathbf{Alg}(\mathrm{B}_S)$.

Definition 4.4. We denote by $\mathbf{BTh}_{\mathrm{f}}(S)$ the category of finitary many-sorted algebraic theories of Bénabou. By a *finitary many-sorted algebraic theory of Bénabou* is meant a pair $\boldsymbol{\mathcal{B}} = (\mathbf{B}, p^{\boldsymbol{\mathcal{B}}})$, where $\mathbf{B}$ is a category that has as objects the words on S, and $p^{\boldsymbol{\mathcal{B}}}$ a family $(p^{\boldsymbol{\mathcal{B}},w})_{w\in S^\star}$ such that, for every word $w \in S^\star$, $p^{\boldsymbol{\mathcal{B}},w}$ is a family $(p_i^{\boldsymbol{\mathcal{B}},w}\colon w \longrightarrow (w_i))_{i\in|w|}$ of morphisms in $\mathbf{B}$, the *projections* for w (where (w_i) is the word of length 1 on S whose only letter is w_i) such that $(w, p^{\boldsymbol{\mathcal{B}},w})$ is a product in $\mathbf{B}$ of the family of words $((w_i))_{i\in|w|}$. A morphism $F\colon \boldsymbol{\mathcal{B}} = (\mathbf{B}, p^{\boldsymbol{\mathcal{B}}}) \longrightarrow \boldsymbol{\mathcal{B}}' = (\mathbf{B}', p^{\boldsymbol{\mathcal{B}}'})$ of finitary many-sorted algebraic theories of Bénabou is a functor F from $\mathbf{B}$ to $\mathbf{B}'$ such that the object mapping of F is the identity and the morphism mapping of F preserves the projections, i.e., for every $w \in S^{\star}$ and $i \in |w|$, it happens that $F(p_i^{\boldsymbol{\mathcal{B}},w}) = p_i^{\boldsymbol{\mathcal{B}}',w}$.

Proposition 4.5. *There exists an isomorphism from the category* $\mathbf{Alg}(\mathrm{B}_S)$ *to the category* $\mathbf{BTh}_{\mathrm{f}}(S)$.

Proof. See [4]. $\square$

Next we state that the categories $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$ are equivalent.

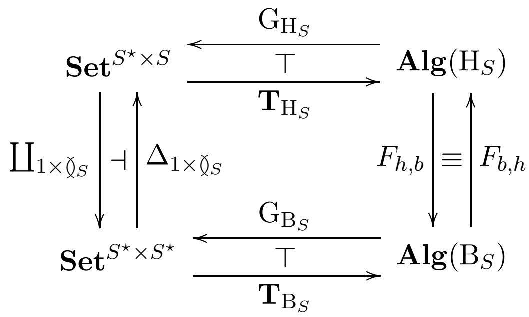
Proposition 4.6. *For every set of sorts S, the categories* $\mathbf{Alg}(\mathrm{H}_S)$ *and* $\mathbf{Alg}(\mathrm{B}_S)$ *are equivalent.*

Proof. See [5]. $\square$

Later on, after having defined the morphisms and transformations of Fujiwara which will allow us to obtain the corresponding 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, of specifications, we will get such an equivalence (which can be considered as having a semantic character) as a consequence, on the one hand, of the existence of a more basic (syntactic) equivalence between the specifications of Hall and Bénabou in $\mathbf{Spf}_{\mathfrak{pd}}$ and, on the other hand, of the existence of a pseudo-functor from $\mathbf{Spf}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$.

In the following proposition, for a set of sorts S, we state some relations among the just stated equivalence between the categories $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$, the adjunctions $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$ and $\mathbf{T}_{\mathrm{B}_S} \dashv \mathrm{G}_{\mathrm{B}_S}$, and the adjunction $\coprod_{1\times \between_S} \dashv \Delta_{1\times \between_S}$ determined by the mapping $1\times \between_S$ from $S^{\star} \times S$ to $S^{\star} \times S^{\star}$ which sends a pair (w, s) in $S^{\star}\times S$ to the pair $(w, (s))$ in $S^{\star}\times S^{\star}$ is isomorphic to $\mathbf{BT}_S(\Sigma)$.

Proposition 4.7. *Let S be a set of sorts. Then for the diagram*



we have that $\Delta_{1\times\between_S} \circ \mathrm{G}_{\mathrm{B}_S} = \mathrm{G}_{\mathrm{H}_S} \circ F_{b,h}$ *and* $\mathbf{T}_{\mathrm{B}_S} \circ \coprod_{1\times\between_S} \cong F_{h,b} \circ \mathbf{T}_{\mathrm{H}_S}$.

Proof. See [5]. □

Corollary 4.8. *Let* Σ *be an* S*-sorted signature. Then the free Bénabou algebra* $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)$ *on* $\coprod_{1\times\between_S}\Sigma$ *is isomorphic to the Bénabou algebra* $\mathbf{BT}_S(\Sigma)$ *for* (S,Σ).

Proof. See [5]. □

This corollary enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{B}_S}(\coprod_{1\times\between_S}\Sigma)$, and, on the other hand, to give, in the fifth section, an alternative, but equivalent, definition of the concept of morphism of Fujiwara between signatures.

5. Morphisms of Fujiwara.

In Mathematics it is standard to compare pairs of objects by means of homomorphisms, i.e., mappings from one of them to the other which relate, in a predetermined way, the primitive operations on the source object to the corresponding primitive operations on the target object. But there are natural examples of comparisons between objects, e.g., the derivations in ring theory (see [17], pp. 169–172), which can only be stated by relating the primitive operations on the source object to corresponding (*families of*) *derived* operations on the target object, thus showing the necessity of conveniently generalizing the ordinary concept of homomorphism.

In this section, following the work by Fujiwara in [8], we generalize the morphisms in **Sig** in such a way that a signature morphism from a signature into another, to be called henceforth a *morphism of Fujiwara*, or more briefly, a *polyderivor*, will consist of two suitably related mappings: On the one hand, a mapping which relates the sets of sorts of the signatures and assigns to each sort in the first, a derived sort in the second, i.e., a word on the set of sorts of the second, and, on the other hand, a mapping which assigns to each formal operation in the first, a family of terms in the second, all in such a way that both transformations are compatible. This type of signature morphism, from which we will get a category $\mathbf{Sig}_{\mathfrak{pd}}$, with the same objects that **Sig**, will allow us to generalize, concordantly, the morphisms between algebras.

We will also prove that the category $\mathbf{Sig}_{\mathfrak{pd}}$ is isomorphic to the Kleisli category for a monad in **Sig**, and that fact will confirm, to some extend, the naturalness of the concept of polyderivor. Furthermore, the contravariant functor $\mathrm{Alg}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ will be lifted to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ and, by applying the EG-construction, we will get the category $\mathbf{Alg}_{\mathfrak{pd}}$ of algebras and algebra morphisms that will have the polyderivors as a component.

Next we define the concept of polyderivor, and we warn the reader about the convenience of looking at the notational conventions stated in the last paragraph of the introduction that have to do with the notion of monoid.

Definition 5.1. Let $\mathbf{\Sigma} = (S, \Sigma)$ and $\mathbf{\Lambda} = (T, \Lambda)$ be signatures. A *polyderivor from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ is a pair $\mathbf{d} = (\varphi, d)$, where $\varphi\colon S \longrightarrow T^{\star}$ while $d\colon \Sigma \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp \times \varphi}$.

Therefore, if $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ is a polyderivor, then, for every $(w, s) \in S^{\star} \times S$, we have that

$$d_{w,s}\colon \Sigma_{w,s} \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp(w),\varphi(s)} (= \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)}),$$

and, given that $\Delta_{\varphi^\sharp \times \varphi} = \Delta_{1 \times \between_S} \circ \Delta_{\varphi^\sharp \times \varphi^\sharp}$ and the functor $\coprod_{1 \times \between_S}$ is left adjoint to the functor $\Delta_{1 \times \between_S}$, d is, essentially, an $S^{\star} \times S^{\star}$-sorted mapping

$$\theta^{1 \times \between_S}(d)\colon \textstyle\coprod_{1 \times \between_S} \Sigma \longrightarrow \mathrm{BT}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}.$$

Henceforth, for every polyderivor $\mathbf{d}$, we identify d and $\theta^{1 \times \between_S}(d)$.

For every signature $\mathbf{\Lambda} = (T, \Lambda)$, $\mathrm{BT}_T(\Lambda)$ is the underlying many-sorted set of $\mathbf{BT}_T(\Lambda)$, the Bénabou algebra for $\mathbf{\Lambda}$. But $\mathbf{BT}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{\mathrm{B}_T}(\coprod_{1 \times \between_T} \Lambda)$, by Corollary 4.8. Hence the polyderivors can also be defined as pairs $\mathbf{d} = (\varphi, d)$, where φ is a mapping from S to $T^{\star}$ while d is an $S^{\star} \times S$-sorted mapping from Σ to $\mathrm{T}_{\mathrm{B}_T}(\coprod_{1 \times \between_T} \Lambda)_{\varphi^\sharp \times \varphi}$, or, equivalently, an $S^{\star} \times S^{\star}$-sorted mapping from $\coprod_{1 \times \between_S} \Sigma$ to $\mathrm{T}_{\mathrm{B}_T}(\coprod_{1 \times \between_T} \Lambda)_{\varphi^\sharp \times \varphi^\sharp}$.

Example. Let $\mathbf{\Sigma}$ be a signature and $p \in \mathbb{N}$. Then taking

1. As $\varphi\colon S \longrightarrow S^{\star}$ the mapping which sends $s \in S$ to the word $\Yup_{\mu \in p}(s)$ and,
2. For $(w, s) \in S^{\star} \times S$, as $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))_s^p$ (since, in this case, $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))_{\varphi(s)} = \mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))_s^p$), which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_p^{w_1}, \ldots, v_{(|w|-1)p}^{w_{|w|-1}}), \ldots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \ldots, v_{|w|p-1}^{w_{|w|-1}})),$$

in $\mathrm{T}_{\boldsymbol{\Sigma}}(\downarrow\varphi^{\sharp}(w))^{p}_{s}$, we have that $\mathbf{d} = (\varphi, d)$ is an endopolyderivor of $\boldsymbol{\Sigma}$.

We refer to the last section of this article for additional examples of polyderivors. Actually, in the seventh section we consider, clear and natural, polyderivors between the (many-sorted) signatures of Hall and Bénabou.

Example. Let $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ and $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$ be two single-sorted signatures and (Φ, P), with $\Phi = \{\, \varphi_\mu \mid \mu \in p \,\}$, a family of basic mapping-formulas from Σ to Λ as defined by Fujiwara in [8], p. 155. Then by associating

1. To the single-sorted signatures Σ and Λ, the signatures $(1, (\Sigma_{n,0})_{(n,0)\in 1^{\star}\times 1})$ and $(1, (\Lambda_{n,0})_{(n,0)\in 1^{\star}\times 1})$, respectively, where, for every $n \in 1^{\star} \cong \mathbb{N}$, $\Sigma_{n,0} = \Sigma_n$ and $\Lambda_{n,0} = \Lambda_n$, and

2. To the morphism (Φ, P) the pair (κ_p, d), where κ_p is the mapping from 1 to $1^{\star}$ which sends $0 \in 1$ to $p \in 1^{\star}$ and d the $1^{\star}\times 1$-sorted mapping from $(\Sigma_{n,0})_{(n,0)\in 1^{\star}\times 1}$ to

$$(\mathrm{T}_{\boldsymbol{\Lambda}}(\downarrow\kappa_p^{\sharp}(n))_{\kappa_p(0)})_{(n,0)\in 1^{\star}\times 1} \cong (\mathrm{T}_{\boldsymbol{\Lambda}}(\Phi \times \downarrow v_n)^{p})_{n\in\mathbb{N}}$$

   which, for $n \in 1^{\star}$, sends $\sigma \in \Sigma_n$ to $d_{n,0}(\sigma) = (P^{n}_{\varphi_0,\sigma}, \ldots, P^{n}_{\varphi_{p-1},\sigma})$,

we have that the families of basic mapping-formulas defined by Fujiwara for the single-sorted case fall under the concept of polyderivor. Consequently, all the examples provided by Fujiwara in [8], pp. 155–156, once reformulated as just said, are also examples of polyderivors.

Example. Let (φ, d) be a standard signature morphism from a signature (S, Σ) into another (T, Λ). Let (φ, d) be a standard signature morphism from a signature (S, Σ) into another (T, Λ). Then from the mapping $\varphi\colon S \longrightarrow T$ we get the mapping $\between_T \circ\varphi\colon S \longrightarrow T^{\star}$, and from the $S^{\star}\times S$-sorted mapping $d\colon \Sigma \longrightarrow \Lambda_{\varphi^{\star}\times\varphi}$, because there exists a canonical embedding of $\Lambda_{\varphi^{\star}\times\varphi}$ into $(\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^{\sharp}\times(\between_T\circ\varphi)}$, we get the composite mapping

$$\Sigma \xrightarrow{\ d\ } \Lambda_{\varphi^{\star}\times\varphi} \to (\textstyle\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^{\sharp}\times(\between_T\circ\varphi)} \to \mathrm{T}_{\mathrm{B}_T}(\textstyle\coprod_{1\times\between_T}\Lambda)_{(\between_T\circ\varphi)^{\sharp}\times(\between_T\circ\varphi)}.$$

Thus the standard signature morphisms fall under the concept of polyderivor.

Our next goal is to define the composition of polyderivors in order to get the category $\mathbf{Sig}_{\mathfrak{pd}}$, of signatures and polyderivors. To attain the just stated goal we need to recall beforehand the concept of derivor from a signature into another, which was defined by Goguen, Thatcher, and Wagner in [13], p. 86, and to set out some of its properties.

Definition 5.2. Let $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ be signatures. Then a *derivor from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ is a pair $\mathbf{d} = (\varphi, d)$, with $\varphi\colon S \longrightarrow T$ and $d\colon \Sigma \longrightarrow \mathrm{HT}_T(\Lambda)_{\varphi^\star \times \varphi}$.

Therefore, if $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ is a derivor, then, for every $(w, s) \in S^\star \times S$, we have that

$$d_{w,s}\colon \Sigma_{w,s} \longrightarrow \mathrm{HT}_T(\Lambda)_{\varphi^\star(w),\varphi(s)} (= \mathrm{T}_\Lambda(\downarrow\varphi^\star(w))_{\varphi(s)})$$

sends a formal operation $\sigma\colon w \longrightarrow s$ to a term $d_{w,s}(\sigma)\colon \varphi^\star(w) \longrightarrow \varphi(s)$, i.e., a term for Λ of type $(\downarrow\varphi^\star(w), \varphi(s))$, and all in such a way that the arities and coarities are preserved, modulus the correspondence between the sorts given by the mapping φ.

For every signature $\mathbf{\Lambda}$ we have that $\mathrm{HT}_T(\Lambda)$ is the underlying many-sorted set of $\mathbf{HT}_T(\Lambda)$, the Hall algebra for (T, Λ). But $\mathbf{HT}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_T}(\Lambda)$, the free H_T-algebra on Λ, by Proposition 4.2. Consequently the derivors can be defined, alternatively, but equivalently, as pairs $\mathbf{d} = (\varphi, d)$ with $\varphi\colon S \longrightarrow T$ and $d\colon \Sigma \longrightarrow \mathrm{T}_{\mathrm{H}_T}(\Lambda)$. Thus, taking into account the equivalence between the categories $\mathbf{Alg}(\mathrm{H}_T)$ and $\mathbf{Alg}(\mathrm{B}_T)$, we can state the following

Corollary 5.3. *Every derivor is a polyderivor (although, obviously, not every polyderivor is a derivor).*

An example of derivor originating from the field of propositional logic is that provided by Gödel (see [10]) in his work about an interpretation of the intuitionistic propositional logic into a modal extension of the classical propositional logic.

Example. Let $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ be a single-sorted signature such that $\Sigma_1 = \{\neg_{\mathrm{i}}\}$, $\Sigma_2 = \{\wedge_{\mathrm{i}}, \vee_{\mathrm{i}}, \rightarrow_{\mathrm{i}}\}$ and $\Sigma_n = \varnothing$, if $n \neq 1, 2$, $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$ a single-sorted signature such that $\Lambda_1 = \{\neg_{\mathrm{c}}, \Box\}$, $\Lambda_2 = \{\wedge_{\mathrm{c}}, \vee_{\mathrm{c}}, \rightarrow_{\mathrm{c}}\}$, and $\Lambda_n = \varnothing$, if $n \neq 1, 2$, and $g = (g_n)_{n\in\mathbb{N}}$ the family defined, for $n \neq 1, 2$, as the unique mapping from $\varnothing$ to $\mathrm{T}_\Lambda(\downarrow v_n)$, and, for $n = 1, 2$, as follows:

1. $g_1(\neg_{\mathrm{i}}) = \neg_{\mathrm{c}}\Box v_0$.

2. $g_2(\wedge_{\mathrm{i}}) = v_0 \wedge_{\mathrm{c}} v_1$.

3. $g_2(\vee_{\mathrm{i}}) = \Box v_0 \vee_{\mathrm{c}} \Box v_1$.

4. $g_2(\rightarrow_{\mathrm{i}}) = \Box v_0 \rightarrow_{\mathrm{c}} \Box v_1$.

Then d is a derivor from Σ to Λ. This derivor defines the intuitionistic connectives in terms of the classical connectives together with $\Box$, the operator of necessity.

Next we proceed to define the composition of derivors and to prove that the corresponding category of signatures and derivors, denoted by $\mathbf{Sig}_{\mathfrak{d}}$, is isomorphic to the Kleisli category for a monad $\mathbb{T}_{\mathfrak{d}}$ in $\mathbf{Sig}$. By proceeding in this way we, on the one hand, move one step forward, from the standpoint of category theory, into the investigation of some of the most notable positive properties of the category $\mathbf{Sig}_{\mathfrak{d}}$, with regard to what has been done in [13], and, on the other hand, get a model on which to base the subsequent work that we have to do concerning polyderivors.

We point out that the definition of the composition of derivors, in strong contrast with that of polyderivors below, is based on the standard specification morphisms between Hall specifications. Actually, if instead of starting from a mapping $\varphi\colon S \longrightarrow T^\star$, as is the case for the polyderivors, we start from an ordinary mapping $\varphi\colon S \longrightarrow T$, then, as we state next, we get a functor $(\varphi^\star \times \varphi, h^\varphi)^*$ from $\mathbf{Alg}(\mathrm{H}_T)$ to $\mathbf{Alg}(\mathrm{H}_S)$ (and the existence of such a functor will follow from that of a specification morphism from $(S^\star \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S})$ to $(T^\star \times T, \Sigma^{\mathrm{H}_T}, \mathcal{E}^{\mathrm{H}_T})$). This functor, in its turn, will allows us to endow to the many-sorted set $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ with a Hall algebra structure for S, from which the composition of derivors will be defined explicitly.

Proposition 5.4. *Let* $\varphi\colon S \longrightarrow T$ *be a mapping. Then the* $S^\star\times S$*-sorted mapping* $h^\varphi\colon \Sigma^{\mathrm{H}_S} \longrightarrow \Sigma^{\mathrm{H}_T}_{\varphi^\star\times\varphi}$ *defined as follows:*

1. *For every* $w \in S^\star$ *and* $i \in |w|$, $h^\varphi(\pi^w_i) = \pi^{\varphi^\star(w)}_i$,

2. *For every* u, $w \in S^\star$ *and* $s \in S$, $h^\varphi(\xi_{u,w,s}) = \xi_{\varphi^\star(u),\varphi^\star(w),\varphi(s)}$,

is such that $(\varphi^\star \times \varphi, h^\varphi)\colon (S^\star \times S, \Sigma^{\mathrm{H}_S}, \mathcal{E}^{\mathrm{H}_S}) \longrightarrow (T^\star \times T, \Sigma^{\mathrm{H}_T}, \mathcal{E}^{\mathrm{H}_T})$ *is a specification morphism. Thus* $\varphi\colon S \longrightarrow T$ *induces a functor* $(\varphi^\star \times \varphi, h^\varphi)^*$

from $\mathbf{Alg}(\mathrm{H}_T)$ *to* $\mathbf{Alg}(\mathrm{H}_S)$ *which sends* $\mathbf{HT}_T(\Lambda)$*, the free Hall algebra on a* T*-sorted signature* Λ*, to a Hall algebra for* S*, with* $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ *as underlying* $S^\star \times S$*-sorted set.*

For a derivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the many-sorted mapping d from Σ to $\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$ can be lifted to a homomorphism of Hall algebras $d^\sharp$ from $\mathbf{HT}_S(\Sigma)$ to $\mathbf{HT}_T(\Lambda)_{\varphi^\star\times\varphi}$, whose underlying $S^\star \times S$-sorted mapping determines a translation of terms for $\mathbf{\Sigma}$ to terms for $\mathbf{\Lambda}$. In particular, for every $(w, s) \in S^\star \times S$, $d^\sharp_{w,s}$ assigns to terms in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow w)_s$ terms in $\mathrm{T}_{\mathbf{\Lambda}}(\downarrow \varphi^\sharp(w))_{\varphi(s)}$.

Before we define immediately below the composition of derivors and the identities we recall that $\mathbf{\Sigma}$, $\mathbf{\Lambda}$, $\mathbf{\Omega}$, and $\mathbf{\Xi}$ denote the signatures (S, Σ), (T, Λ), (U, Ω), and (X, Ξ), respectively, and $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{h}$ denote the derivors (φ, d), (ψ, e), and (γ, h), respectively.

Definition 5.5. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$ be derivors. Then $\mathbf{e} \circ \mathbf{d}$, the *composition of* $\mathbf{d}$ *and* $\mathbf{e}$, is the derivor $(\psi \circ \varphi, e^\sharp_{\varphi^\star\times\varphi} \circ d)$, where $e^\sharp_{\varphi^\star\times\varphi} \circ d$ is obtained from

$$
\begin{array}{ccc}
\Lambda & \xrightarrow{\eta^{\mathrm{H}_T}_{\Lambda}} & \mathrm{HT}_T(\Lambda) \\
& {}_{e}\searrow & \downarrow{\scriptstyle e^\sharp} \\
& & \mathrm{HT}_U(\Omega)_{\psi^\star\times\psi}
\end{array}
\qquad \text{as} \qquad
\begin{array}{ccc}
\mathrm{HT}_T(\Lambda)_{\varphi^\star\times\varphi} & \xleftarrow{\;d\;} & \Sigma \\
\downarrow{\scriptstyle e^\sharp_{\varphi^\star\times\varphi}} & & \\
\mathrm{HT}_U(\Omega)_{\psi^\star\times\psi_{\varphi^\star\times\varphi}} & &
\end{array}
$$

where $e^\sharp$ is the extension of e to the free Hall algebra on Λ. On the other hand, for every signature $\mathbf{\Sigma}$, the *identity at* $\mathbf{\Sigma}$ is $(\mathrm{id}_S, \eta^{\mathrm{H}_S}_{\Sigma})$.

The preceding definition allows us to get a corresponding, and explicit, category of signatures and derivors.

Proposition 5.6. *The signatures together with the derivors determine a category, that we denote by* $\mathbf{Sig}_{\mathfrak{d}}$*.*

Following this we state that $\mathbf{Sig}_{\mathfrak{d}}$ can be obtained, naturally, as an isomorphic copy of the Kleisli category for a monad in $\mathbf{Sig}$. This is founded on the fact that, for every set of sorts S, we have the adjunction $\mathbf{T}_{\mathrm{H}_S} \dashv \mathrm{G}_{\mathrm{H}_S}$, from which we get the monad $\mathbb{T}_{\mathrm{H}_S} = (\mathrm{T}_{\mathrm{H}_S}, \eta^{\mathrm{H}_S}, \mu^{\mathrm{H}_S})$ in $\mathbf{Set}^{S^\star\times S}$, that canonically induces the monad in $\mathbf{Sig}$ at issue.

Proposition 5.7. *The triple* $\mathbb{T}_{\mathfrak{d}} = (\mathfrak{d}, \eta^{\mathfrak{d}}, \mu^{\mathfrak{d}})$, *where* $\mathfrak{d}$ *is the endofunctor of* **Sig** *which sends a signature* $\mathbf{\Sigma}$ *to the signature* $(S, \mathrm{T}_{\mathrm{H}_S}(\Sigma))$, *and a signature morphism* $\mathbf{d}$ *from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ *to the signature morphism* $(\varphi, d^{\sharp})$ *from* $(S, \mathrm{T}_{\mathrm{H}_S}(\Sigma))$ *to* $(T, \mathrm{T}_{\mathrm{H}_T}(\Lambda))$; $\eta^{\mathfrak{d}}_{\mathbf{\Sigma}} = (\mathrm{id}_S, \eta^{\mathrm{H}_S}_{\Sigma})$, *with* $\eta^{\mathrm{H}_S}_{\Sigma}$ *the value at* Σ *of the unit* η^{H_S} *of the monad* $\mathbb{T}_{\mathrm{H}_S}$; *and* $\mu^{\mathfrak{d}}_{\mathbf{\Sigma}} = (\mathrm{id}_S, \mu^{\mathrm{H}_S}_{\Sigma})$, *with* $\mu^{\mathrm{H}_S}_{\Sigma}$ *the value at* Σ *of the multiplication* μ^{H_S} *of the monad* $\mathbb{T}_{\mathrm{H}_S}$, *is a monad in* **Sig** *and the categories* $\mathbf{Sig}_{\mathfrak{d}}$ *and* $\mathbf{Kl}(\mathbb{T}_{\mathfrak{d}})$ *are isomorphic.*

Remark. Almost all the results about the categories **Sig**, **Alg** and **Spf** established in the second and third section, suitably extended, are also valid for the corresponding categories $\mathbf{Sig}_{\mathfrak{d}}$, $\mathbf{Alg}_{\mathfrak{d}}$ and $\mathbf{Spf}_{\mathfrak{d}}$. But the derivors being a particular case of the polyderivors, we restrict ourselves to unfold those results only for the polyderivors.

Our next goal is to define the composition of two polyderivors. To attain it we begin by stating that every mapping φ from S to $T^{\star}$ determines a functor $(\varphi^{\sharp}\times\varphi^{\sharp}, b^{\varphi})^{*}$ from $\mathbf{Alg}(\mathrm{B}_T)$ to $\mathbf{Alg}(\mathrm{B}_S)$ (observe that such a functor is induced not by a standard specification morphism between Bénabou specifications, but by a *derivor* b^{φ} between the corresponding Bénabou signatures). This functor, in its turn, will allow us to endow the many-sorted set $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ with a Bénabou algebra structure for S, from which the definition of the composition of polyderivors will follow.

Proposition 5.8. *Let* $\varphi\colon S \longrightarrow T^{\star}$ *be a mapping. Then* b^{φ} *which is the* $((S^{\star})^2)^{\star}\times(S^{\star})^2$*-sorted mapping* $b^{\varphi}\colon \Sigma^{\mathrm{B}_S} \longrightarrow \mathrm{HT}_{T^{\star}\times T^{\star}}(\Sigma^{\mathrm{B}_T})_{(\varphi^{\sharp}\times\varphi^{\sharp})^{\star}\times(\varphi^{\sharp}\times\varphi^{\sharp})}$ *defined as follows:*

1. *For every* $w \in S^{\star}$ *and* $\alpha \in |w|$, $b^{\varphi}(\pi^{w}_{\alpha})$ *is the* $\mathbf{\Sigma}^{\mathrm{B}_T}$*-term*
$$\langle \pi^{\varphi^{\sharp}(w)}_{\sum_{\beta\in\alpha} p_{\beta}}, \ldots, \pi^{\varphi^{\sharp}(w)}_{\sum_{\beta\in\alpha+1} p_{\beta}-1}\rangle_{\varphi^{\sharp}(w),\varphi(w_{\alpha})}$$
of type $\lambda \longrightarrow (\varphi^{\sharp}(w), (\varphi(w_{\alpha})))$,

2. *For every* u, $w \in S^{\star}$, $b^{\varphi}(\langle\,\rangle_{u,w})$ *is the* $\mathbf{\Sigma}^{\mathrm{B}_T}$*-term*
$$\begin{aligned}\langle &\pi^{\varphi(w_0)}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_0))}_{0}, \ldots, \pi^{\varphi(w_0)}_{|\varphi(w_0)|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_0))}_{0}, \ldots,\\ &\pi^{\varphi(w_i)}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_i))}_{i}, \ldots, \pi^{\varphi(w_i)}_{|\varphi(w_i)|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_i))}_{i}, \ldots,\\ &\pi^{\varphi(w_{|w|-1})}_{0} \circ v^{(\varphi^{\sharp}(u),\varphi(w_{|w|-1}))}_{|w|-1}, \ldots, \pi^{\varphi(w_{|w|-1})}_{|\varphi(w_{|w|-1})|-1} \circ v^{(\varphi^{\sharp}(u),\varphi(w_{|w|-1}))}_{|w|-1}\rangle\end{aligned}$$
of type $((\varphi^{\sharp}(u), \varphi(w_0)), \ldots, (\varphi^{\sharp}(u), \varphi(w_{|w|-1}))) \longrightarrow (\varphi^{\sharp}(u), \varphi^{\sharp}(w))$,

3. For every u, x, $w \in S^{\star}$, $b^{\varphi}(\circ_{u,x,w})$ is the $\mathbf{\Sigma}^{\mathrm{B}_T}$-term

$$\circ_{\varphi^{\sharp}(u),\varphi^{\sharp}(x),\varphi^{\sharp}(w)}(v_0^{(\varphi^{\sharp}(u),\varphi^{\sharp}(x))}, v_1^{(\varphi^{\sharp}(x),\varphi^{\sharp}(w))})$$

of type $((\varphi^{\sharp}(u),\varphi^{\sharp}(x)),(\varphi^{\sharp}(x),\varphi^{\sharp}(w))) \longrightarrow (\varphi^{\sharp}(u),\varphi^{\sharp}(w))$,

is such that $(\varphi^{\sharp}\times\varphi^{\sharp}, b^{\varphi})\colon (S^{\star}\times S^{\star}, \Sigma^{\mathrm{B}_S}, \mathcal{E}^{\mathrm{B}_S}) \longrightarrow (T^{\star}\times T^{\star}, \Sigma^{\mathrm{B}_T}, \mathcal{E}^{\mathrm{B}_T})$ is a specification morphism. Thus $\varphi\colon S \longrightarrow T^{\star}$ induces a functor $(\varphi^{\sharp}\times\varphi^{\sharp}, b^{\varphi})^{*}$ from $\mathbf{Alg}(\mathrm{B}_T)$ to $\mathbf{Alg}(\mathrm{B}_S)$ which sends $\mathbf{BT}_T(\Lambda)$, the free Bénabou algebra on the T-sorted signature Λ, to a Bénabou algebra for S, with $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ as underlying $S^{\star}\times S^{\star}$-sorted set.

For a polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, we can extend the $S^{\star}\times S^{\star}$-sorted mapping d from $\coprod_{1\times\between_S}\Sigma$ to $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$ to a homomorphism of Bénabou algebras $d^{\sharp}$ from $\mathrm{BT}_S(\Sigma)$ to $\mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}}$, whose underlying $S^{\star}\times S^{\star}$-mapping determines a translation of terms for $\mathbf{\Sigma}$ into terms for $\mathbf{\Lambda}$.

We define next the composition of polyderivors and the identities.

Definition 5.9. Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$ be polyderivors. Then the *composition of* $\mathbf{d}$ *and* $\mathbf{e}$, henceforth denoted by $\mathbf{e}\circ\mathbf{d}$, is the morphism $(\psi^{\sharp}\circ\varphi, e^{\sharp}_{\varphi^{\sharp}\times\varphi^{\sharp}}\circ d)$, where the first component $\psi^{\sharp}\circ\varphi$ is a mapping from S to $U^{\star}$ and $e^{\sharp}_{\varphi^{\sharp}\times\varphi^{\sharp}}\circ d$ is obtained from

$$\begin{array}{ccc} \coprod_{1\times\between_T}\Lambda & \xrightarrow{\eta^{\mathrm{B}_T}_{\coprod_{1\times\between_T}\Lambda}} & \mathrm{BT}_T(\Lambda) \\ & {\scriptstyle e}\searrow & \downarrow{\scriptstyle e^{\sharp}} \\ & & \mathrm{BT}_U(\Omega)_{\psi^{\sharp}\times\psi^{\sharp}} \end{array} \quad \text{as} \quad \begin{array}{ccc} \mathrm{BT}_T(\Lambda)_{\varphi^{\sharp}\times\varphi^{\sharp}} & \xleftarrow{\ d\ } & \coprod_{1\times\between_S}\Sigma \\ \downarrow{\scriptstyle e^{\sharp}_{\varphi^{\sharp}\times\varphi^{\sharp}}} & & \\ \mathrm{BT}_U(\Omega)_{\psi^{\sharp}\times\psi^{\sharp}\,\varphi^{\sharp}\times\varphi^{\sharp}} & & \end{array}$$

On the other hand, for every signature $\mathbf{\Sigma}$, the *identity at* $\mathbf{\Sigma}$ is the polyderivor $(\between_S, \eta^{\mathrm{B}_S}_{\Sigma})$.

Proposition 5.10. *The signatures together with the polyderivors determine a category, that we denote by* $\mathbf{Sig}_{\mathfrak{pd}}$.

Following this we prove that the category $\mathbf{Sig}_{\mathfrak{pd}}$ can be obtained as an isomorphic copy of the Kleisli category for some monad in $\mathbf{Sig}$. However, the process we should follow to determine such a monad is more complicated than the one, relatively simple, we have followed for the derivors. This is due to the fact that, for a signature $\mathbf{\Sigma} = (S, \Sigma)$, the pair $(S^\star \times S^\star, \mathrm{BT}_S(\Sigma))$ is not a signature, because $\mathrm{BT}_S(\Sigma)$ is an $S^\star \times S^\star$-sorted set, but not an $((S^\star)^2)^\star \times (S^\star)^2$-sorted set.

Proposition 5.11. *There exists a monad* $\mathbb{T}_{\mathfrak{pd}} = (\mathfrak{pd}, \eta^{\mathfrak{pd}}, \mu^{\mathfrak{pd}})$ *in* $\mathbf{Sig}$ *such that the categories* $\mathbf{Sig}_{\mathfrak{pd}}$ *and* $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ *are isomorphic.*

Proof. Let $\mathfrak{pd}$ be the endofunctor of $\mathbf{Sig}$ defined as follows: its object mapping sends each signature $\mathbf{\Sigma}$ to $(S^\star, \mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times \between_S} \Sigma)_{\curlywedge_S \times 1}))$; its arrow mapping sends each signature morphism $\mathbf{d}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ to

$$(\varphi^\star, (d^\sharp)_{\curlywedge_S \times 1})\colon (S^\star, \mathrm{T}_{\mathrm{B}_S}(\textstyle\coprod_{1\times \between_S} \Sigma)_{\curlywedge_S \times 1})) \longrightarrow (T^\star, \mathrm{T}_{\mathrm{B}_T}(\textstyle\coprod_{1\times \between_T} \Lambda)_{\curlywedge_T \times 1})),$$

where $\mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times \between_S} \Sigma)_{\curlywedge_S \times 1}$ is the value at Σ of the functor

$$\mathbf{Set}^{S^\star \times S} \xrightarrow{\ \coprod_{1\times \between_S}\ } \mathbf{Set}^{S^\star \times S^\star} \xrightarrow{\ \mathrm{T}_{\mathrm{B}_S}\ } \mathbf{Set}^{S^\star \times S^\star} \xrightarrow{\ \Delta_{\curlywedge_S \times 1}\ } \mathbf{Set}^{S^{\star\star} \times S^\star}.$$

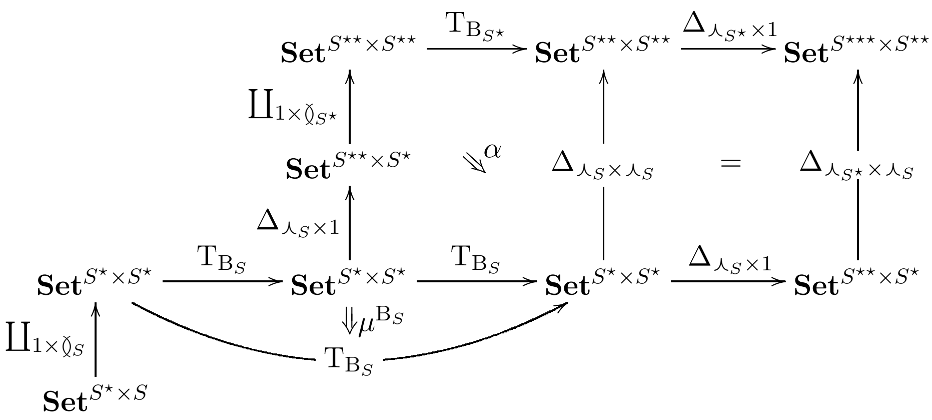
After having defined the endofunctor $\mathfrak{pd}$ of $\mathbf{Sig}$, we proceed to define the unit $\eta^{\mathfrak{pd}}$ and multiplication $\mu^{\mathfrak{pd}}$ of the monad $\mathbb{T}_{\mathfrak{pd}}$.

Let $\mathbf{\Sigma}$ be a signature. Then we have that $\eta^{\mathfrak{pd}}_{\mathbf{\Sigma}}$, the component of the unit $\eta^{\mathfrak{pd}}$ of the purported monad $\mathbb{T}_{\mathfrak{pd}}$ in $\mathbf{\Sigma}$, is the signature morphism $(\between_S, \eta^{\mathrm{B}_S}_\Sigma)$, i.e., the value at Σ of the unit of the monad $\mathbb{T}_{\mathrm{B}_S} = (\mathrm{T}_{\mathrm{B}_S}, \eta^{\mathrm{B}_S}, \mu^{\mathrm{B}_S})$ in $\mathbf{Set}^{S^\star \times S^\star}$, obtained from the adjunction $\mathbf{T}_{\mathrm{B}_S} \dashv \mathrm{G}_{\mathrm{B}_S}$. On the other hand, we want $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$, the component of the multiplication $\mu^{\mathfrak{pd}}$ of the purported monad $\mathbb{T}_{\mathfrak{pd}}$ in $\mathbf{\Sigma}$, to be a morphism as in the following diagram

$$\begin{array}{c} (S^{\star\star}, \mathrm{T}_{\mathrm{B}_{S^\star}}(\coprod_{1\times \between_{S^\star}}(\mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times \between_S}\Sigma)_{\curlywedge_S \times 1}))_{\curlywedge_{S^\star} \times 1}) \\ \Big\downarrow {\scriptstyle \mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}} \\ (S^\star, \mathrm{T}_{\mathrm{B}_S}(\coprod_{1\times \between_S}\Sigma)_{\curlywedge_S \times 1}) \end{array}$$

The first coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ is $\curlywedge_S$, the multiplication of the monad $\mathbb{T}_\star$. To get the second coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ we have to define a natural transformation

α as in the following diagram



Let Θ be an $S^{\star} \times S^{\star}$-sorted set. Then $\mathrm{T}_{\mathrm{B}_S}(\Theta)_{\curlywedge_S \times \curlywedge_S}$ has a natural $\mathbf{\Sigma}^{\mathrm{B}_{S^{\star}}}$-algebra structure, obtained from the $(S^{\star\star} \times S^{\star\star})^{\star} \times (S^{\star\star} \times S^{\star\star})$-sorted mapping

$$b^{\curlywedge_S} \colon \Sigma^{\mathrm{B}_{S^{\star}}} \longrightarrow \mathrm{Ter}_{S^{\star}\times S^{\star}}(\Sigma^{\mathrm{B}_S})_{(\curlywedge_S \times \curlywedge_S)^{\star} \times (\curlywedge_S \times \curlywedge_S)}$$

by applying Proposition 5.8 to the mapping $\curlywedge_S \colon S^{\star\star} \longrightarrow S^{\star}$.

On the other hand, for every $S^{\star} \times S^{\star}$-sorted set Θ, we have an $S^{\star\star} \times S^{\star\star}$-sorted mapping f_{Θ} from $\coprod_{1\times \between_{S^{\star}}}(\Delta_{\curlywedge_S \times 1}(\Theta))$ to $\Delta_{\curlywedge_S \times \curlywedge_S}(\mathrm{T}_{\mathrm{B}_S}(\Theta))$ which, for every $(\overline{u}, \overline{w}) \in S^{\star\star} \times S^{\star\star}$, assigns to an element P, the image of P under the inclusion $\eta^{\mathrm{B}_S}_{\Theta}$ of Θ into $\mathrm{T}_{\mathrm{B}_S}(\Theta)$. The definition is sound because, in this case, $\overline{w}$ has the form (w), P is in $\Theta_{\curlywedge_S u, w}$ and $\eta^{\mathrm{B}_S}_{\Theta}(P)$ belongs to $\Delta_{\curlywedge_S \times \curlywedge_S}(\mathrm{T}_{\mathrm{B}_S}(\Theta))$.

Then the extension $f^{\sharp}_{\Theta}$ of f_{Θ} to $\mathrm{T}_{\mathrm{B}_{S^{\star}}}(\coprod_{1\times\between_{S^{\star}}}(\Delta_{\curlywedge_S \times 1}(\Theta)))$ is the component at Θ of the natural transformation α.

Therefore, the second coordinate of $\mu^{\mathfrak{pd}}_{\mathbf{\Sigma}}$ is the value at Σ of the natural transformation $(\Delta_{\curlywedge_{S^{\star}} \times \curlywedge_S} * \Delta_{\curlywedge_S \times 1} * \mu^{\mathrm{B}_S} * \coprod_{1\times\between_S}) \circ (\Delta_{\curlywedge_{S^{\star}} \times 1} * \alpha * \mathrm{T}_{\mathrm{B}_S} * \coprod_{1\times\between_S})$.

Finally we prove that $\mathbf{Sig}_{\mathfrak{pd}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ are isomorphic.

A morphism $\mathbf{d} \colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ in $\mathbf{Kl}(\mathbb{T}_{\mathfrak{pd}})$ is a morphism $\mathbf{d} \colon \mathbf{\Sigma} \longrightarrow \mathfrak{pd}(\mathbf{\Lambda})$ in $\mathbf{Sig}$, hence $\varphi \colon S \longrightarrow T^{\star}$ and

$$\begin{aligned} d \colon \Sigma &\longrightarrow \Delta_{\varphi^{\star}\times\varphi}(\mathrm{T}_{\mathrm{B}_T}(\textstyle\coprod_{1\times\between_T}\Lambda)_{\curlywedge_T \times 1}) \\ &= \Delta_{\varphi^{\sharp}\times\varphi}(\mathrm{T}_{\mathrm{B}_T}(\textstyle\coprod_{1\times\between_T}\Lambda)) \\ &\cong \Delta_{\varphi^{\sharp}\times\varphi}(\mathrm{BT}_T(\Lambda)), \end{aligned}$$

that is exactly the definition of polyderivor in $\mathbf{Sig}_{\mathfrak{pd}}$. □

From Proposition 3.5, pp. 122–123, in [4], it follows that $\mathbf{Sig}_{\mathfrak{pd}}$ has coproducts.

Our next goal is to lift the contravariant functor $\mathrm{Alg}\colon \mathbf{Sig} \longrightarrow \mathbf{Cat}$ to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$, that will allow us, by applying the EG-construction, to get a new category of algebras $\mathbf{Alg}_{\mathfrak{pd}}$ into which is embedded the category $\mathbf{Alg}$. But to achieve the just stated objective we should define beforehand some auxiliary functors and natural transformations.

Proposition 5.12. *Let S be a set of sorts. Then we have that*

1. *There exists an* expansion *functor* $(\cdot)^{\natural_S}$ *from* $\mathbf{Set}^S$ *to* $\mathbf{Set}^{S^\star}$ *which sends an S-sorted set $A=(A_s)_{s\in S}$ to the $S^\star$-sorted set $A^{\natural_S}=(A_w)_{w\in S^\star}$, and an S-sorted mapping f from A to B to the $S^\star$-sorted mapping $f^{\natural_S} = (f_w)_{w\in S^\star}$ from $(A_w)_{w\in S^\star}$ to $(B_w)_{w\in S^\star}$. If A is an S-sorted set and $f\colon A \longrightarrow B$ an S-sorted mapping, then we say that $A^{\natural_S}$ and $f^{\natural_S}$ are the* expansions of *A and f, respectively,* to the words on *S and, to simplify notation, we write $A^\natural$ and $f^\natural$ instead of $A^{\natural_S}$ and $f^{\natural_S}$, respectively.*

2. *From the contravariant functor* MSet*, from* $\mathbf{Set}$ *to* $\mathbf{Cat}$*, to the contravariant functor* $\mathrm{MSet} \circ \mathrm{T}_\star^{\mathrm{op}}$ *between the same categories, where* $\mathrm{T}_\star^{\mathrm{op}}$ *is the composite of* $\mathbf{T}_\star^{\mathrm{op}}$ *(the dual of the free monoid functor* $\mathbf{T}_\star$ *from* $\mathbf{Set}$ *to* $\mathbf{Mon}$*, the category of monoids), and* $G_{\mathbf{Mon}}$ *(the forgetful functor from* $\mathbf{Mon}$ *to* $\mathbf{Set}$*), there exists a natural transformation* $(\cdot)^\natural$ *which sends a set S to the expansion functor* $(\cdot)^{\natural_S}$ *from* $\mathbf{Set}^S$ *to* $\mathbf{Set}^{S^\star}$.

3. *There exists a natural isomorphism ι_S from the functor* $(\cdot)^{\natural_{S^\star}} \circ (\cdot)^{\natural_S}$ *to the functor* $\Delta_{\curlywedge_S} \circ (\cdot)^{\natural_S}$*, both from the category* $\mathbf{Set}^S$ *to the category* $\mathbf{Set}^{S^{\star\star}}$.

Proof. We restrict ourselves to prove the second and third parts of the proposition.

(2) $(\cdot)^\natural$ is a natural transformation from MSet to $\mathrm{MSet} \circ \mathrm{T}_\star^{\mathrm{op}}$ since, for a mapping $\varphi\colon S \longrightarrow T$, the functors $(\cdot)^{\natural_S} \circ \Delta_\varphi$ and $\Delta_{\varphi^\star} \circ (\cdot)^{\natural_T}$ from $\mathbf{Set}^T$ to $\mathbf{Set}^{S^\star}$ are identical. Observe, in particular, that for a T-sorted set B, we have that $(B_\varphi)^{\natural_S} = (B^{\natural_T})_{\varphi^\star}$.

(3) It is enough to define, for every S-sorted set A, the component $(\iota_S)_A$ of ι_S at A, as the $S^{\star\star}$-isomorphism $(\iota_S)_A\colon A^{\natural\natural} \longrightarrow (A^{\natural})_{\curlywedge}$ that has as $\overline{w}$-th coordinate, for $\overline{w} = (w_\alpha)_{\alpha\in|\overline{w}|} \in S^{\star\star}$, the canonical isomorphism

$$A^{\natural\natural}_{\overline{w}} = \prod\nolimits_{\alpha\in|\overline{w}|}\prod\nolimits_{j\in|w_\alpha|} A_{w_{\alpha j}} \xrightarrow{\langle \mathrm{pr}_{\alpha_j}\circ\,\mathrm{pr}_\alpha\rangle_{\alpha\in|\overline{w}|,j\in|w_\alpha|}} \prod\nolimits_{\alpha\in|\overline{w}|,\,j\in|w_\alpha|} A_{w_{\alpha j}} = A^{\natural}_{\curlywedge\overline{w}},$$

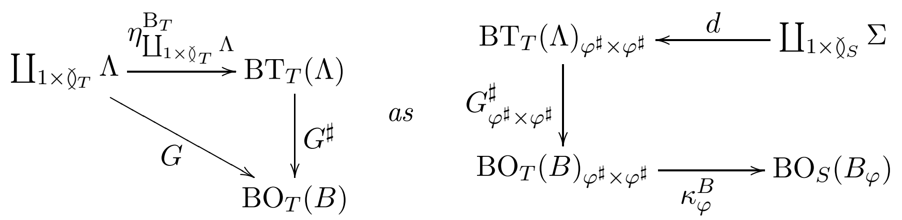
where $\mathrm{pr}_\alpha\colon A_{\overline{w}} \longrightarrow A_{w_\alpha}$ and $\mathrm{pr}_{\alpha_j}\colon A_{w_\alpha} \longrightarrow A_{w_{\alpha j}}$ are the canonical projections. To simplify notation we let ι^A stand for $(\iota_S)_A$. □

Corollary 5.13. *Let $\varphi\colon S \longrightarrow T^\star$ and $\psi\colon T \longrightarrow U^\star$ be mappings. Then, for every T-sorted set B and U-sorted set C, we have that*

1. *$((B^{\natural_T})^{\natural_{T^\star}})_{\varphi^\star}$, denoted by $B_{\varphi^\star}$, and $(B^{\natural_T})_{\varphi^\sharp}$, denoted by $B_{\varphi^\sharp}$, are isomorphic $S^\star$-sorted sets.*
2. *$(((C^{\natural_U})_\psi)^{\natural_T})_\varphi$, denoted by C_{ψ_φ}, and $(C^{\natural_U})_{\psi^\sharp\circ\varphi}$, denoted by $C_{\psi^\sharp\circ\varphi}$, are isomorphic S-sorted sets.*
3. *There exists an isomorphism $\kappa^B_\varphi\colon \mathrm{BO}_T(B)_{\varphi^\sharp\times\varphi^\sharp} \longrightarrow \mathrm{BO}_S(B_\varphi)$, where, to simplify notation, we let B_φ stand for $(B^{\natural_T})_\varphi$.*

We state in the following proposition that the polyderivors between signatures determine functors, in the opposite direction, from the category of algebras associated to the target signature to the category of algebras associated to the source signature. These functors will be the components of the morphism mapping of the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$.

Proposition 5.14. *Let $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ be a morphism in $\mathbf{Sig}_{\mathfrak{pd}}$. Then there exists a functor $\mathrm{Alg}_{\mathfrak{pd}}(\mathbf{d}) = \mathbf{d}^*_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$ defined as follows: its object mapping sends each $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ to the $\mathbf{\Sigma}$-algebra $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$, where $G^{\mathbf{d}}$ is $\kappa^B_\varphi \circ G^\sharp_{\varphi^\sharp\times\varphi^\sharp} \circ d$, which is obtained from*



its arrow mapping sends each $\mathbf{\Lambda}$*-homomorphism* f *from* $\mathbf{B}$ *to* $\mathbf{B}'$ *to the* $\mathbf{\Sigma}$*-homomorphism* $\mathbf{d}^*_{\mathfrak{pd}}(f) = f_\varphi$ *from* $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ *to* $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B}')$.

Given a polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, a $\mathbf{\Lambda}$-algebra $\mathbf{B} = (B, G)$ and an operation $\sigma \in \Sigma_{w,s}$, if we agree that w is the word $(s_i)_{i\in m}$, that, for every $i \in m$, $\varphi(s_i)$ is the word $(t_{i,j})_{j\in n_i}$, and that $\varphi(s)$ is the word $(t_k)_{k\in p}$, then we have that $\varphi^\sharp(w)$ is the word $(t_{0,0}, \ldots, t_{0,n_0-1}, \ldots, t_{m-1,0}, \ldots, t_{m-1,n_{m-1}-1})$ and that $d(\sigma)\colon \varphi^\sharp(w) \longrightarrow \varphi(s)$ is a family of terms $P = (P_0, \ldots, P_{p-1})$ such that, for every $k \in p$, $P_k\colon \varphi^\sharp(w) \longrightarrow t_k$. Therefore $G^\sharp_{\varphi^\sharp \times \varphi^\sharp}(P)$, the realization of $d(\sigma)$ in $\mathbf{B}$, is precisely the term operation $P^{\mathbf{B}} = \langle P^{\mathbf{B}}_0, \ldots, P^{\mathbf{B}}_{p-1}\rangle$, of type

$$B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}} \times \cdots \times B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}} \longrightarrow B_{t_0} \times \cdots \times B_{t_{p-1}},$$

that by composition with the isomorphism from B_{φ_w} to $B_{\varphi^\sharp(w)}$ provides the operation $G^{\mathbf{d}}_\sigma$

$$\begin{array}{c}
(B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}}) \times \cdots \times (B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}}) \\
\Big\downarrow \iota^B_{(\varphi(s_0),\ldots,\varphi(s_{m-1}))} \\
B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}} \times \cdots \times B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}} \\
\Big\downarrow P^{\mathbf{B}} \\
B_{t_0} \times \cdots \times B_{t_{p-1}}
\end{array}$$

It is now when we can state that the contravariant functor Alg from **Sig** to **Cat**, defined in the second section, can be lifted to a contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to **Cat**.

Proposition 5.15. *There exists a contravariant pseudo-functor* $\mathrm{Alg}_{\mathfrak{pd}}$ *from* $\mathbf{Sig}_{\mathfrak{pd}}$ *to the* 2*-category* **Cat** *given by the following data: its object mapping sends each signature* $\mathbf{\Sigma}$ *to* $\mathrm{Alg}_{\mathfrak{pd}}(\mathbf{\Sigma}) = \mathbf{Alg}(\mathbf{\Sigma})$*; its arrow mapping sends each polyderivor* $\mathbf{d}$ *from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ *to* $\mathbf{d}^*_{\mathfrak{pd}}\colon \mathbf{Alg}(\mathbf{\Lambda}) \longrightarrow \mathbf{Alg}(\mathbf{\Sigma})$*; for every* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *and* $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$*, the natural isomorphism* $\gamma^{\mathbf{d},\mathbf{e}}$ *from* $\mathbf{e}^*_{\mathfrak{pd}} \circ \mathbf{d}^*_{\mathfrak{pd}}$ *to* $(\mathbf{e} \circ \mathbf{d})^*_{\mathfrak{pd}}$ *is that which is defined, for every* $\mathbf{\Omega}$*-algebra* $\mathbf{C}$*, as the isomorphism* $\iota^C_{\psi^\star \circ \varphi}$*; for every* $\mathbf{\Sigma}$*, the natural isomorphism* $\nu^{\mathbf{\Sigma}}$ *from* $\mathrm{Id}_{\mathbf{Alg}(\mathbf{\Sigma})}$ *to* $(\between_S, \eta^{\mathrm{B}_S}_{\Sigma})^*_{\mathfrak{pd}}$ *is that which is defined, for every* $\mathbf{\Sigma}$*-algebra* $\mathbf{A}$*, as the canonical isomorphism* $\delta^A_S\colon A \longrightarrow (A_{(s)})_{s\in S}$.

By applying the EG-construction to the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ we get the category $\mathbf{Alg}_{\mathfrak{pd}}$.

Definition 5.16. The category $\mathbf{Alg}_{\mathfrak{pd}}$ is given by $\mathbf{Alg}_{\mathfrak{pd}} = \int^{\mathbf{Sig}_{\mathfrak{pd}}} \mathrm{Alg}_{\mathfrak{pd}}$. Therefore the category $\mathbf{Alg}_{\mathfrak{pd}}$ has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, with $\mathbf{\Sigma}$ a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs $(\mathbf{d}, h)$, with $\mathbf{d}$ a polyderivor from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and h a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$.

Example. Let $\mathbf{\Sigma}$ be a signature, $p \in \mathbb{N}$, and $\mathbf{d} = (\varphi, d)$ the endopolyderivor of $\mathbf{\Sigma}$, where $\varphi\colon S \longrightarrow S^\star$ is the mapping which sends $s \in S$ to the word $\curlywedge_{\mu\in p}(s)$ and, for $(w, s) \in S^\star \times S$, $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$ which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_p^{w_1}, \ldots, v_{(|w|-1)p}^{w_{|w|-1}}), \ldots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \ldots, v_{|w|p-1}^{w_{|w|-1}})),$$

in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$. Then, for the polyderivor $\mathbf{d}$ and two $\mathbf{\Sigma}$-algebras $\mathbf{A}$ and $\mathbf{B}$, we have that $(\mathbf{d}, \langle h^\mu\rangle_{\mu\in p})$, where, for every $\mu \in p$, $h^\mu = (h^\mu_s)_{s\in S}$ is a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$, is a morphism from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Sigma}, \mathbf{B})$, because $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B}) = \mathbf{B}^p$.

Additional examples related to computer sciences can be found, e.g., in [13].

We define next some auxiliary functors and natural transformations that we will use afterwards to prove, on the one hand, that there exists a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from the category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, which generalizes the pseudo-functor Ter from the category $\mathbf{Sig}$ to the 2-category $\mathbf{Cat}$, and, on the other hand, that the category $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts.

Proposition 5.17. *Let S be a set of sorts. Then we have that*

1. *There exists a* compression *functor $(\cdot)^{\dagger_S}$ from $\mathbf{Set}^{S^\star}$ to $\mathbf{Set}^S$, left adjoint to the expansion functor $(\cdot)^{\natural_S}$; its object mapping is defined as $C_s^{\dagger_S} = \bigcup_{w\in S^\star \;\&\; w^{-1}[s]\neq\varnothing} (C_w\times\{w\}\times w^{-1}[s])$, for each $S^\star$-sorted set C and $s \in S$; its arrow mapping is defined, for each $S^\star$-mapping $f\colon C \longrightarrow C'$, $s \in S$, and (c, w, i) in $C_s^{\dagger_S}$, as $f_s^{\dagger_S}(c, w, i) = (f_w(c), w, i)$.*
2. *From the contravariant functor $\mathrm{MSet} \circ \mathrm{T}_\star^{\mathrm{op}}$, from $\mathbf{Set}$ to $\mathbf{Cat}$, to the contravariant functor MSet between the same categories, there exists a natural transformation $(\cdot)^\dagger$ which sends a set S to the compression functor $(\cdot)^{\dagger_S}$ from $\mathbf{Set}^{S^\star}$ to $\mathbf{Set}^S$.*

3. *There exists a natural isomorphism* ζ_S *from the functor* $(\cdot)^{\dagger_S} \circ (\cdot)^{\dagger_{S^\star}}$ *to the functor* $(\cdot)^{\dagger_S} \circ \coprod_{\curlywedge_S}$.

If $\varphi\colon S \longrightarrow T^\star$ is a mapping, then from the adjunctions $\coprod_\varphi \dashv \Delta_\varphi$ and $(\cdot)^{\dagger_T} \dashv (\cdot)^{\natural_T}$, we get the adjunction $\coprod_\varphi^\dagger \dashv \Delta_\varphi^\natural$ where, to simplify notation, we let $\coprod_\varphi^\dagger$ stand for $(\cdot)^{\dagger_T} \circ \coprod_\varphi$ and let $\Delta_\varphi^\natural$ stand for $\Delta_\varphi \circ (\cdot)^{\natural_T}$. Furthermore, we write $\theta_\varphi^{\dagger\natural}$, $\eta_\varphi^{\dagger\natural}$, and $\varepsilon_\varphi^{\dagger\natural}$, respectively, for the natural isomorphism, the unit, and the counit of this composite adjunction.

What we want to establish now is that $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts and for this we begin by proving that, for every polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, the functor $\mathbf{d}^*_{\mathfrak{pd}}$ from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$ has a left adjoint $\mathbf{d}_*^{\mathfrak{pd}}$.

Proposition 5.18. *Let* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *be a polyderivor. Then there exists a functor* $\mathbf{d}_*^{\mathfrak{pd}}$ *from* $\mathbf{Alg}(\mathbf{\Sigma})$ *to* $\mathbf{Alg}(\mathbf{\Lambda})$ *that is left adjoint to the functor* $\mathbf{d}^*_{\mathfrak{pd}}$ *from* $\mathbf{Alg}(\mathbf{\Lambda})$ *to* $\mathbf{Alg}(\mathbf{\Sigma})$.

Proof. We restrict ourselves to define the action of $\mathbf{d}_*^{\mathfrak{pd}}$ on the objects since the verification of the remaining parts is straightforward. Let $\mathbf{A}$ be a $\mathbf{\Sigma}$-algebra. Then $\mathbf{d}_*^{\mathfrak{pd}}(\mathbf{A})$ is the $\mathbf{\Lambda}$-algebra defined as $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi^\dagger A)/\overline{R}^{\mathbf{A}}$, where $\overline{R}^{\mathbf{A}}$ is the congruence on $\mathbf{T}_{\mathbf{\Lambda}}(\coprod_\varphi^\dagger A)$ generated by the T-sorted relation $R^{\mathbf{A}}$, defined, for every $t \in T$, as

$$R_t^{\mathbf{A}} = \left\{ \big((F_\sigma^{\mathbf{A}}(a_i \mid i \in |w|), s, \varphi(s), j), d(\sigma)_j(\mathbf{a})\big) \;\middle|\; \begin{array}{c} j \in \varphi(s)^{-1}[t],\, w \in S^\star, \\ s \in S,\, \sigma \in \Sigma_{w,s},\, a \in A_w \end{array} \right\},$$

$\mathbf{a}$ being the matrix

$$\mathbf{a} = \begin{pmatrix} {\scriptstyle (a_0, w_0, \varphi(w_0), 0)} & \cdots & {\scriptstyle (a_0, w_0, \varphi(w_0), |\varphi(w_0)|-1)} \\ \vdots & \ddots & \vdots \\ {\scriptstyle (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), 0)} & \cdots & {\scriptstyle (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), |\varphi(w_{|w|-1})|-1)} \end{pmatrix},$$

and $d(\sigma)_j(\mathbf{a})$ the result of replacing the variables in the term $d(\sigma)_j$ with the entries in the matrix $\mathbf{a}$ (recall that, for $\sigma \in \Sigma_{w,s}$, we have agreed that $d(\sigma) = d_{w,s}(\sigma)$, where $d_{w,s}(\sigma) \in \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)}$, hence, for every $j \in |\varphi(s)|$, $d(\sigma)_j \in \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^\sharp(w))_{\varphi(s)_j}$). $\square$

Proposition 5.19. *The category* $\mathbf{Alg}_{\mathfrak{pd}}$ *has coproducts.*

Proof. The category $\mathbf{Sig}_{\mathfrak{pd}}$ has coproducts. For every signature $\mathbf{\Sigma}$, the category $\mathbf{Alg}(\mathbf{\Sigma})$ has coproducts. The functor $\mathrm{Alg}_{\mathfrak{pd}}$ is locally reversible.

Therefore, by a particular case of Theorem 2, pp. 250–251, in [25], the category $\mathbf{Alg}_{\mathfrak{pd}}$ has coproducts. $\square$

Our next goal is to state that every polyderivor induces a functor between the associated categories of terms as was the case for the signature morphisms.

Proposition 5.20. *Let $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ be a polyderivor. Then there exists a functor $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ from $\mathbf{Ter}(\boldsymbol{\Sigma})$ to $\mathbf{Ter}(\boldsymbol{\Lambda})$. Its object mapping assigns to each S-sorted set X the T-sorted set $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(X) = \coprod^{\dagger}_{\varphi} X$; its morphism mapping assigns to each morphism P from X to Y in $\mathbf{Ter}(\boldsymbol{\Sigma})$ the morphism $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P) = (\theta^{\dagger\natural}_{\varphi})^{-1}(\eta^{\mathbf{d}}_{X} \circ P)$ from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\varphi} Y$, where $\theta^{\dagger\natural}_{\varphi}$ is the natural isomorphism of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$, $\eta^{\mathbf{d}}_{X}$ the $\boldsymbol{\Sigma}$-homomorphism from $\mathbf{T}_{\boldsymbol{\Sigma}}(X)$ to $\Delta^{\natural}_{\varphi}(\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X))$ that extends the S-sorted mapping $\Delta^{\natural}_{\varphi}(\eta_{\coprod^{\dagger}_{\varphi} X}) \circ (\eta^{\dagger\natural}_{\varphi})_X$ from X to $\Delta^{\natural}_{\varphi}(\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X))$, and $\eta^{\dagger\natural}_{\varphi}$ the unit of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$.*

Proof. The proof is founded on the fact that for every term $P\colon X \longrightarrow Y$ the term $\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)\colon \coprod^{\dagger}_{\varphi} X \longrightarrow \coprod^{\dagger}_{\varphi} Y$ is the composition of the morphisms in the following diagram

$$
\begin{array}{ccccc}
\coprod^{\dagger}_{\varphi} Y & \xrightarrow{\ \coprod^{\dagger}_{\varphi} P\ } & \coprod^{\dagger}_{\varphi} \mathrm{T}_{\boldsymbol{\Sigma}}(X) & \xrightarrow{\ \coprod^{\dagger}_{\varphi} \eta^{\mathbf{d}}_{X}\ } & \coprod^{\dagger}_{\varphi} \Delta^{\natural}_{\varphi}(\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X)) \\
 & \searrow^{\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)} & & & \Big\downarrow (\varepsilon^{\dagger\natural}_{\varphi})_{\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X)} \\
 & & & & \mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X)
\end{array}
$$

where $(\varepsilon^{\dagger\natural}_{\varphi})_{\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X)}$ is the value at $\mathrm{T}_{\boldsymbol{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$ of the counit of the adjunction $\coprod^{\dagger}_{\varphi} \dashv \Delta^{\natural}_{\varphi}$. $\square$

Before we state that the above construction can be lifted to a pseudofunctor from the category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, we point out that the relation of satisfaction is also invariant under polyderivor change, i.e., that for every polyderivor $\mathbf{d}\colon \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$, if (P, Q) is a $\boldsymbol{\Sigma}$-equation of type (X, Y) and $\mathbf{A}$ a $\boldsymbol{\Lambda}$-algebra, then

$$\mathbf{d}^{*}_{\mathfrak{pd}}(\mathbf{A}) \models^{\boldsymbol{\Sigma}}_{X,Y} (P, Q) \text{ if and only if } \mathbf{A} \models^{\boldsymbol{\Lambda}}_{\coprod^{\dagger}_{\varphi} X, \coprod^{\dagger}_{\varphi} Y} (\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P), \mathbf{d}^{\mathfrak{pd}}_{\diamond}(Q)).$$

This follows from the invariant character under signature change through the polyderivors of the realization of terms as term operations in arbitrary, but fixed, algebras.

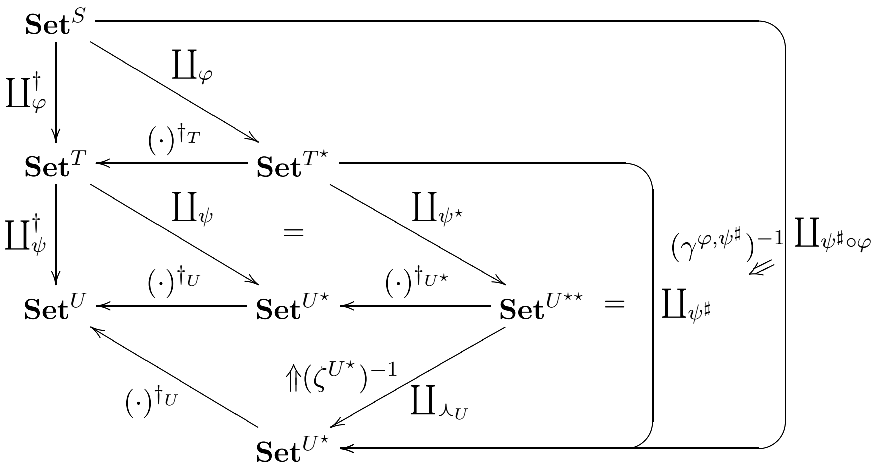
It is now when we can properly state that the pseudo-functor Ter from **Sig** to the 2-category **Cat**, defined in the second section, can be lifted to a pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category **Cat**.

Proposition 5.21. *There exists a pseudo-functor* $\mathrm{Ter}_{\mathfrak{pd}}$ *from* $\mathbf{Sig}_{\mathfrak{pd}}$ *to the* 2*-category* **Cat** *given by the following data*

1. *The object mapping of* $\mathrm{Ter}_{\mathfrak{pd}}$ *is that which sends* $\mathbf{\Sigma}$ *in* $\mathbf{Sig}_{\mathfrak{pd}}$ *to* $\mathrm{Ter}_{\mathfrak{pd}}(\mathbf{\Sigma}) = \mathbf{Ter}(\mathbf{\Sigma})$.
2. *The morphism mapping of* $\mathrm{Ter}_{\mathfrak{pd}}$ *is that which sends* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *in* $\mathbf{Sig}_{\mathfrak{pd}}$ *to* $\mathbf{d}_{\diamond}^{\mathfrak{pd}}\colon \mathbf{Ter}(\mathbf{\Sigma}) \longrightarrow \mathbf{Ter}(\mathbf{\Lambda})$.
3. *For* $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ *and* $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$, *the natural isomorphism* $\gamma^{\mathbf{d},\mathbf{e}}$ *from the composite* $\mathbf{e}_{\diamond}^{\mathfrak{pd}} \circ \mathbf{d}_{\diamond}^{\mathfrak{pd}}$ *to* $(\mathbf{e} \circ \mathbf{d})_{\diamond}^{\mathfrak{pd}}$ *is that which is defined, for every* S*-sorted set* X, *as the isomorphism* $\gamma_X^{\mathbf{d},\mathbf{e}}\colon \coprod_{\psi}^{\dagger} \coprod_{\varphi}^{\dagger} X \longrightarrow \coprod_{\psi^{\sharp}\circ\varphi}^{\dagger} X$ *in* $\mathbf{Ter}(\mathbf{\Omega})$ *that corresponds to the* U*-sorted mapping*

$$\textstyle\coprod_{\psi^{\sharp}\circ\varphi}^{\dagger} X \xrightarrow{\ \rho_X\ } \coprod_{\psi}^{\dagger} \coprod_{\varphi}^{\dagger} X \xrightarrow{\ \eta_{\coprod_{\psi}^{\dagger} \coprod_{\varphi}^{\dagger} X}\ } \mathrm{T}_{\mathbf{\Omega}}(\coprod_{\psi}^{\dagger} \coprod_{\varphi}^{\dagger} X),$$

where ρ *is the isomorphism obtained from the following diagram*



and γ *the isomorphism associated to the pseudo-functor* $\mathrm{MSet}^{\coprod}$.

4. For $\mathbf{\Sigma}$*, the natural isomorphism* $\nu^{\mathbf{\Sigma}}$ *from* $\mathrm{Id}_{\mathbf{Ter}(\mathbf{\Sigma})}$ *to* $(\between_S, \eta_{\Sigma}^{\mathrm{B}_S})_{\diamond}^{\mathfrak{pd}}$ *is that which is defined, for an* S*-sorted set* X*, as the isomorphism* $\nu_X^{\mathbf{\Sigma}}$ *from* X *to* $\coprod_{\between_S}^{\dagger} X$ *that corresponds to the* S*-sorted mapping* $\eta_X \circ \tau_X^S$ *from* $\coprod_{\between_S}^{\dagger} X$ *to* $\mathrm{T}_{\mathbf{\Omega}}(X)$*, where* τ^S *is the natural isomorphism from* $(\cdot)^{\dagger_S} \circ \coprod_{\between_S}$ *to* $\mathrm{Id}_{\mathbf{Set}^S}$ *defined, for an* S*-sorted set* X*, as the* S*-sorted mapping whose* s*-th coordinate, for* $s \in S$*, sends an* $((a,s),(s),0) \in (\coprod_{\between_S}^{\dagger} X)_s$ *to* $(\tau_X^S)_s((a,s),(s),0) = a$*.*

To finalize this section we notice that the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\mathbf{\Sigma}})_{\mathbf{\Sigma} \in \mathbf{Sig}_{\mathfrak{pd}}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Sig}_{\mathfrak{pd}})}$, with $\theta_{\mathbf{A},X}^{\mathbf{d}} = \theta_{X,A}^{\dagger\natural}$, is a pseudo-extranatural transformation from the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ to the functor $\mathrm{K}_{\mathbf{Set}}$.

6. Transformations of Fujiwara.

We recall that one of the aims of this article is to prove the equivalence between many-sorted clones and many-sorted algebraic theories. But to attain such an aim it is necessary to begin by defining a convenient 2-category of specifications $\mathbf{Spf}_{\mathfrak{pd}}$, and to get it, and this we will do in the section following this one, we need to state several concepts and operations. On the one hand, the concept of morphism from a specification into another, on the other hand, for two morphisms $\mathbf{d}$, $\mathbf{e}$ from a specification $(\mathbf{\Sigma}, \mathcal{E})$ to another $(\mathbf{\Lambda}, \mathcal{H})$, the concept of transformation from $\mathbf{d}$ to $\mathbf{e}$, and, finally, operations of vertical and horizontal compositions for these transformations. However, to succeed in obtaining the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ we should begin, as we do in this section, by transforming the category $\mathbf{Sig}_{\mathfrak{pd}}$ into a 2-category by adding to it as 2-cells the adequate transformation between polyderivors.

The transformations between polyderivors that we define below are a first step in the process of generalization (to the many-sorted case) of the concept of transformation between families of basic mapping-formulas, as stated by Fujiwara in [9]. And this is so because the polyderivors being, simply, morphisms from a signature into another, and not morphisms between specifications (which, in addition to signatures, include also equations), they will only satisfy, for every formal operation, a strict equation, instead of an equation modulus a set of equations for the target signature as it is the case in [9]. However, after defining, in a second step, in the last section of this article, the adequate morphisms between specifications, called there $\mathfrak{pd}$-specification morphisms (through the polyderivors between the underlying signatures of the specifications), we get the full generalization of the theory of Fujiwara in [9]. Moreover, by adding to the specifications and the $\mathfrak{pd}$-specification morphisms the so-called transformations between $\mathfrak{pd}$-specification morphisms, we obtain a 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ as announced at the beginning of this section.

In this section we also prove that the transformations between polyderivors determine natural transformations between the functors associated to the polyderivors. This fact allows us to lift: (1) the contravariant pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, to a 2-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, contravariant in the morphisms and covariant in the 2-cells, and hence to get a 2-category $\mathbf{Alg}_{\mathfrak{pd}}$, and (2) the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$ to a 2-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$, covariant in the morphisms and the 2-cells. We notice that the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$, will be used in the last section to prove the equivalence between many-sorted clones and many-sorted algebraic theories.

In order to define and investigate the transformations between polyderivors it is convenient to make use of some derived operations in the Bénabou algebras of terms for the different signatures, concretely of those in the following definition.

Definition 6.1. Let S be a set of sorts.

1. For every $\overline{w} \in S^{\star\star}$ and $\alpha \in |\overline{w}|$, let $\pi^{\overline{w}}_{\alpha}$ be the derived operation of type $\lambda \longrightarrow (\curlywedge\overline{w}, \overline{w}_{\alpha})$ defined as $\langle \pi^{\curlywedge\overline{w}}_{\sum_{\beta\in\alpha} p_\beta}, \dots, \pi^{\curlywedge\overline{w}}_{\sum_{\beta\in\alpha+1} p_\beta - 1} \rangle_{\curlywedge\overline{w}, \overline{w}_\alpha}$, where $\overline{w}$ is of the form
$$((\cdot, \dots, \cdot), \dots, \overbrace{(\underset{\sum_{\beta\in\alpha} p_\beta}{\cdot}, \dots, \underset{\sum_{\beta\in\alpha+1} p_\beta - 1}{\cdot})}^{\overline{w}_\alpha}, \dots, (\cdot, \dots, \cdot)),$$
and, for every $\alpha \in |\overline{w}|$, $p_\alpha = |\overline{w}_\alpha|$.

2. For every $u \in S^{\star}$ and $\overline{w} \in S^{\star\star}$, let $\langle\,\rangle_{u,\overline{w}}$ be the derived operation of type $((u, \overline{w}_0), \dots, (u, \overline{w}_{|\overline{w}|-1})) \longrightarrow (u, \curlywedge\overline{w})$ defined as
$$\langle P_0, \dots, P_{|\overline{w}|-1} \rangle_{u,\overline{w}} = \langle \pi^{\overline{w}_0}_0 \circ P_0, \dots, \pi^{\overline{w}_0}_{|\overline{w}_0|-1} \circ P_0, \dots, \pi^{\overline{w}_{|\overline{w}|-1}}_0 \circ P_{|\overline{w}|-1}, \dots, \pi^{\overline{w}_{|\overline{w}|-1}}_{|\overline{w}_{|\overline{w}|-1}|-1} \circ P_{|\overline{w}|-1} \rangle_{u,\curlywedge\overline{w}}.$$

3. For every $n \in \mathbb{N}$, and $\overline{u}$, $\overline{w} \in S^{\star n}$, let $\curlywedge_{\overline{u},\overline{w}}$ be the derived operation of type $((\overline{u}_0, \overline{w}_0), \ldots, (\overline{u}_{n-1}, \overline{w}_{n-1})) \longrightarrow (\curlywedge\overline{u}, \curlywedge\overline{w})$ defined as

$$\curlywedge_{\overline{u},\overline{w}}(P_0, \ldots, P_{n-1}) = \langle P_0 \circ \pi^{\overline{u}}_0, \ldots, P_{n-1} \circ \pi^{\overline{u}}_{n-1}\rangle_{\curlywedge\overline{u},\overline{w}}.$$

Henceforth, to simplify notation, we will omit some subscripts in the expressions. Moreover, for the operations of the form $\curlywedge_{\overline{u},\overline{w}}$ we adopt the infix notation, and we will write $P_0 \curlywedge \cdots \curlywedge P_{n-1}$ instead of $\curlywedge_{\overline{u},\overline{w}}(P_0, \ldots, P_{n-1})$, the type, in its turn, will be $\overline{u}_0 \curlywedge \cdots \curlywedge \overline{u}_{n-1} \longrightarrow \overline{w}_0 \curlywedge \cdots \curlywedge \overline{w}_{n-1}$.

For the algebras of terms $\mathbf{BT}_S(\Sigma)$, the operations $\curlywedge_{\overline{u},\overline{w}}$ are, essentially, the result of gathering into a family the corresponding terms, relabeling adequately the variables.

Recalling that the Bénabou algebras are, up to isomorphism, the finitary many-sorted algebraic theories of Bénabou (see Proposition 4.5) we will represent, henceforth, the composition of terms diagrammatically, and the equality of two coterminal paths composed of terms by asserting the commutativity of the appropriate diagram.

Definition 6.2. Let $\mathbf{d}$ and $\mathbf{e}$ be polyderivors from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$. A *transformation from* $\mathbf{d}$ *to* $\mathbf{e}$ is a choice function ξ for

$$(\mathrm{BT}_T(\Lambda)_{\varphi(s),\psi(s)})_{s\in S} = (\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))_{\psi(s)})_{s\in S},$$

such that, for every operation $\sigma\colon w \longrightarrow s$, the following diagram commutes

$$\begin{array}{ccc}
1 & \xrightarrow{\langle \xi_s, d(\sigma)\rangle} & \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))_{\psi(s)} \times \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\varphi(s)} \\
{\scriptstyle \langle e(\sigma), \xi_w\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \circ} \\
\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\psi^{\sharp}(w))_{\psi(s)} \times \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\psi^{\sharp}(w)} & \xrightarrow[\circ]{} & \mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi^{\sharp}(w))_{\psi(s)}
\end{array}$$

or more briefly, such that $\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w$, where ξ_w is $\xi_{w_0} \curlywedge \cdots \curlywedge \xi_{w_{|w|-1}}$. From now on, we write $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ to denote that ξ is a transformation from $\mathbf{d}$ to $\mathbf{e}$.

Example. Let $\mathbf{\Sigma}$ be a signature, $p, q \in \mathbb{N}$, $\mathbf{d} = (\varphi, d)$ the endopolyderivor of $\mathbf{\Sigma}$, where $\varphi\colon S \longrightarrow S^{\star}$ is the mapping which sends $s \in S$ to the word $\curlywedge_{\mu\in p}(s)$ and, for $(w,s)\in S^\star\times S$, $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi^\sharp(w))^p_s$ which sends $\sigma\in\Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0},v_p^{w_1},\ldots,v_{(|w|-1)p}^{w_{|w|-1}}),\ldots,\sigma(v_{p-1}^{w_0},v_{2p-1}^{w_1},\ldots,v_{|w|p-1}^{w_{|w|-1}})),$$

and $\mathbf{e}=(\psi,e)$ the endopolyderivor of $\mathbf{\Sigma}$, where $\psi\colon S\longrightarrow S^\star$ is the mapping which sends $s\in S$ to the word $\curlywedge_{\nu\in q}(s)$ and, for $(w,s)\in S^\star\times S$, $e_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\psi^\sharp(w))^q_s$ which sends $\sigma\in\Sigma_{w,s}$ to $(\sigma(v_0^{w_0},v_q^{w_1},\ldots,v_{(|w|-1)q}^{w_{|w|-1}}),\ldots,\sigma(v_{q-1}^{w_0},v_{2q-1}^{w_1},\ldots,v_{|w|q-1}^{w_{|w|-1}}))$. Then, for an arbitrary, but fixed, mapping $f=(f(\nu))_{\nu\in q}$ from the natural number q to the natural number p, taking as ξ the element of $\prod_{s\in S}\mathrm{T}_{\mathbf{\Lambda}}(\downarrow\varphi(s))^q_s$ defined, for every $s\in S$, as $\xi_s=(v^s_{f(0)},\ldots,v^s_{f(q-1)})$, where, to simplify notation, we have identified the variables in $\downarrow\varphi(s)$ with their images in $\mathrm{T}_{\mathbf{\Sigma}}(\downarrow\varphi(s))$ under $\eta_{\downarrow\varphi(s)}$, we have that ξ is a transformation from $\mathbf{d}$ to $\mathbf{e}$. We point out that the working out of all the details of this example, even if a little troublesome, helps to grasp the functioning of the polyderivors and the transformations between them.

For more examples of transformations between polyderivors we refer to the last section of this article.

The commutativity condition in the above definition of transformation from a polyderivor into another can be extended to the terms, as proved in the following proposition.

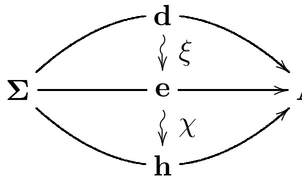
Proposition 6.3. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be polyderivors from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$ *and* $\xi\colon\mathbf{d}\rightsquigarrow\mathbf{e}$ *a transformation. Then* $\xi_w\circ d^\sharp(P)=e^\sharp(P)\circ\xi_u$*, for each term* $P\colon u\longrightarrow w$ *in* $\mathrm{BT}_S(\Sigma)$.

Proof. By algebraic induction in the Bénabou algebra $\mathrm{BT}_S(\Sigma)$. □

What we want now is to endow the category $\mathbf{Sig}_{\mathfrak{pd}}$ with a 2-category structure. For this we provide in the following proposition the definitions of the horizontal and vertical composition of the transformations between polyderivors, prove the law of Godement, and define the identity transformations at the polyderivors.

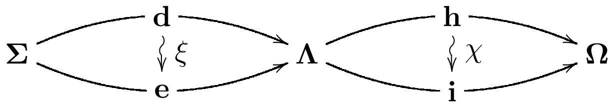
Proposition 6.4. *The signatures together with the polyderivors and the transformations between the polyderivors have a 2-category structure, denoted as* $\mathbf{Sig}_{\mathfrak{pd}}$.

Proof. *Definition of the vertical composition.* Given the configuration



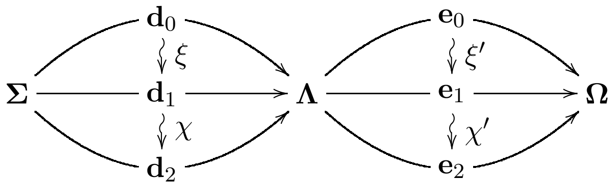
the vertical composition of ξ and χ, defined as $\chi \circ \xi = (\chi_s \circ \xi_s)_{s\in S}$, is a transformation from $\mathbf{d}$ to $\mathbf{h}$.

Definition of the horizontal composition. Given the configuration



the horizontal composition of ξ and χ, defined as $\chi * \xi = (\chi_{\psi(s)} \circ h^\sharp(\xi_s))_{s\in S}$, or, what is equivalent, as $(i^\sharp(\xi_s) \circ \chi_{\varphi(s)})_{s\in S}$, is a transformation from $\mathbf{h}\circ\mathbf{d}$ to $\mathbf{i}\circ\mathbf{e}$. We have to show that $\chi * \xi$ is a transformation from $(\gamma^\sharp \circ \varphi, h^\sharp_{\varphi^\sharp\times\varphi^\sharp} \circ d)$ to $(\nu^\sharp \circ \psi, i^\sharp_{\psi^\sharp\times\psi^\sharp} \circ e)$, i.e., that, for every $\sigma\colon w \longrightarrow s$, we have that the following equation holds $(\chi * \xi)_s \circ h^\sharp(d(\sigma)) = i^\sharp(e(\sigma)) \circ (\chi * \xi)_w$. But this happens since ξ, χ are transformations and $h^\sharp$, $i^\sharp$ morphisms.

Law of Godement. Given the configuration



we have, after the definitions of the vertical and horizontal compositions, that

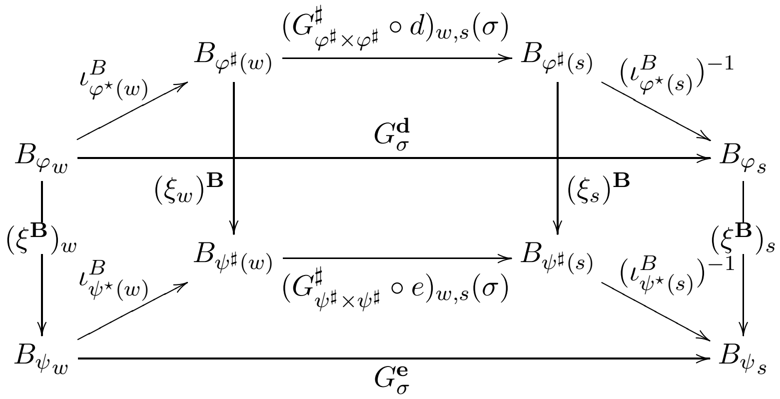
$$(\chi' * \chi) \circ (\xi' * \xi) = (\chi' \circ \xi') * (\chi \circ \xi).$$

Identities. Finally, given polyderivor $\mathbf{d}\colon \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$ and $\mathbf{e}\colon \mathbf{\Lambda} \longrightarrow \mathbf{\Omega}$ it is obvious that the S-family $(\langle \pi_0^{\varphi(s)}, \ldots, \pi_{|\varphi(s)|-1}^{\varphi(s)} \rangle_{\varphi(s),\varphi(s)})_{s\in S}$, denoted by $\mathrm{id}_{\mathbf{d}}$, is the identity transformation at $\mathbf{d}$, and that $\mathrm{id}_{\mathbf{e}} * \mathrm{id}_{\mathbf{d}} = \mathrm{id}_{\mathbf{e}\circ\mathbf{d}}$. □

Our next goal is to prove that the transformations between polyderivors from a signature into another, determine natural transformations between the functors between the categories of algebras associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivor $\mathbf{d}$ to another $\mathbf{e}$, both from a signature $\boldsymbol{\Sigma}$ to a signature $\boldsymbol{\Lambda}$, determines, for a given $\boldsymbol{\Lambda}$-algebra $\mathbf{B}$, a $\boldsymbol{\Sigma}$-homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$.

Proposition 6.5. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be polyderivors from* $\boldsymbol{\Sigma}$ *to* $\boldsymbol{\Lambda}$, $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *a transformation in* $\mathbf{Sig}_{\mathfrak{pd}}$, *and, for a* $\boldsymbol{\Lambda}$-*algebra* $\mathbf{B} = (B, G)$, *let* $\xi^{\mathbf{B}}$ *be the* S-*sorted mapping* $(\xi^{\mathbf{B}}_s)_{s\in S}$ *from* B_φ *to* B_ψ, *where we have that, for each* $s \in S$, $\xi^{\mathbf{B}}_s = G^\sharp_{\varphi(s),\psi(s)}(\xi_s)\colon B_{\varphi(s)} \longrightarrow B_{\psi(s)}$. *Then* $\xi^{\mathbf{B}}$ *is a* $\boldsymbol{\Sigma}$-*homomorphism from* $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ *to* $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$.

Proof. For every operation $\sigma\colon w \longrightarrow s$, in Σ, we have to prove that $G^{\mathbf{e}}_\sigma \circ \xi^{\mathbf{B}}_w = \xi^{\mathbf{B}}_s \circ G^{\mathbf{d}}_\sigma$, and for this it is enough to prove that every face, except at most the frontal one, in the following diagram commutes



from which it follows, necessarily, that the frontal face also commutes.

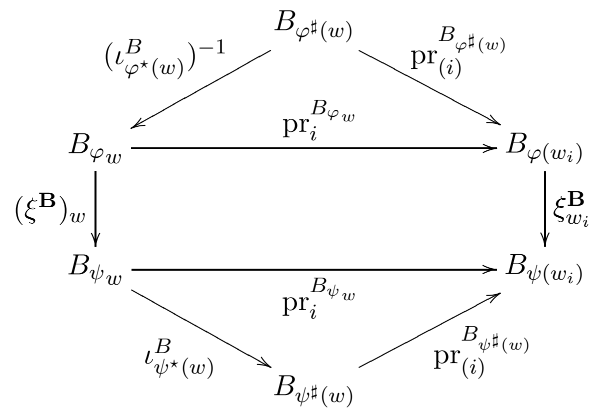
The top and bottom faces commute by definition. The back face commutes because, ξ being a transformation from $\mathbf{d}$ to $\mathbf{e}$, from $\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w$ it follows that

$$
\begin{aligned}
(\xi_s)^{\mathbf{B}} \circ (G^\sharp_{\varphi^\sharp\times\varphi^\sharp} \circ d)_{w,s}(\sigma) &= G^\sharp_{\varphi(s),\psi(s)}(\xi_s) \circ G^\sharp_{\varphi^\sharp(w),\varphi(s)}(d_{w,s}(\sigma))\\
&= G^\sharp_{\varphi^\sharp(w),\psi(s)}(\xi_s \circ d_{w,s}(\sigma))\\
&= G^\sharp_{\varphi^\sharp(w),\psi(s)}(e_{w,s}(\sigma) \circ \xi_w)\\
&= G^\sharp_{\psi^\sharp(w),\psi(s)}(e_{w,s}(\sigma)) \circ G^\sharp_{\varphi^\sharp(w),\psi^\sharp(w)}(\xi_w)\\
&= (G^\sharp_{\psi^\sharp\times\psi^\sharp} \circ e)_{w,s}(\sigma) \circ (\xi_w)^{\mathbf{B}}.
\end{aligned}
$$

Related to the lateral faces, let us verify, e.g., that the left one commutes. For this it suffices to prove that $(\xi_w)^{\mathbf{B}} = \iota^B_{\psi^\star(w)} \circ \xi^{\mathbf{B}}_w \circ (\iota^B_{\varphi^\star(w)})^{-1}$. But we have that

$$\begin{aligned}
(\xi_w)^{\mathbf{B}} &= G^\sharp_{\varphi^\sharp(w),\psi^\sharp(w)}(\xi_{w_0} \curlywedge \cdots \curlywedge \xi_{w_{|w|-1}}) \\
&= G^\sharp_{\varphi(w_0),\psi(w_0)}(\xi_{w_0}) \curlywedge \cdots \curlywedge G^\sharp_{\varphi(w_{|w|-1},\psi(w_{|w|-1})}(\xi_{w_{|w|-1}}) \\
&= \xi^{\mathbf{B}}_{w_0} \curlywedge \cdots \curlywedge \xi^{\mathbf{B}}_{w_{|w|-1}} \\
&= \langle \xi^{\mathbf{B}}_{w_0} \circ \mathrm{pr}^{B_{\varphi^\sharp(w)}}_{(0)}, \ldots, \xi^{\mathbf{B}}_{w_{|w|-1}} \circ \mathrm{pr}^{B_{\varphi^\sharp(w)}}_{(|w|-1)} \rangle.
\end{aligned}$$

Hence it suffices to prove that, for every $i \in |w|$, the following equation holds $\mathrm{pr}^{B_{\psi^\sharp(w)}}_{(i)} \circ \iota^B_{\psi^\star(w)} \circ \xi^{\mathbf{B}}_w \circ (\iota^B_{\varphi^\star(w)})^{-1} = \xi^{\mathbf{B}}_{w_i} \circ \mathrm{pr}^{B_{\varphi^\sharp(w)}}_{(i)}$. But this follows from the commutativity of the following diagram



□

After having proved, for two polyderivors $\mathbf{d}$ and $\mathbf{e}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, that every transformation ξ from $\mathbf{d}$ to $\mathbf{e}$, induces, for every $\mathbf{\Lambda}$-algebra $\mathbf{B}$, a $\mathbf{\Sigma}$-homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}^*_{\mathfrak{pd}}(\mathbf{B})$ to $\mathbf{e}^*_{\mathfrak{pd}}(\mathbf{B})$, we prove in the following proposition the naturalness of the involved procedure.

Proposition 6.6. *Let* $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *be a transformation with* $\mathbf{d}$ *and* $\mathbf{e}$ *polyderivors from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$. *Then the family* $(\xi^{\mathbf{B}})_{\mathbf{B}\in\mathbf{Alg}(\mathbf{\Lambda})}$, *denoted by* $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$, *is a natural transformation from the functor* $\mathbf{d}^*_{\mathfrak{pd}}$ *to the functor* $\mathbf{e}^*_{\mathfrak{pd}}$, *both from* $\mathbf{Alg}(\mathbf{\Lambda})$ *to* $\mathbf{Alg}(\mathbf{\Sigma})$.

Proof. We have to prove that, for every $\mathbf{\Lambda}$-algebras $\mathbf{B} = (B, G)$, $\mathbf{C} = (C, H)$ and morphism $f\colon \mathbf{B} \longrightarrow \mathbf{C}$ in $\mathbf{Alg}(\mathbf{\Lambda})$, the $\mathbf{\Sigma}$-homomorphism $\xi^{\mathbf{C}} \circ f_\varphi$

and $f_\psi \circ \xi^{\mathbf{B}}$ from $(B_\varphi, G^{\mathbf{d}})$ to $(C_\psi, H^{\mathbf{e}})$ are identical. But this is immediate since, for every $s \in S$, $\xi_s^{\mathbf{B}}$ and $\xi_s^{\mathbf{C}}$ being the realizations of the term ξ_s in the respective algebras, the mappings $\xi_s^{\mathbf{C}} \circ f_{\varphi(s)}$ and $f_{\psi(s)} \circ \xi_s^{\mathbf{B}}$ from $B_{\varphi(s)}$ to $C_{\psi(s)}$ necessarily coincide. □

Once stated that the transformations between polyderivors from a signature into another, induce natural transformations among the functors between the categories of algebras associated to the signatures, we can properly lift the pseudo-functor $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}} \colon \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}} \longrightarrow \mathbf{Cat}$ to the 2-cells in the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$.

Proposition 6.7. *There exists a pseudo-functor* $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}$*, contravariant in the morphisms and covariant in the* 2*-cells, from the* 2*-category* $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ *to the* 2*-category* $\mathbf{Cat}$*, together with the accompanying natural isomorphisms* $\gamma^{\mathbf{d},\mathbf{e}}$ *and* $\nu^{\mathbf{\Sigma}}$*, as defined in Proposition 5.15.*

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the 2-category structure of $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$. □

On the basis of this last proposition we can lift the category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ to a 2-category as in the following definition.

Definition 6.8. We denote by $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}} = \iint^{\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}} \mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}$ the 2-category which has as objects (0-cells) the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and $\mathbf{A}$ a $\mathbf{\Sigma}$-algebra; as morphisms (1-cells) from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$ the pairs $(\mathbf{d}, f)$, where $\mathbf{d}$ is a polyderivor from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$-homomorphism from $\mathbf{A}$ to $\mathbf{d}^*_{\mathfrak{p}\mathfrak{d}}(\mathbf{B})$; and as 2-cells from $(\mathbf{d}, f)$ to $(\mathbf{e}, g)$, where $(\mathbf{d}, f)$ and $(\mathbf{e}, g)$ are morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the 2-cells $\xi \colon \mathbf{\Sigma} \rightsquigarrow \mathbf{\Lambda}$ in $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ such that $\xi^{\mathbf{B}} \circ f = g$.

As was the case above for algebras and transformations, our goal now is to prove that the transformations between polyderivors from a signature into another, also determine natural transformations between the functors between the categories of terms associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivor $\mathbf{d}$ to another one $\mathbf{e}$, both from a signature $\mathbf{\Sigma}$ to a signature $\mathbf{\Lambda}$, determines, for a given S-sorted set X, a morphism ξ_X, in the category $\mathbf{Ter}(\mathbf{\Lambda})$, from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} X$.

Proposition 6.9. *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be polyderivors from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$, $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *a transformation in* $\mathbf{Sig}_{\mathfrak{pd}}$*, and, for an* S*-sorted set* X*, let*

$$\xi_X\colon \coprod^{\dagger}_{\psi} X \longrightarrow \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$$

be the T*-sorted mapping defined, for each* $t \in T$ *and each* $(x, s, \psi(s), i) \in (\coprod^{\dagger}_{\psi} X)_t$*, as follows*

$$(\xi_X)_t(x, s, \psi(s), i) = (\xi_s)_i(v_j^{\varphi(s)_j}/(x, s, \varphi(s), j) \mid j \in |\varphi(s)|).$$

Then the mapping ξ_X *is a morphism, in the category* $\mathbf{Ter}(\mathbf{\Lambda})$*, from* $\coprod^{\dagger}_{\varphi} X$ *to* $\coprod^{\dagger}_{\psi} X$*.*

Proof. The definition of $\xi_X\colon \coprod^{\dagger}_{\psi} X \longrightarrow \mathrm{T}_{\mathbf{\Lambda}}(\coprod^{\dagger}_{\varphi} X)$ is sound since, for every j in $|\varphi(s)|$, we have that $(x, s, \varphi(s), j) \in (\coprod^{\dagger}_{\varphi} X)_{\varphi(s)_j}$ and $(\xi_s)_i \in \mathrm{T}_{\mathbf{\Lambda}}(\varphi(s))_{\psi(s)_i}$, hence $(\xi_X)_t(x, s, \varphi(s), i)$ is a term for $\mathbf{\Lambda}$ of type $\psi(s)_i = t$. □

After having proved, for two polyderivors $\mathbf{d}$ and $\mathbf{e}$ from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$, that every transformation ξ from $\mathbf{d}$ to $\mathbf{e}$, induces, for every S-sorted set X, a morphism ξ_X from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} X$, we prove in the following proposition that they are the components of a natural transformation.

Proposition 6.10. *Let* $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *be a transformation in* $\mathbf{Sig}_{\mathfrak{pd}}$*, with* $\mathbf{d}$*,* $\mathbf{e}$ *polyderivors from* $\mathbf{\Sigma}$ *to* $\mathbf{\Lambda}$*. Then* $\mathrm{Ter}_{\mathfrak{pd}}(\xi) = (\xi_X)_{X \in \mathbf{Ter}(\mathbf{\Sigma})}$ *is a natural transformation from* $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ *to* $\mathbf{e}^{\mathfrak{pd}}_{\diamond}$*.*

Proof. Because, for a morphism $P\colon X \longrightarrow Y$ in $\mathbf{Ter}(\mathbf{\Sigma})$, the T-sorted mappings $\xi_Y \circ \mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)$ and $\mathbf{e}^{\mathfrak{pd}}_{\diamond}(P) \circ \xi_X$ from $\coprod^{\dagger}_{\varphi} X$ to $\coprod^{\dagger}_{\psi} Y$ are identical. □

Let us observe that this last proposition is analogous to Proposition 6.3 but for derived operations with variables in arbitrary many-sorted sets.

Once stated that the transformations between polyderivors from a signature into another, induce natural transformations among the functors between the categories of terms associated to the signatures, we can properly lift the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}\colon \mathbf{Sig}_{\mathfrak{pd}} \longrightarrow \mathbf{Cat}$ to the 2-cells of the 2-category $\mathbf{Sig}_{\mathfrak{pd}}$.

Proposition 6.11. *There exists a pseudo-functor* $\mathrm{Ter}_{\mathfrak{pd}}$ *from the* 2*-category* $\mathbf{Sig}_{\mathfrak{pd}}$ *to* $\mathbf{Cat}$*, covariant in the morphisms and the* 2*-cells, together with the accompanying natural isomorphisms* $\gamma^{\mathbf{d},\mathbf{e}}$ *and* $\nu^{\boldsymbol{\Sigma}}$*, as defined in* Proposition 5.21*.*

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the 2-category structure of $\mathbf{Sig}_{\mathfrak{pd}}$. □

We notice that the family of functors $\mathrm{Tr} = (\mathrm{Tr}^{\boldsymbol{\Sigma}})_{\boldsymbol{\Sigma}\in\mathbf{Sig}_{\mathfrak{pd}}}$ together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d}\in\mathrm{Mor}(\mathbf{Sig}_{\mathfrak{pd}})}$, where $\theta^{\mathbf{d}}_{\mathbf{A},X} = \theta^{\dagger\natural}_{X,A}$, is a pseudo-extranatural transformation from the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}_{\mathfrak{pd}}(\cdot)$ to the functor $\mathrm{K}_{\mathbf{Set}}$.

7. Equivalence of the specifications of Hall and Bénabou.

In this section we define a 2-category of specifications, $\mathbf{Spf}_{\mathfrak{pd}}$, with objects the specifications, morphisms from a specification into another the polyderivors between the underlying signatures of the specifications that are compatible with the equations, and 2-cells from a morphism into another a convenient class of transformations between the polyderivors. In such a 2-category we prove, for every set of sorts S, the equivalence of the specifications of Hall and Bénabou for S, from which, through the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$, the equivalence between the corresponding categories of algebras, $\mathbf{Alg}(\mathrm{H}_S)$ and $\mathbf{Alg}(\mathrm{B}_S)$, is obtained as an easy corollary.

For a polyderivor $\mathbf{d}\colon \Sigma \longrightarrow \Lambda$, the functor $\mathbf{d}^{\mathfrak{pd}}_{\diamond}$ of translation from $\mathbf{Ter}(\boldsymbol{\Sigma})$ to $\mathbf{Ter}(\boldsymbol{\Lambda})$ enables us to define the concept of $\mathfrak{pd}$-specification morphism from a specification into another.

Definition 7.1. Let $(\boldsymbol{\Sigma},\mathcal{E})$ and $(\boldsymbol{\Lambda},\mathcal{H})$ be specifications. A $\mathfrak{pd}$-*specification morphism from* $(\boldsymbol{\Sigma},\mathcal{E})$ *to* $(\boldsymbol{\Lambda},\mathcal{H})$ is a polyderivor $\mathbf{d}\colon \Sigma \longrightarrow \Lambda$ such that $(\mathbf{d}^{\mathfrak{pd}}_{\diamond})^2[\mathcal{E}] \subseteq \mathrm{Cn}_{\boldsymbol{\Lambda}}(\mathcal{H})$. We denote by $\mathbf{Spf}_{\mathfrak{pd}}$ the corresponding category.

Given two $\mathfrak{pd}$-specification morphisms $\mathbf{d}$ and $\mathbf{e}$ from $(\boldsymbol{\Sigma},\mathcal{E})$ to $(\boldsymbol{\Lambda},\mathcal{H})$, since $\mathbf{d}$ and $\mathbf{e}$ are, in particular, polyderivors from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$, we have, in principle, at our disposal all the transformations $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ from $\mathbf{d}$ to $\mathbf{e}$ as potential candidates for a concept of transformation between these $\mathfrak{pd}$-specification morphisms. However, the condition of commutativity for the transformations between polyderivors is too much strict, because it requires, for every formal operation $\sigma\colon w \longrightarrow s$ in $\Sigma_{w,s}$, the strict equality $\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w$, and, actually, what could happen (and probably the most one reasonably can hope for), as was pointed out by Fujiwara in [9], is that, under the presence of equations, such a type of equation holds only modulus the congruence generated by the equations in the target specification. Therefore, for the $\mathfrak{pd}$-specification morphisms, the notion of transformation that we adopt, following the example of Fujiwara in [9], is that one where the strict equality between terms is replaced by the equality between them but relative to the congruence generated by the equations in the target specification. These transformations, in its turn, allow us to endow the category $\mathbf{Spf}_{\mathfrak{pd}}$ with a 2-category structure.

Definition 7.2. Let $\mathbf{d}$ and $\mathbf{e}\colon (\boldsymbol{\Sigma}, \mathcal{E}) \longrightarrow (\boldsymbol{\Lambda}, \mathcal{H})$ be $\mathfrak{pd}$-specification morphisms. A *transformation from* $\mathbf{d}$ *to* $\mathbf{e}$ is a choice function ξ for $(\mathrm{BT}_T(\Lambda)_{\varphi(s),\psi(s)})_{s\in S}$, such that, for every formal operation $\sigma\colon w \longrightarrow s$, we have that $\xi_s \circ d(\sigma) \equiv_{\overline{\mathcal{H}}} e(\sigma) \circ \xi_w$.

Proposition 7.3. *The specifications, the* $\mathfrak{pd}$*-specification morphisms, and the transformations between* $\mathfrak{pd}$*-specification morphisms determine a* 2*-category* $\mathbf{Spf}_{\mathfrak{pd}}$.

Before we prove that the specifications of Bénabou and Hall are equivalent in the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$, we notice that the pseudo-functor $\mathrm{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ has a lifting $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$.

Proposition 7.4. *There exists a pseudo-functor* $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ *from* $\mathbf{Spf}_{\mathfrak{pd}}$ *to* $\mathbf{Cat}$ *defined as follows*

1. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a specification* $(\boldsymbol{\Sigma}, \mathcal{E})$ *to the category* $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\boldsymbol{\Sigma}, \mathcal{E}) = \mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$ *of its models, i.e., the full subcategory of* $\mathbf{Alg}(\boldsymbol{\Sigma})$ *determined by those* $\boldsymbol{\Sigma}$*-algebras that satisfy all the equations in* $\mathcal{E}$*.*
2. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a* $\mathfrak{pd}$*-specification morphism* $\mathbf{d}$ *from* $(\boldsymbol{\Sigma}, \mathcal{E})$ *to* $(\boldsymbol{\Lambda}, \mathcal{H})$ *to the functor* $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d}) = \mathbf{d}^*_{\mathfrak{pd}}$ *from* $\mathbf{Alg}(\boldsymbol{\Lambda}, \mathcal{H})$ *to* $\mathbf{Alg}(\boldsymbol{\Sigma}, \mathcal{E})$*, obtained from the functor* $\mathbf{d}^*_{\mathfrak{pd}}$ *from* $\mathbf{Alg}(\boldsymbol{\Lambda})$ *to* $\mathbf{Alg}(\boldsymbol{\Sigma})$ *by bi-restriction.*
3. $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a transformation* $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *from* $\mathbf{d}$ *to* $\mathbf{e}$ *to the natural transformation* $\mathrm{Alg}_{\mathfrak{pd}}(\xi)$ *from* $\mathbf{d}^*_{\mathfrak{pd}}$ *to* $\mathbf{e}^*_{\mathfrak{pd}}$*.*

It is also true that the pseudo-functor $\mathrm{Ter}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to $\mathbf{Cat}$ can also be lifted to the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ as stated in the following proposition.

Proposition 7.5. *There exists a pseudo-functor* $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ *from* $\mathbf{Spf}_{\mathfrak{pd}}$ *to* $\mathbf{Cat}$ *defined as follows*

1. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a specification* $(\mathbf{\Sigma},\mathcal{E})$ *to the category* $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{\Sigma},\mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$, *where* $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$ *is the quotient category* $\mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$.
2. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a* $\mathfrak{pd}$*-specification morphism* $\mathbf{d}$ *from* $(\mathbf{\Sigma},\mathcal{E})$ *to* $(\mathbf{\Lambda},\mathcal{H})$ *to the functor* $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})$ *from the quotient category* $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E}) = \mathbf{Ter}(\mathbf{\Sigma})/\overline{\mathcal{E}}$ *to the quotient category* $\mathbf{Ter}(\mathbf{\Lambda},\mathcal{H}) = \mathbf{Ter}(\mathbf{\Lambda})/\overline{\mathcal{H}}$, *which assigns to a morphism* $[P]_{\overline{\mathcal{E}}}$ *from* X *to* Y *in* $\mathbf{Ter}(\mathbf{\Sigma},\mathcal{E})$ *the morphism*
$$\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}^{\mathfrak{pd}}_{\diamond}(P)]_{\overline{\mathcal{H}}}\colon \textstyle\coprod^{\dagger}_{\varphi} X \longrightarrow \coprod^{\dagger}_{\varphi} Y$$
in $\mathbf{Ter}(\mathbf{\Lambda},\mathcal{H})$.
3. $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}$ *sends a transformation* $\xi\colon \mathbf{d} \rightsquigarrow \mathbf{e}$ *from* $\mathbf{d}$ *to* $\mathbf{e}$ *to the natural transformation* $\mathrm{Ter}_{\mathfrak{pd}}(\xi)$ *from* $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{d})$ *to* $\mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\mathbf{e})$.

We notice that from the 2-category $\mathbf{Spf}^{\mathrm{op}}_{\mathfrak{pd}} \times \mathbf{Spf}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$ there exists a pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot)$ and a pseudo-extranatural transformation $(\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}})$ from $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot) \times \mathrm{Ter}^{\mathrm{sp}}_{\mathfrak{pd}}(\cdot)$ to $\mathrm{K}_{\mathbf{Set}}$.

Finally, we prove that the specifications of Bénabou and Hall are equivalent in the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$.

Proposition 7.6. *For every set of sorts* S, *the specifications* B_S, *of Bénabou for* S, *and* H_S, *of Hall for* S, *are equivalent in the 2-category* $\mathbf{Spf}_{\mathfrak{pd}}$.

Proof. Let S be a set of sorts. From the signature $\mathbf{\Sigma}^{\mathrm{B}_S}$ to the signature $\mathbf{\Sigma}^{\mathrm{H}_S}$, we have the polyderivor $\mathbf{d} = (\varphi, d)$, where φ is the mapping from $S^{\star} \times S^{\star}$ to $(S^{\star} \times S)^{\star}$ which sends (u,v) to $((u,v_0),\dots,(u,v_{|v|-1}))$, while $d\colon \Sigma^{\mathrm{B}_S} \longrightarrow \mathrm{BT}_{S^{\star}\times S}(\Sigma^{\mathrm{H}_S})_{\varphi^{\sharp}\times\varphi}$ is defined as

1. For every $w \in S^{\star}$, and $i \in |w|$, $d(\pi^w_i) = (\pi^w_i)$.
2. For every u, $w \in S^{\star}$, $d(\langle\,\rangle_{u,w}) = (v^{u,w_0}_0, \dots, v^{u,w_{|w|-1}}_{|w|-1})$.

3. For every u, v, $w \in S^{\star}$,

$$d(\circ_{u,v,w}) = (\xi_{u,v,w_0}(v_{|v|}^{u,w_0}, v_0^{u,v_0}, \ldots, v_{|v|-1}^{u,v_{|v|-1}}), \ldots, \xi_{u,v,w_{|w|-1}}(v_{|v|+|w|-1}^{u,w_{|w|-1}}, v_0^{u,v_0}, \ldots, v_{|v|-1}^{u,v_{|v|-1}})).$$

Now we prove that the definition of $\mathbf{d}$ is sound. For the operations $\pi_i^w \in \Sigma^{\mathrm{B}_S}_{\lambda,(w,(w_i))}$, we have that

$$\begin{aligned} d(\pi_i^w) &\in \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{\varphi^\sharp(\lambda),\varphi(w,(w_i))} \\ &= \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{\lambda,((w,(w_i)))} \\ &= \mathrm{T}_{\Sigma^{\mathrm{H}_S}}(\downarrow\lambda)_{((w,(w_i)))}, \end{aligned}$$

because $d(\pi_i^w)$ is a word of length 1 that has as its unique component an operation of coarity $(w,(w_i))$.

For the operations $\langle\,\rangle_{u,w} \in \Sigma^{\mathrm{B}_S}_{((u,(w_0)),\ldots,(u,(w_{|w|-1}))),(u,w)}$, we have that

$$\begin{aligned} d(\langle\,\rangle_{u,w}) &\in \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{\varphi^\sharp((u,w_0),\ldots,(u,(w_{|w|-1}))),\varphi(u,w)} \\ &= \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{((u,w_0),\ldots,(u,w_{|w|-1})),((u,w_0),\ldots,(u,w_{|w|-1}))} \\ &= \mathrm{T}_{\Sigma^{\mathrm{H}_S}}(\downarrow((u,w_0),\ldots,(u,w_{|w|-1})))_{((u,w_0),\ldots,(u,w_{|w|-1}))}, \end{aligned}$$

because $d(\langle\,\rangle_{u,w})$ is a word of length $|w|$ that, for every $i \in |w|$, has as i-th component a term of coarity $(u,(w_i))$. For the operations $\circ_{u,v,w} \in \Sigma^{\mathrm{B}_S}_{((u,v),(v,w)),(u,w)}$, we have that

$$\begin{aligned} d(\circ_{u,v,w}) &\in \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{\varphi^\sharp((u,v),(v,w)),\varphi(u,w)} \\ &= \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{H}_S})_{((u,v_0),\ldots,(u,v_{|v|-1}),(v,w_0),\ldots,(v,w_{|w|-1})),((u,w_i)\mid i\in|w|)} \\ &= \mathrm{T}_{\Sigma^{\mathrm{H}_S}}(\downarrow(((u,v_j)\mid j\in|v|),((v,w_i)\mid i\in|w|))_{((u,w_i)\mid i\in|w|)}, \end{aligned}$$

because $d(\circ_{u,v,w})$ is a word of length $|w|$ that, for every $i \in |w|$, has as i-th component a term of coarity (u,w_i).

From the signature $\mathbf{\Sigma}^{\mathrm{H}_S}$ to the signature $\mathbf{\Sigma}^{\mathrm{B}_S}$ we have the polyderivor $\mathbf{e} = (\psi, e)$, where ψ is the mapping from $S^\star \times S$ to $(S^\star \times S^\star)^\star$ which sends (w,s) to $((w,(s)))$, while $e\colon \Sigma^{\mathrm{H}_S} \longrightarrow \mathrm{BT}_{S^\star\times S}(\Sigma^{\mathrm{B}_S})_{\psi^\sharp\times\psi}$ is defined as

1. For every $w \in S^\star$ and $i \in |w|$, $e(\pi_i^w) = (\pi_i^w)$.
2. For every u, $w \in S^\star$ and $s \in S$, $e(\xi_{u,w,s}) = (v_0^{w,s} \circ \langle v_1^{u,w_0}, \ldots, v_{|w|}^{u,w_{|w|-1}}\rangle)$.

The polyderivors **d** and **e** are, obviously, compatible with the respective equations, hence are $\mathfrak{pd}$-specification morphisms.

Finally we should prove that there are invertible transformations between the identity at $\mathbf{\Sigma}^{\mathrm{B}_S}$ and the polyderivor $\mathbf{e}\circ\mathbf{d}$, as well as between the identity at $\mathbf{\Sigma}^{\mathrm{H}_S}$ and the polyderivor $\mathbf{d}\circ\mathbf{e}$. Since both proofs are analogous, we restrict ourselves to present only the first one.

From the identity at $\mathbf{\Sigma}^{\mathrm{B}_S}$ into $\mathbf{e}\circ\mathbf{d}$ we have the transformation χ, defined, for every $(u,w)\in S^\star\times S^\star$, as the term

$$\chi_{(u,w)} = (\pi^w_0\circ v_0,\dots,\pi^w_{|w|-1}\circ v_0)\in \mathrm{T}_{\mathbf{\Sigma}^{\mathrm{B}_S}}(\downarrow((u,w)))_{((u,(w_0)),\dots,(u,(w_{|w|-1})))},$$

and from $\mathbf{e}\circ\mathbf{d}$ into the identity at $\mathbf{\Sigma}^{\mathrm{B}_S}$ we have the transformation ρ, defined, for every $(u,w)\in S^\star\times S^\star$, as the term

$$\rho_{(u,w)} = \langle v_0,\dots,v_{|w|-1}\rangle_{u,w}\in \mathrm{T}_{\mathbf{\Sigma}^{\mathrm{B}_S}}(\downarrow((u,(w_0)),\dots,(u,(w_{|w|-1}))))_{((u,w))}.$$

Then $\rho_{(u,w)}\circ\chi_{(u,w)}$ is the term $\langle\pi^w_0\circ v_0,\dots,\pi^w_{|w|-1}\circ v_0\rangle_{u,w} = v_0$, and $\chi_{(u,w)}\circ\rho_{(u,w)}$ is the term

$$(\pi^w_0\circ\langle v_0,\dots,v_{|w|-1}\rangle_{u,w},\dots,\pi^w_{|w|-1}\circ\langle v_0,\dots,v_{|w|-1}\rangle_{u,w}) = (v_0,\dots,v_{|w|-1}),$$

hence ξ and ρ are inverses. □

Corollary 7.7. *For every set of sorts S, the category* $\mathbf{Alg}(\mathrm{H}_S)$*, of Hall algebras for S, is equivalent to the category* $\mathbf{Alg}(\mathrm{B}_S)$*, of Bénabou algebras for S.*

Proof. It follows from the existence of the pseudo-functor $\mathrm{Alg}^{\mathrm{sp}}_{\mathfrak{pd}}$ from the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$ to the 2-category $\mathbf{Cat}$ and from the fact that the specifications $(\mathbf{\Sigma}^{\mathrm{B}_S},\mathcal{E}^{\mathrm{B}_S})$ and $(\mathbf{\Sigma}^{\mathrm{H}_S},\mathcal{E}^{\mathrm{H}_S})$ are equivalent in the 2-category $\mathbf{Spf}_{\mathfrak{pd}}$. □

References

[1] J. Bénabou, *Structures algebriques dans les categories*, Cahiers de topologie et géometrie différentielle **10** (1968), pp. 1–126.

[2] G. Birkhoff and J. D. Lipson, *Heterogeneous algebras*, J. Combinatorial Theory **8** (1970), pp. 115–133.

[3] F. Borceux and G. Janelidze, Galois theories, Cambridge University Press. Cambridge Studies in Advanced Mathematics 72. Cambridge, 2001.

[4] J. Climent and J. Soliveres, *The completeness theorem for monads in categories of sorted sets*, Houston Journal of Mathematics **31** (2005), pp. 103–129.

[5] J. Climent and J. Soliveres, *On the completeness theorem of many-sorted equational logic and the equivalence between Hall algebras and Bénabou theories*, Reports on Mathematical Logic **40** (2006), pp. 127–158.

[6] P. Cohn, Universal algebra, D. Reidel Publishing Company, Dordrecht: Holland / Boston: U.S.A. / London: England, 1981.

[7] Ch. Ehresmann, Catégories et structures, Dunod, Paris, 1965.

[8] T. Fujiwara, *On mappings between algebraic systems*, Osaka Math. J., **11** (1959), pp. 153–172.

[9] T. Fujiwara, *On mappings between algebraic systems, II*, Osaka Math. J. **12** (1960), pp. 253–268.

[10] K. Gödel, *Eine interpretation des intuitionistischen Aussagenkalküls*, Ergebnisse eines mathematischen Kolloquiums **4** (1933), pp. 39–40.

[11] J. Goguen and R. Burstall, *Introducing institutions*. In E. Clarke and D. Kozen, editors, Logics of programs. Proceedings of the fourth workshop held at Carnegie-Mellon University (Pittsburgh, Pa., June 6–8, 1983). Springer-Verlag, Berlin, 1984, pp. 221–256.

[12] J. Goguen and J. Meseguer, *Completeness of many-sorted equational logic*, Houston Journal of Mathematics **11** (1985), pp. 307–334.

[13] J. Goguen, J. Thatcher and E. Wagner, *An initial algebra approach to the specification, correctness, and implementation of abstract data types*, IBM Thomas J. Watson Research Center, Tecnical Report RC 6487, October 1976.

[14] G. Grätzer, Universal algebra, 2nd ed., Springer-Verlag, New York · Heidelberg · Berlin, 1979.

[15] A. Grothendieck, *Catégories fibrées et descente (Exposé VI)*. In A. Grothendieck, editor, Revêtements Étales et Groupe Fondamental (SGA 1), Springer-Verlag, Berlin · Heidelberg · New York, 1971, pp. 145–194.

[16] P.J. Higgins, *Algebras with a scheme of operators*, Mathematische Nachrichten **27** (1963), pp. 115–132.

[17] N. Jacobson, Structure of rings. Amer. Math. Soc. Colloq. Pub., vol. 37. Revised edition. American Math. Soc., Providence, R. I., 1968.

[18] H. Kleisli, *Every standard construction is induced by a pair of adjoint functors*, Proc. Amer. Math. Soc. **16** (1965), pp. 544–546.

[19] F. W. Lawvere, *Functorial semantics of algebraic theories*, Dissertation. Columbia University, 1963.

[20] M. Lazard, *Lois de groupes et analyseurs*, Ann. Sci. Ecole Norm. Sup. **72** (1955), pp. 299–400.

[21] S. Mac Lane, Categories for the working mathematician. 2nd ed., Springer-Verlag, New York · Berlin · Heidelberg, 1998.

[22] G. Matthiessen, *Theorie der Heterogenen Algebren*, Mathematik-Arbeits-Papiere, Nr. 3, Universität, Bremen, 1976.

[23] E. Post. *Introduction to a general theory of elementary propositions*, Amer. J. Math. **43** (1921), pp. 163–185.

[24] E. Post, The two-valued iterative systems of mathematical logic. Annals of Math. Studies, no. 5., Princeton Univ. Press, Princeton, N. J., 1941.

[25] A. Tarlecki, R. Burstall and J. Goguen, *Some fundamental algebraic tools for the semantics of computation: Part 3. Indexed categories*, Theoretical Computer Science **91** (1991), pp. 239–264.

Universidad de Valencia
Departamento de Lógica y Filosofía de la Ciencia
E-46010 Valencia, Spain

Juan.B.Climent@uv.es, Juan.Soliveres@uv.es