REPORTS ON MATHEMATICAL LOGIC
44 (2009), 31–63

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ON SOME PROPERTIES OF QUASI-MV ALGEBRAS AND $\sqrt{'}$ QUASI-MV ALGEBRAS

A b s t r a c t. We investigate some properties of two varieties of algebras arising from quantum computation - quasi-MV algebras and $\sqrt{'}$ quasi-MV algebras - first introduced in [13], [12] and tightly connected with fuzzy logic. We establish the finite model property and the congruence extension property for both varieties; we characterize the quasi-MV reducts and subreducts of $\sqrt{'}$ quasi-MV algebras; we give a representation of semisimple $\sqrt{'}$ quasi-MV algebras in terms of algebras of functions; finally, we describe the structure of free algebras with one generator in both varieties.

1. Introduction

Unlike classical computation, quantum computation [17] allows one to represent two atomic information bits in parallel. Here, in fact, the appropriate

Received 27 July 2006

counterpart of a classical bit is the *qubit*, defined as a unit vector in the 2-dimensional Hilbert space C^2:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,,$$

where a, b are complex numbers s.t. $|a|^2 + |b|^2 = 1$, while $|0\rangle, |1\rangle$ are the unit vectors $\langle 1, 0\rangle, \langle 0, 1\rangle$, respectively. Supposing that, in analogy with the classical case, $|0\rangle$ and $|1\rangle$ represent maximal and precise pieces of information, the *superposition* state $|\psi\rangle$ corresponds to an *uncertain* information: as dictated by the Born rule, $|a|^2$ yields the probability of the information described by the pure state $|0\rangle$, while $|b|^2$ yields the probability of the information described by the pure state $|1\rangle$. A system of n qubits, also called a $n-$*quregister*, is represented by a unit vector in the n-fold tensor product Hilbert space $\otimes^n C^2$. Qubits and quregisters encode possibly uncertain, yet *maximal* information. Non-maximal information pieces are matched, on a mathematical level, by *qumixes*, i.e. density operators in C^2 or in an appropriate tensor product $\otimes^n C^2$ of C^2, for which we will sometimes use the variables $\rho, \sigma, ...$

Similarly to the classical case, we can introduce and study the behaviour of a number of *quantum logical gates* (hereafter *quantum gates* for short) operating on such information units. These gates are mathematically represented by unitary operators on the appropriate Hilbert spaces. In this way, we end up defining an array of *quantum computational logics* ([3], [8]). Here are some significant examples of quantum gates, whose behaviour is at first described in the framework of quregisters.

Example 1. For any $n \geq 1$, the *negation* on $\otimes^n C^2$ is the unitary operator $\mathtt{Not}^{(\mathtt{n})}$ such that, for every element $|a_1, ..., a_n\rangle$ of the computational basis[1] $\mathcal{B}^{(n)}$,

$$\mathtt{Not}^{(\mathtt{n})}(|a_1, ..., a_n\rangle) = |a_1, ..., a_{n-1}\rangle \otimes |1 - a_n\rangle\,.$$

Example 2. For any $n, m \geq 1$, the *Petri-Toffoli gate* on $\otimes^{n+m+1} C^2$ is the unitary operator $T^{(n,m,1)}$ such that, for every element

$$|a_1, ..., a_n\rangle \otimes |b_1, ..., b_m\rangle \otimes |c\rangle$$

[1]By $\mathcal{B}^{(n)}$ we denote the set $\{|a_1, ..., a_n\rangle : a_i \in \{0, 1\}\}$, which is an orthonormal basis for the space $\otimes^n C^2$.

of the computational basis $\mathcal{B}^{(n+m+1)}$ (shortened as $|\overrightarrow{a}\rangle \otimes \left|\overrightarrow{b}\right\rangle \otimes |c\rangle$),

$$T^{(n,m,1)}(|\overrightarrow{a}\rangle \otimes \left|\overrightarrow{b}\right\rangle \otimes |c\rangle) = |\overrightarrow{a}\rangle \otimes \left|\overrightarrow{b}\right\rangle \otimes \left|a_n b_m \widehat{+} c\right\rangle,$$

where $\widehat{+}$ represents sum modulo 2. The conjunction $\mathtt{And}(|\overrightarrow{a}\rangle, \left|\overrightarrow{b}\right\rangle)$ can be defined as $T^{(n,m,1)}(|\overrightarrow{a}\rangle \otimes \left|\overrightarrow{b}\right\rangle \otimes |0\rangle)$.

One can easily verify that, when applied to classical bits, Not and And behave as the standard Boolean truth functions. However, the quantum computational And is, unlike classical conjunction, *reversible*: one can retrieve the input values from the output with no loss of information.

Example 3. For any $n \geq 1$, the *square root of the negation* on $\otimes^n C^2$ is the unitary operator $\sqrt{\mathtt{Not}}^{(\mathtt{n})}$ such that, for every element $|a_1, ..., a_n\rangle$ of the computational basis $\mathcal{B}^{(n)}$,

$$\sqrt{\mathtt{Not}}^{(\mathtt{n})}(|a_1, ..., a_n\rangle) = |a_1, ..., a_{n-1}\rangle \otimes \frac{1}{2}\left((1+i)\,|a_n\rangle + (1-i)\,|1-a_n\rangle\right),$$

where i is the imaginary unit.

The basic property of $\sqrt{\mathtt{Not}}^{(\mathtt{n})}$ is the following: for any $|\overrightarrow{a}\rangle$ in $\otimes^n C^2$, $\sqrt{\mathtt{Not}}^{(\mathtt{n})}\left(\sqrt{\mathtt{Not}}^{(\mathtt{n})}(|\overrightarrow{a}\rangle)\right) = \mathtt{Not}^{(\mathtt{n})}(|\overrightarrow{a}\rangle)$. From a logical point of view, the square root of the negation can be regarded as a kind of "tentative partial negation" that transforms precise pieces of information into maximally uncertain ones. True to form, this gate has no Boolean counterpart.

Next, we provide an example of an (irreversible) quantum gate whose behaviour is described in the framework of qumixes. We parenthetically observe that, when applied to qumixes, the previously mentioned gates are usually written in capital letters, while the superscript indicating the Hilbert space where the operator "lives" is dropped; for example, $\sqrt{\mathtt{Not}}^{(\mathtt{n})}$ becomes $\sqrt{\mathtt{NOT}}$.[2]

Example 4. Let τ, σ be density operators of $\otimes^n C^2$ and $\otimes^m C^2$, respectively. The *Łukasiewicz disjunction* of τ and σ is the operator $\oplus$: $\otimes^n C^2 \times \otimes^m C^2 \to \otimes^{n+m} C^2$ defined by:

$$\tau \oplus \sigma = (1 - (\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)))\,\kappa P_0^{(n+m)} + (\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma))\,\kappa P_1^{(n+m)},$$

[2] In order to guarantee closure of the set of density operators w.r.t. the above-mentioned gates, a special device must be adopted. The exact definition of $\sqrt{\mathtt{NOT}}\rho$, for example, is $\sqrt{\mathtt{Not}}^{(\mathtt{n})}\rho\sqrt{\mathtt{Not}}^{(\mathtt{n})*}$, where $\sqrt{\mathtt{Not}}^{(\mathtt{n})*}$ is the adjoint operator of $\sqrt{\mathtt{Not}}$.

where:

- $\kappa = \frac{1}{2^{n+m-1}}$ is a normalisation factor;
- $P_0^{(n+m)}$ (respectively, $P_1^{(n+m)}$) is the projection operator onto the subspace spanned by the vectors of the computational basis $\mathcal{B}^{(n+m)}$ whose last figure is 0 (respectively, 1) and acts as a mathematical representative of the "falsity" property (respectively, "truth" property) in $\otimes^{n+m}C^2$;
- $\mathtt{p}(\rho)$ is the probability[3] of the density operator ρ;
- $\oplus$ is the usual Łukasiewicz truncated sum.

  Thus, if $n = m = 1$,

$$\tau \oplus \sigma = \frac{(1 - (\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma)))}{2} P_0^{(2)} + \frac{(\mathtt{p}(\tau) \oplus \mathtt{p}(\sigma))}{2} P_1^{(2)}.$$

Although the preceding examples were given in the general framework of arbitrary n-fold tensor products of C^2, it can be shown [4] that - from a logical viewpoint - it is unnecessary to consider information quantities in Hilbert spaces other than C^2: in fact, the algebra whose universe is the set of all qumixes of C^2 and whose operations correspond to appropriate extensions of the quantum logical gates generates the same logical consequence relation (in the sense of [3]) as the algebra over the set of all qumixes of arbitrary n-fold tensor products of C^2. This result smooths things out to a considerable extent, since density operators of C^2 are amenable to the well-known matrix representation

$$\frac{1}{2}\left(\mathbb{I} + r_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + r_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

(where $\mathbb{I}$ is the identity 2×2 matrix, while r_1, r_2, r_3 are real numbers s.t. $r_1^2 + r_2^2 + r_3^2 \leq 1$). In other words, every density operator ρ of C^2 can be represented as a special triple $\langle r_1, r_2, r_3 \rangle$ of real numbers, namely a point in the Bloch-Poincaré sphere $\mathbf{D}^3$. The third element of the triple determines

[3] Recall that the probability of the density operator ρ of $\otimes^n C^2$ is simply the trace of the operator $P_1^{(n)}\rho$.

the probability of ρ, while the second element of the triple determine the probability of $\sqrt{\mathtt{NOT}}\rho$. In fact, an easy calculation shows that

$$\mathtt{p}\,(\rho) = \frac{1+r_3}{2},\ \mathtt{p}\left(\sqrt{\mathtt{NOT}}\rho\right) = \frac{1-r_2}{2}$$

It follows that, if we are concerned only with the probability of ρ and with the probability of $\sqrt{\mathtt{NOT}}\rho$, we can shift down by one dimension: the triple $\langle r_1, r_2, r_3\rangle$ shrinks to the pair $\langle a, b\rangle$, where a represents the probability of ρ and b represents the probability of $\sqrt{\mathtt{NOT}}\rho$. Clearly, the elements a, b must satisfy the condition that $a^2 + b^2 \leq 1$; that is, they must belong to the closed disc $\mathbf{D}^2$. To make computations easier, however, it is more convenient to transpose the disc to the first quadrant, scaling it down by one half: after such a move, qumixes are represented (modulo a neglection of the first component) by points of the closed disc with centre $\left\langle \frac{1}{2}, \frac{1}{2}\right\rangle$ and radius $\frac{1}{2}$ - which correspond to the subset $\left\{\langle a, b\rangle \in R \times R : (1-2a)^2 + (1-2b)^2 \leq 1\right\}$ of the set of all *complex numbers*. In this way, quantum logical gates are transformed into operations on such a set of complex numbers. So, we can obtain some *standard algebras* over the complex numbers, sharing the same universe but having different signatures according to the set of logical gates under examination ([4], [9]).

In ([13]) we considered, for a start, the standard algebra whose fundamental operations included the counterparts of the negation gate and of the Łukasiewicz disjunction gate (see Examples 1 and 4) and two distinguished elements, $\left\langle 0, \frac{1}{2}\right\rangle$ and $\left\langle 1, \frac{1}{2}\right\rangle$ (standing, respectively, for the "falsity" property and for the "truth" property). Such a structure, therefore, has the similarity type of Chang's MV algebras ([6]) and satisfies all of the MV algebraic axioms except that $\left\langle 0, \frac{1}{2}\right\rangle$ is not a neutral element for truncated sum. We then introduced the notion of *quasi-MV algebra* with the following analogy in mind: the above-mentioned algebra should play w.r.t. quasi-MV algebras the same role as the standard algebra over the real closed unit interval plays w.r.t. MV algebras. Our choice of axioms was bolstered, indeed, by a completeness theorem to the effect that an equation in the appropriate language holds in all quasi-MV algebras iff it holds in the standard algebra over the complex numbers.

In a subsequent paper we focussed on $\sqrt{'}$ *quasi-MV algebras*, i.e. quasi-MV algebras expanded by an operation of *square root of the inverse* ($\sqrt{'}$), which is the algebraic counterpart of the logical gate of square root of negation introduced in Example 3. The square root of the inverse can therefore be seen as a kind of "tentative inversion": by applying it twice to a given element a, we obtain the inverse a' of the element itself. In the standard algebra, for example, we have

$$\begin{aligned} \sqrt{'}\,\langle a,b\rangle &= \langle b,1-a\rangle\,; \\ \sqrt{'}\sqrt{'}\,\langle a,b\rangle &= \langle 1-a,1-b\rangle = \langle a,b\rangle'\,. \end{aligned}$$

Also this variety, as a matter of fact, turns out to be generated by the standard algebra over the complex numbers.

Once we forget about their original computational motivation, these varieties lend themselves to a purely algebraic investigation. As already mentioned, some progress towards this goal was made already in [13], [12]. In this paper we inquire further into this subject: we establish the finite model property and the congruence extension property for both varieties; we characterize the quasi-MV reducts and subreducts of $\sqrt{'}$ quasi-MV algebras; we give a representation of semisimple $\sqrt{'}$ quasi-MV algebras in terms of algebras of functions; finally, we describe the structure of free algebras with one generator in both varieties.

To keep the paper self-contained, the next two sections will contain a brief précis of [13], [12].

2. Quasi-MV algebras

Definition 5. A *quasi-MV algebra* (for short, qMV algebra[4]) is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ of type $\langle 2,1,0,0\rangle$ satisfying the following equations:

[4]A different notation was used in [13], [12]: "quasi-MV algebra" was shortened into "QMV algebra" (with uppercase Q). The present notational change is motivated by the desire to avoid any possible confusion with *quantum MV algebras* (also abbreviated as QMV algebras: see e.g. [11]).

A1. $x \oplus (y \oplus z) \approx (x \oplus z) \oplus y$
A2. $x'' \approx x$
A3. $x \oplus 1 \approx 1$
A4. $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$
A5. $(x \oplus 0)' \approx x' \oplus 0$
A6. $(x \oplus y) \oplus 0 \approx x \oplus y$
A7. $0' \approx 1$

We can think of a qMV algebra as identical to an MV algebra, except for the fact that 0 need not be a neutral element for the truncated sum $\oplus$. Of course, a qMV algebra is an MV algebra iff it satisfies the additional equation $x \oplus 0 \approx x$.

An immediate consequence of Definition 5 is the fact that the class of qMV algebras is a *variety* in its signature. Henceforth, such a variety will be denoted by $q\mathbb{MV}$. The subvariety of MV algebras will be denoted by $\mathbb{MV}$.

Definition 6. We introduce the following abbreviations:

$$x \otimes y = (x' \oplus y')';$$
$$x \Cup y = x \oplus (x' \otimes y);$$
$$x \Cap y = x \otimes (x' \oplus y).$$

As already remarked, every MV algebra is an example of qMV algebra. Examples of "pure" qMV algebras, i.e. qMV algebras that are *not* MV algebras, are given by the next two structures over the complex numbers, **S** (for *square*) and **D** (for *disc*, already mentioned in the introduction).

Example 7. (standard quasi-MV algebras). We introduce two *standard quasi-MV algebras*. **S** is the algebra $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}}, {}'^{\mathbf{S}}, 0^{\mathbf{S}}, 1^{\mathbf{S}} \right\rangle$, where:

- $\langle a, b \rangle \oplus^{\mathbf{S}} \langle c, d \rangle = \left\langle \min(1, a+c), \frac{1}{2} \right\rangle$;
- $\langle a, b \rangle'^{\mathbf{S}} = \langle 1-a, 1-b \rangle$;
- $0^{\mathbf{S}} = \left\langle 0, \frac{1}{2} \right\rangle$;
- $1^{\mathbf{S}} = \left\langle 1, \frac{1}{2} \right\rangle$.

Remark that $\langle a,b\rangle \oplus^{\mathbf{S}} \left\langle 0,\frac{1}{2}\right\rangle \neq \langle a,b\rangle$ *whenever* $b \neq \frac{1}{2}$.
$\mathbf{D}$ *is the subalgebra of* $\mathbf{S}$ *whose universe is the set*

$$\{\langle a,b\rangle : a,b \in R \text{ and } (1-2a)^2 + (1-2b)^2 \leq 1\}.$$

We now list some very simple properties of qMV algebras.

Lemma 8. *The following equations are satisfied in every qMV algebra:*

(i) $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$;	*(v)* $0 \oplus 0 \approx 0$;
(ii) $x \oplus y \approx y \oplus x$;	*(vi)* $x \oplus 0 \approx x \Cap x$;
(iii) $x \oplus x' \approx 1$;	*(vii)* $x \Cap y \approx y \Cap x$.
(iv) $x \otimes x' \approx 0$;	*(viii)* $x \Cup y \approx y \Cup x$.

It is well-known (see e.g. [6]) that it is possible to introduce a *lattice* order on any MV algebra by simply taking $a \leq b$ to hold whenever $a \otimes (a' \oplus b) = a$. This condition is obviously equivalent to $a \Cap b = a \oplus 0$ in an MV algebraic setting, yet it is no longer such in a quasi-MV algebraic one. We define:

Definition 9. Let $\mathbf{A}$ be a qMV algebra. For all $a, b \in A$:

$$a \leq b \text{ iff } a \Cap b = a \oplus 0.$$

Lemma 10. *Let* $\mathbf{A}$ *be a qMV algebra. (i) For all* $a, b \in A$, $a \leq b$ *iff* $1 = a' \oplus b$*; (ii)* $\leq$ *is a preordering, but not necessarily a partial ordering, of* A.

Our preordering relation enjoys some standard properties, including a few monotonicity properties:

Lemma 11. *Let* $\mathbf{A}$ *be a qMV algebra. For all* $a, b, c, d \in A$*:*

(i) $a \oplus 0 \leq b \oplus 0, b \oplus 0 \leq a \oplus 0 \Rightarrow a \oplus 0 = b \oplus 0$;
(ii) $a \leq b, c \leq d \Rightarrow a \oplus c \leq b \oplus d$;
(iii) $a \leq b, c \leq d \Rightarrow a \otimes c \leq b \otimes d$;
(iv) $a \leq b, c \leq d \Rightarrow a \Cap c \leq b \Cap d$;
(v) $a \leq b, c \leq d \Rightarrow a \Cup c \leq b \Cup d$;
(vi) $a \leq a \oplus 0, a \oplus 0 \leq a$;
(vii) $a \otimes b \leq c \Leftrightarrow a \leq b' \oplus c$;
(viii) $a \leq b \Rightarrow b' \leq a'$;
(ix) $0 \leq a, a \leq 1$;
(x) $a \oplus 0 = b \oplus 0 \Rightarrow a \oplus c = b \oplus c$.

Some elements in a qMV algebra (at least one indeed, i.e. 0) are "well-behaved" in that they satisfy the equation $x \oplus 0 \approx x$; we call them *regular*. Of course, MV algebras contain nothing but regular elements. Pure qMV algebras, on the contrary, also have *irregular* elements which fail to satisfy that equation.

Definition 12. Let $\mathbf{A}$ be a qMV algebra and let $a \in A$. We call a *regular* just in case $a \oplus 0 = a$. We denote by $\mathcal{R}(\mathbf{A})$ the set of all regular elements of $\mathbf{A}$.

The relations $\chi^{\mathbf{A}}$ and $\tau^{\mathbf{A}}$ on A defined by

$$a\chi^{\mathbf{A}}b \text{ iff } a \leq^{\mathbf{A}} b \text{ and } b \leq^{\mathbf{A}} a \text{ (iff } a \oplus^{\mathbf{A}} 0 = b \oplus^{\mathbf{A}} 0)$$
$$a\tau^{\mathbf{A}}b \text{ iff } a, b \in \mathcal{R}(\mathbf{A}) \text{ or } a = b$$

are congruences on any qMV algebra $\mathbf{A}$; we drop the superscripts whenever it is clear which algebra is at issue. Moreover, we call *clouds* the elements of A/χ. We have that:

Lemma 13. *Let* $\mathbf{A}$ *be a qMV algebra. The algebra*

$$\mathbf{R_A} = \left\langle \mathcal{R}(\mathbf{A}), \oplus^{\mathbf{R}}, '^{\mathbf{R}}, 0^{\mathbf{R}}, 1^{\mathbf{R}} \right\rangle$$

where, for any functor f*,* $f^{\mathbf{R}}$ *is the restriction to* $\mathcal{R}(\mathbf{A})$ *of* $f^{\mathbf{A}}$*, is an MV-subalgebra of* $\mathbf{A}$*, lattice ordered by the restriction to* $\mathcal{R}(\mathbf{A})$ *of* $\leq^{\mathbf{A}}$*, and isomorphic to* $\mathbf{A}/\chi$*.*

QMV algebras consisting of just one cloud are called *flat*; they correspond to the subvariety of qMV algebras whose equational basis w.r.t. $q\mathbb{MV}$ is the single equation $0 \approx 1$. For any qMV algebra $\mathbf{A}$, $\mathbf{A}/\tau$ is a flat algebra.

Definition 14. A qMV algebra $\mathbf{F}$ is called *flat* iff it satisfies the equation $0 \approx 1$. The subvariety of flat qMV algebras will be denoted by $\mathbb{F}q\mathbb{MV}$.

Any qMV algebra can be thought of as composed by an MV algebraic component and a flat component. More precisely:

Theorem 15. *For every qMV algebra* $\mathbf{Q}$*, there exist an MV algebra* $\mathbf{M}$ *and a flat qMV algebra* $\mathbf{F}$ *such that* $\mathbf{Q}$ *can be embedded into the direct product* $\mathbf{M} \times \mathbf{F}$*.*

Proof. Indeed, choosing $\mathbf{A}/\chi$ for $\mathbf{M}$ and $\mathbf{A}/\tau$ for $\mathbf{F}$ does the trick. □

As a corollary to Theorem 15, to Chang's completeness theorem for MV algebras and to the completeness of flat qMV algebras w.r.t. a standard flat algebra over the complex numbers, we get the desired completeness result w.r.t. both the square and the disc:

Theorem 16. *If t, s are terms in the language of qMV algebras, the following are equivalent:*

(i) $q\mathbb{MV} \models t \approx s$;
(ii) $\mathbf{S} \models t \approx s$;
(iii) $\mathbf{D} \models t \approx s$.

3. Adding square roots of the inverse

We now enrich quasi-MV algebras by an additional unary operation of *square root of the inverse* and by a constant k, which realises in the standard algebra the element $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$, corresponding to the "absolutely undetermined" density operator[5].

Definition 17. A $\sqrt{'}$ *quasi-MV algebra* (for short, $\sqrt{'}$*qMV algebra)* is an algebra $\mathbf{A} = \left\langle A, \oplus, \sqrt{'}, 0, 1, k \right\rangle$ of type $\langle 2, 1, 0, 0, 0 \rangle$ such that, upon defining $a' = \sqrt{'}\sqrt{'}a$ for all $a \in A$, the following conditions are satisfied:

SQ1. $\langle A, \oplus, ', 0, 1 \rangle$ is a quasi-MV algebra;
SQ2. $k = \sqrt{'}k$;
SQ3. $\sqrt{'}(a \oplus b) \oplus 0 = k$ for all $a, b \in A$.

The axiom SQ3 may at first look somewhat puzzling. The reason why it is there lies in two facts: first, the output of a Łukasiewicz disjunction between any density operator ρ and $\left\langle 0, \frac{1}{2} \right\rangle$ has the same probability as ρ, while the probability of its square root of the negation is the constant value $\frac{1}{2}$; second, the probability of the square root of the negation $\sqrt{\mathtt{NOT}}\,(\rho \oplus \sigma)$ of the Łukasiewicz disjunction $\rho \oplus \sigma$ of any two density operators ρ, σ turns out to assume the constant value $\frac{1}{2}$.

[5] An interesting physical model for this operator is given by the so-called *semi-transparent mirror*, i.e. a half-silvered mirror which reflects exactly one half of any light beam incident at a $\frac{\pi}{4}$ angle.

$\sqrt{'}$qMV algebras form a variety in their own similarity type, hereafter named $\sqrt{'}q\mathbb{MV}$. We remark in passing that it is impossible to add a square root of the inverse to a nontrivial MV algebra: letting b be 0 in SQ3, for all $a \in A$ we would have $\sqrt{'}a = k$, whence by SQ2 $a' = \sqrt{'}\sqrt{'}a = \sqrt{'}k = k$ and so $a = a'' = k' = \sqrt{'}\sqrt{'}k = k$.

Examples of $\sqrt{'}$qMV algebras are the following expansions of the standard qMV algebras over the complex numbers; finite examples of $\sqrt{'}$qMV algebras can be found in [12].

Example 18. (standard $\sqrt{'}$qMV algebras). We introduce two *standard* $\sqrt{'}$*qMV algebras.* $\mathbf{S}_r$ is the algebra $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}_r}, \sqrt{'}^{\mathbf{S}_r}, 0^{\mathbf{S}_r}, 1^{\mathbf{S}_r}, k^{\mathbf{S}_r} \right\rangle$, where:

- $\left\langle [0,1] \times [0,1], \oplus^{\mathbf{S}_r}, '^{\mathbf{S}_r}, 0^{\mathbf{S}_r}, 1^{\mathbf{S}_r} \right\rangle$ is the qMV algebra $\mathbf{S}$ of Example 7;
- $\sqrt{'}^{\mathbf{S}_r} \langle a, b\rangle = \langle b, 1-a\rangle$;
- $k^{\mathbf{S}_r} = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$.

$\mathbf{D}_r$ *is the subalgebra of* $\mathbf{S}_r$ *whose universe is the set*

$$\{\langle a, b\rangle : a, b \in R \text{ and } (1-2a)^2 + (1-2b)^2 \leq 1\}.$$

In $\sqrt{'}$qMV algebras we have not only regular elements, but also *coregular* elements, i.e. elements whose square roots of the inverse are regular. In other words, a is coregular just in case $\sqrt{'}a \oplus 0 = \sqrt{'}a$. We denote by $\mathcal{COR}(\mathbf{A})$ the set of all coregular elements of $\mathbf{A}$.

Lemma 19. *The following equations are satisfied in every* $\sqrt{'}$*qMV algebra:*

(i) $k \approx k'$; *(iii)* $\sqrt{'}(x') \approx (\sqrt{'}x)'$;
(ii) $k \approx k \oplus 0$; *(iv)* $\sqrt{'}(x \oplus y) \oplus \sqrt{'}(z \oplus w) \approx 1$.

Lemma 20. *Let* $\mathbf{A}$ *be a* $\sqrt{'}$*qMV algebra and let* $a \in \mathcal{COR}(\mathbf{A})$. *Then: (i)* $a \oplus 0 = k$; *(ii)* $a \oplus k = a \oplus \sqrt{'}0 = a \oplus \sqrt{'}1 = 1$.

$\sqrt{'}$qMV algebras whose universes contain nothing but regular or coregular elements are worth a special label. For a reason that will become clear presently, we dub them *strongly cartesian*. Indeed, any $\sqrt{'}$qMV algebra has a strongly cartesian subalgebra:

Lemma 21. *Let* $\mathbf{A}$ *be a* $\sqrt{\prime}$*qMV algebra. The algebra*

$$\mathbf{N_A} = \left\langle \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A}), \oplus^{\mathbf{N}}, \sqrt{\prime}^{\mathbf{N}}, 0^{\mathbf{N}}, 1^{\mathbf{N}}, k^{\mathbf{N}} \right\rangle$$

where, for any functor f*,* $f^{\mathbf{N}}$ *is the restriction to* $\mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})$ *of* $f^{\mathbf{A}}$*, is a subalgebra of* $\mathbf{A}$*.*

Such a subalgebra can always be embedded into a quotient algebra of $\mathbf{A}$ modulo a congruence relation which we are now going to define.

Definition 22. Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We set:

$$a\lambda^{\mathbf{A}}b \text{ iff } a \oplus^{\mathbf{A}} 0 = b \oplus^{\mathbf{A}} 0 \text{ and } \sqrt{\prime}a \oplus^{\mathbf{A}} 0 = \sqrt{\prime}b \oplus^{\mathbf{A}} 0$$

or, equivalently,

$$a\lambda^{\mathbf{A}}b \text{ iff } a \leq^{\mathbf{A}} b, b \leq^{\mathbf{A}} a, \sqrt{\prime}a \leq^{\mathbf{A}} \sqrt{\prime}b \text{ and } \sqrt{\prime}b \leq^{\mathbf{A}} \sqrt{\prime}a.$$

It turns out that $\lambda^{\mathbf{A}}$ is a congruence on every $\sqrt{\prime}$qMV algebra. We call the relation $\lambda^{\mathbf{A}}$ the *cartesian* congruence on a given $\sqrt{\prime}$qMV algebra, and drop once again the superscripts whenever it is clear which algebra is at issue. It is easily seen that:

Lemma 23. *Let* $\mathbf{A}$ *be a* $\sqrt{\prime}$*qMV algebra. The subalgebra* $\mathbf{N_A}$ *of Lemma 21 is embeddable into* $\mathbf{A}/\lambda$*.*

Likewise, we introduce a congruence which we call the *flat* congruence on a $\sqrt{\prime}$ qMV algebra. Omitting superscripts from the very beginning, we put:

Definition 24. Let $\mathbf{A}$ be a $\sqrt{\prime}$qMV algebra and let $a, b \in A$. We define:

$$a\mu b \text{ iff } a = b \text{ or } a, b \in \mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A}).$$

λ is orthogonal to μ (i.e., their intersection is the identity relation). We now introduce two special classes of $\sqrt{\prime}$ qMV algebras: *cartesian* algebras, where λ is the identity relation Δ, and *flat* algebras, where λ is the universal relation ∇.

Definition 25. A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *cartesian* iff $\lambda = \Delta$, i.e. iff it satisfies the quasi-equation

$$x \oplus 0 \approx y \oplus 0 \wedge \sqrt{\prime}x \oplus 0 \approx \sqrt{\prime}y \oplus 0 \Rightarrow x \approx y$$

A $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is called *flat* iff $\lambda = \nabla$. We denote by $\mathbb{F}$ the class of flat $\sqrt{\prime}$ qMV algebras, and by $\mathbb{C}$ the class of cartesian $\sqrt{\prime}$ qMV algebras.

As a consequence of the definition, the only $\sqrt{'}$qMV algebra which is both cartesian and flat is the trivial one-element algebra. Strongly cartesian $\sqrt{'}$qMV algebras are cartesian, but not always conversely; for example, the algebras of Example 18 are cartesian but not strongly cartesian. The algebras in the next example, on the other hand, are flat.

Example 26. $\mathbf{F}_{100}$ is the algebra[6] whose universe is the 2-element set $\{0, b\}$, s.t. all truncated sums equal 0, while $\sqrt{'}0 = 0$ and $\sqrt{'}b = b$. $\mathbf{F}_{020}$ is the algebra whose universe is the 3-element set $\{0, a, b\}$, whose semigroup reduct is again the constant 0-valued semigroup and whose table for $\sqrt{'}$ is given by

$\sqrt{'}$	
$\mathbf{0}$	0
$\mathbf{a}$	b
$\mathbf{b}$	a

Both $\mathbf{F}_{100}$ and $\mathbf{F}_{020}$ are flat; moreover, $\mathbf{F}_{100}$ is simple, while $\mathbf{F}_{020}$ is a nonsimple subdirectly irreducible algebra having three congruences: $\Delta, \lambda = \nabla$ and the monolith θ whose cosets are $\{a, b\}$ and $\{0\}$.

$\mathbb{C}$ and $\mathbb{F}$ do not exhaust the variety of $\sqrt{'}$qMV algebras: for example, the direct product of the Cross (Example 20 in [12]) and $\mathbf{F}_{100}$ is neither cartesian nor flat. It is also worth noticing that $\mathbb{F}$ is a variety, whose equational basis relative to $\sqrt{'}q\mathbb{MV}$ is given by the single equation $0 \approx 1$, while $\mathbb{C}$ is a quasivariety which is not a variety.

Lemma 27. *(i)* $\mathbb{F}$ *is a variety; (ii)* $\mathbb{C}$ *is a quasivariety but not a variety.*

Cartesian $\sqrt{'}$qMV algebras are amenable to a clean representation in terms of algebras of *pairs*. Consider the standard $\sqrt{'}$qMV algebra $\mathbf{S}_r$. One may think of it as obtained out of the standard MV algebra $\mathbf{MV}_{[0,1]}$ by taking the cartesian square of its universe and defining the operations in such a way that each component of the result may be extracted out of the components of the argument(s) simply by means of polynomial $\mathbf{MV}_{[0,1]}$-operations. It turns out that this construction can be carried out not just for $\mathbf{MV}_{[0,1]}$, but for an *arbitrary* MV algebra - provided that inverse has a fixpoint - as the next definition shows.

[6] In general, we denote by $\mathbf{F}_{nmp}$ the finite flat algebra which contains n fixpoints for $\sqrt{'}$ beside 0, m fixpoints for the inverse which are not fixpoints for $\sqrt{'}$, and p elements which are not fixpoints under either operation.

Definition 28. Let $\mathbf{A} = \left\langle A, \oplus^{\mathbf{A}}, {}'^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \right\rangle$ be an MV algebra and let $k \in A$ be such that $k = k'$. The *pair algebra* over $\mathbf{A}$ is the algebra

$$\mathcal{P}(\mathbf{A}) = \left\langle A^2, \oplus^{\mathcal{P}(\mathbf{A})}, \sqrt{'}^{\mathcal{P}(\mathbf{A})}, 0^{\mathcal{P}(\mathbf{A})}, 1^{\mathcal{P}(\mathbf{A})}, k^{\mathcal{P}(\mathbf{A})} \right\rangle$$

where:

- $\langle a, b \rangle \oplus^{\mathcal{P}(\mathbf{A})} \langle c, d \rangle = \left\langle a \oplus^{\mathbf{A}} c, k \right\rangle$;
- $\sqrt{'}^{\mathcal{P}(\mathbf{A})} \langle a, b \rangle = \left\langle b, a'^{\mathbf{A}} \right\rangle$;
- $0^{\mathcal{P}(\mathbf{A})} = \left\langle 0^{\mathbf{A}}, k \right\rangle$;
- $1^{\mathcal{P}(\mathbf{A})} = \left\langle 1^{\mathbf{A}}, k \right\rangle$;
- $k^{\mathcal{P}(\mathbf{A})} = \langle k, k \rangle$.

Every pair algebra $\mathcal{P}(\mathbf{A})$ over an MV algebra $\mathbf{A}$ is a cartesian $\sqrt{'}$qMV algebra. Conversely, every cartesian $\sqrt{'}$qMV algebra is embeddable into a pair algebra:

Theorem 29. *Every cartesian $\sqrt{'}$qMV algebra $\mathbf{A}$ is embeddable into the pair algebra $\mathcal{P}(\mathbf{R_A})$ over its MV polynomial subreduct $\mathbf{R_A}$ of regular elements.*

Proof. Let $f : A \to \mathcal{R}(\mathbf{A})^2$ be defined by

$$f(a) = \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle .$$

The quasi-equation of Definition 25 guarantees that f is one-one. It can be checked that it preserves the operations. □

A variant of the direct decomposition for quasi-MV algebras provided by Theorem 15 carries over to our enriched structures:

Theorem 30. *For every $\sqrt{'}$qMV algebra $\mathbf{Q}$, there exist a cartesian algebra $\mathbf{C}$ and a flat algebra $\mathbf{F}$ such that $\mathbf{Q}$ can be embedded into the direct product $\mathbf{C} \times \mathbf{F}$.*

Proof. Let $\mathbf{Q} = \left\langle Q, \oplus^{\mathbf{Q}}, \sqrt{'}^{\mathbf{Q}}, 0^{\mathbf{Q}}, 1^{\mathbf{Q}}, k^{\mathbf{Q}} \right\rangle$ be a $\sqrt{'}$qMV algebra. The ingredients of our representation are the following:

- $\mathbf{C} = \mathcal{P}(\mathbf{R_Q})$, the pair $\sqrt{'}$qMV algebra over the MV algebra $\mathbf{R_Q}$ of regular elements of $\mathbf{Q}$;
- $\mathbf{F} = \mathbf{Q}/\mu$.

Now, let $h : Q \to \mathcal{R}(\mathbf{Q})^2 \times Q$ be given by:

$$h(a) = \begin{cases} \left\langle \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle, k/\mu \right\rangle, \text{ if } a \in \mathcal{R}(\mathbf{Q}) \cup \mathcal{COR}(\mathbf{Q}); \\ \left\langle \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle, a/\mu \right\rangle, \text{ otherwise.} \end{cases}$$

The function h does the required job. □

As regards positive properties of congruence lattices in $\sqrt{'}$qMV algebras, we were able to collect nothing but meagre results. For example, $\sqrt{'}q\mathbb{MV}$ is neither congruence modular nor ideal-determined. Nonetheless, in any $\sqrt{'}$qMV algebra, an important sublattice of the lattice of congruences is isomorphic to the lattice of the ideals of the corresponding MV algebraic subreduct:

Definition 31. Let $\mathbf{A}$ be a $\sqrt{'}$qMV algebra. A congruence θ on $\mathbf{A}$ is called a $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ *congruence* iff $\mathbf{A}/\theta$ is cartesian; in other words, iff for any $a, b \in A$, $a \oplus 0/\theta = b \oplus 0/\theta$ and $\sqrt{'}a \oplus 0/\theta = \sqrt{'}b \oplus 0/\theta$ implies $a/\theta = b/\theta$.

From the previous definition it follows that:

- λ is the smallest $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruence in any $\sqrt{'}$qMV algebra;
- in a flat $\sqrt{'}$qMV algebra, $\lambda = \nabla$ is the unique $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruence;
- in a cartesian $\sqrt{'}$qMV algebra, $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruences are exactly the relative congruences.

Ideals in $\sqrt{'}$qMV algebras are defined exactly as in MV algebras. Let $\mathcal{I}(\mathbf{R_A})$, $\mathcal{I}(\mathbf{A})$ and $\mathcal{C}^{\mathcal{I}}(\mathbf{A})$ denote the lattices of, respectively, the ideals of $\mathbf{R_A}$, the ideals of $\mathbf{A}$, and the $\sqrt{'}q\mathbb{MV} - \mathbb{C}$ congruences of $\mathbf{A}$. We have that:

Theorem 32. *Let* $\mathbf{A}$ *be a* $\sqrt{'}$*qMV algebra. The following lattices are mutually isomorphic:*

- $\mathcal{C}^{\mathcal{I}}(\mathbf{A})$;
- $\mathcal{C}(\mathbf{R_A})$;

- $\mathcal{I}(\mathbf{R_A})$;
- $\mathcal{I}(\mathbf{A})$.

Since $\mathbb{MV}$ is known to be an arithmetical variety, the previous theorem implies that:

Corollary 33. $\mathbb{C}$ *is a relatively congruence distributive and relatively congruence permutable quasivariety.*

A closer scrutiny of strongly cartesian algebras allows one to prove:

Lemma 34. *The universal class of strongly cartesian* $\sqrt{\prime}$ *qMV algebras is closed w.r.t. quotients, but not w.r.t. products.*

Proof. The first claim follows from the fact that the universal formula defining strongly cartesian algebras is a positive formula. As to the second one, consider the Cross $\mathbf{C}$, which is strongly cartesian. In $\mathbf{C} \times \mathbf{C}$, the element $\left\langle \sqrt{\prime}0, 1 \right\rangle$ is neither regular nor coregular, for $\left\langle \sqrt{\prime}0, 1 \right\rangle \oplus \langle 0, 0 \rangle = \langle k, 1 \rangle$, while $\sqrt{\prime} \left\langle \sqrt{\prime}0, 1 \right\rangle \oplus \langle 0, 0 \rangle = \langle 1, k \rangle \neq \left\langle 1, \sqrt{\prime}1 \right\rangle = \sqrt{\prime} \left\langle \sqrt{\prime}0, 1 \right\rangle$. □

It is shown in [12] that the quasivariety of Cartesian $\sqrt{\prime}$qMV algebras generates the whole variety $\sqrt{\prime}q\mathbb{MV}$:

Theorem 35. $\mathbf{V}(\mathbb{C}) = \sqrt{\prime}q\mathbb{MV}$.

Moreover, the standard completeness theorem for qMV algebras carries over to $\sqrt{\prime}q\mathbb{MV}$. We state it in the next section in a more general form than we did in [12].

4. Finite model property

A variety of algebras of type $\mathcal{T}$ has the finite model property (FMP) if every equation of type $\mathcal{T}$ which does not hold in the variety can be falsified in a finite member of the variety. Alternatively, one can say that a variety has the FMP iff it is generated as a variety by its finite members. This is indeed a remarkable property, since e.g. it implies decidability. As it is well-known that the variety $\mathbb{MV}$ has the FMP (see e.g. [15]), it is all too natural to investigate whether $q\mathbb{MV}$ and $\sqrt{\prime}q\mathbb{MV}$ have the property as well. In [12] it is shown that their respective subvarieties $\mathbb{F}q\mathbb{MV}$ (flat qMV algebras) and $\mathbb{F}$ (flat $\sqrt{'}$qMV algebras) have the FMP (actually, an explicit proof is given only for $\mathbb{F}$, yet a proof for $\mathbb{F}q\mathbb{MV}$ can be easily obtained by fiddling).

For a start, we notice that the finite model property for $q\mathbb{MV}$ is little more than a triviality:

Lemma 36. *$q\mathbb{MV}$ has the FMP.*

Proof. Let $t, s \in Term(\langle 2, 1, 0, 0 \rangle)$ be such that the equation $t \approx s$ has a counterexample in a given qMV algebra $\mathbf{A}$. By Theorem 15 it has a counterexample in $\mathbf{A}/\chi \times \mathbf{A}/\tau$, thus either in $\mathbf{A}/\chi$ or in $\mathbf{A}/\tau$. Since both $\mathbb{MV}$ and $\mathbb{F}q\mathbb{MV}$ are subvarieties of $q\mathbb{MV}$ with the FMP, there exists either a finite MV algebra $\mathbf{M}$ or a finite flat qMV algebra $\mathbf{F}$ which falsifies the equation. □

The analogous problem for $\sqrt{'}q\mathbb{MV}$ is somewhat more interesting. To address it, we need to restate the standard completeness theorem for $\sqrt{'}$qMV algebras in a more general form. Thus, let $\mathbf{Q}_r$ be the subalgebra of $\mathbf{S}_r$ whose universe is the set $(\mathbb{Q} \cap [0,1]) \times (\mathbb{Q} \cap [0,1])$.

Theorem 37. *Let $t, s \in Term(\langle 2, 1, 0, 0, 0 \rangle)$. The following are equivalent:*

1. $\mathbf{S}_r \vDash t \approx s$;
2. $\mathbf{Q}_r \vDash t \approx s$;
3. $\sqrt{'}q\mathbb{MV} \vDash t \approx s$;
4. $\mathbf{N}_{\mathbf{Q}_r} \vDash t \approx s$;
5. $\mathbf{N}_{\mathbf{S}_r} \vDash t \approx s$.

Proof. The equivalence of (1) and (3) is just Corollary 53 in [12], while the equivalence of (2) and (3) can be recovered from the proof of Corollary 53 in [12], in the light of the completeness theorem for MV algebras w.r.t. the standard MV algebra over the *rational* numbers. (2) obviously implies (4) and (4) implies (5) for the same reason as above. Thus, it remains to show that (5) implies (1).

We proceed contrapositively. Suppose that $t \approx s$ has a counterexample in $\mathbf{S}_r$. W.l.g. we can restrict ourselves to the case in which t either is a constant preceded by k ($k \geq 0$) square root symbols, or else contains at least an occurrence of $\oplus$, while s is the variable x preceded by k ($k \geq 0$) square root symbols, for equations of the remaining forms are easily seen either to have no counterexample (e.g. $x \approx x$), or to have counterexamples already in $\mathbf{N}_{\mathbf{S}_r}$.

If t is a constant preceded by k ($k \geq 0$) square root symbols, it suffices to assign x the value $\left\langle m, \frac{1}{2} \right\rangle$, with $0 < m < \frac{1}{2}$, to get the desired counterexample. So, let t be a term in the variables $x_1, ..., x_n$ containing at least an occurrence of $\oplus$. It follows that there exist $a_i, b_i, m, n, a \in [0,1]$ and $b, c \neq \frac{1}{2}$ s.t. either:

$$t^{\mathbf{S}_r}\left(\left\langle \overrightarrow{a_i, b_i} \right\rangle\right) = \left\langle a, \tfrac{1}{2} \right\rangle \neq s^{\mathbf{S}_r}\left(\langle b, c \rangle\right) \text{ or}$$
$$t^{\mathbf{S}_r}\left(\left\langle \overrightarrow{a_i, b_i} \right\rangle\right) = \left\langle \tfrac{1}{2}, a \right\rangle \neq s^{\mathbf{S}_r}\left(\langle b, c \rangle\right),$$

where the vectorial notation is self-explanatory. We deal with the former case, the latter being perfectly dual.

First case: $a \neq \frac{1}{2}$. Then there exist terms t_1, t_2 such that t is the term $t_1 \oplus t_2$ preceded by an even number (possibly zero) of square root symbols. Thus, we assign x the value $\left\langle b, \frac{1}{2} \right\rangle$ or $\left\langle \frac{1}{2}, c \right\rangle$ according as it is preceded by an odd or an even number of square root symbols, while the other variables in t can be assigned arbitrary values (say $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$). Let $t^{\mathbf{S}_r}(v_1, ..., v_n)$ be the value obtained by t under this assignment. Then $s^{\mathbf{S}_r}\left(\left\langle b, \frac{1}{2} \right\rangle\right)$ (respectively, $s^{\mathbf{S}_r}\left(\left\langle \frac{1}{2}, c \right\rangle\right)$ is coregular and different from $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$, while $t^{\mathbf{S}_r}(v_1, ..., v_n)$ is regular.

Second case: $a = \frac{1}{2}$. Then there exist terms t_1, t_2 such that t is the term $t_1 \oplus t_2$ preceded by zero or more square root symbols, and $t_1 \oplus t_2^{\mathbf{S}_r}\left(\left\langle \overrightarrow{a_i, b_i} \right\rangle\right) = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$. If the number of square root symbols preceding $t_1 \oplus t_2$ is even, we proceed as in the previous case. If it is odd, we assign x the value $\left\langle b, \frac{1}{2} \right\rangle$ or $\left\langle \frac{1}{2}, c \right\rangle$ according as it is preceded by an even or an odd number of square root symbols, while the other variables in t can be assigned arbitrary values (say $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$). Let $t^{\mathbf{S}_r}(v_1, ..., v_m)$ be the value obtained by t under this assignment. Then $s^{\mathbf{S}_r}\left(\left\langle b, \frac{1}{2} \right\rangle\right)$ (respectively, $s^{\mathbf{S}_r}\left(\left\langle \frac{1}{2}, c \right\rangle\right)$ is regular and different from $\left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$, while $t^{\mathbf{S}_r}(v_1, ..., v_m)$ is coregular. $\square$

This variant of the completeness theorem for $\sqrt{\prime}q\mathbb{MV}$ comes in quite handy since $\mathbf{N}_{\mathbf{Q}_r}$, unlike $\mathbf{S}_r$ or $\mathbf{N}_{\mathbf{S}_r}$, is a locally finite algebra, as we presently show.

Lemma 38. $\mathbf{N}_{\mathbf{Q}_r}$ *is locally finite.*

Proof. Let $a_1, ..., a_n \in \mathcal{R}(\mathbf{Q}_r) \cup \mathcal{COR}(\mathbf{Q}_r)$, and let $\mathtt{Sg}(a_1, ..., a_n)$ be the subalgebra of $\mathbf{N}_{\mathbf{Q}_r}$ generated by $a_1, ..., a_n$. Recall that, if a is coregular, then for any b we have that $b \oplus a = b \oplus k$, while the sum of two coregulars equals 1. Thus, since the MV term subreduct $\mathbf{R}_{\mathbf{Q}_r}$ is locally finite (cp. [7]), closure under truncated sum yields only finitely many new elements beside the generators, and the same can be said for closure under square root of the inverse. Therefore $\mathtt{Sg}(a_1, ..., a_n)$ is finite. □

Theorem 39. $\sqrt{'}q\mathbb{MV}$ *has the FMP.*

Proof. Let $t(x_1, ..., x_n) \approx s(y_1, ..., y_m)$ be an equation which has a counterexample in a $\sqrt{'}$ qMV algebra. By Theorem 37 it has a counterexample in $\mathbf{N}_{\mathbf{Q}_r}$, i.e. there exist $a_1, ..., a_n, b_1, ..., b_m \in \mathcal{R}(\mathbf{Q}_r) \cup \mathcal{COR}(\mathbf{Q}_r)$ s.t.

$$t^{\mathbf{N}_{\mathbf{Q}_r}}(a_1, ..., a_n) \neq s^{\mathbf{N}_{\mathbf{Q}_r}}(b_1, ..., b_m).$$

Consider $\mathtt{Sg}(a_1, ..., a_n, b_1, ..., b_m)$. By Lemma 38, it is a finite subalgebra of $\mathbf{N}_{\mathbf{Q}_r}$, hence a finite $\sqrt{'}$ qMV algebra in its own right. Thus we have the desired countermodel. □

It would be interesting to ascertain whether $q\mathbb{MV}$ and $\sqrt{'}q\mathbb{MV}$ have the *strong* finite model property - i.e. whether they are generated *as quasivarieties* by their finite members. This question is answered in the positive for $\mathbb{MV}$ in [2].

5. Congruence extension property

We say that a variety $\mathcal{V}$ has the *congruence extension property* (CEP) iff for every $\mathbf{A} \in \mathcal{V}$, every subalgebra $\mathbf{B}$ of $\mathbf{A}$, and every congruence θ on $\mathbf{B}$, there is a congruence ψ on the whole of $\mathbf{A}$ which extends θ, i.e. such that $\theta = \psi \cap B^2$. That $\mathbb{MV}$ has the CEP seems to be part of the folklore about the subject: it follows from more general results about residuated lattices ([18]) and is explicitly proved e.g. in [10]. On the other hand, the proof of the CEP for MV algebras - as well as the analogous proofs for commutative residuated lattices or for other varieties of algebras motivated by logic - rests essentially on the 1-1 correspondence between congruences and ideals, a property which is available neither for $q\mathbb{MV}$ nor for $\sqrt{\prime}q\mathbb{MV}$. As a consequence, a different strategy is needed.

For a start, we notice that:

Lemma 40. *A flat qMV algebra has the same congruences as its monounary reduct.*

Proof. There is no way a reflexive binary relation on a flat qMV algebra can disrespect truncated sum. □

As a consequence:

Lemma 41. *The variety* $\mathbb{F}q\mathbb{MV}$ *has the CEP.*

Proof. Let $\mathbf{A}$ be a flat qMV algebra, let $\mathbf{B}$ be a subalgebra of its, and let θ be an arbitrary congruence on $\mathbf{B}$. It follows from Lemma 40 and from simple results about monounary algebras that the relation

$$\theta' = \left\{ \langle a, b \rangle \in A^2 : a\theta b \text{ or } a = b \right\}$$

is a congruence on $\mathbf{A}$ s.t. $\theta = \theta' \cap B^2$. □

Thus, every qMV algebra is directly decomposable in terms of algebras which belong to subvarieties of $q\mathbb{MV}$ with the CEP. We exploit this fact to attain our goal.

Lemma 42. *Let* $\mathbf{A}$ *be a qMV algebra and let* θ *be any congruence on* $\mathbf{A}$. *Then there exist a congruence* θ_1 *on* $\mathbf{A}/\chi$ *and a congruence* θ_2 *on* $\mathbf{A}/\tau$ *such that* $a\theta b$ *iff* $\langle a/\chi, a/\tau \rangle \, \theta_1 \times \theta_2 \, \langle b/\chi, b/\tau \rangle$.

Proof. Let θ be a congruence on $\mathbf{A}$. If f is any injective homomorphism with domain $\mathbf{A}$, it follows that $\{\langle f(a), f(b) \rangle : a\theta b\}$ is a congruence on the range of f. In particular, if f is the embedding provided by Theorem 15, then $\theta' = \{\langle \langle a/\chi, a/\tau \rangle, \langle b/\chi, b/\tau \rangle \rangle : a\theta b\}$ is a congruence s.t. $a\theta b$ iff $f(a)\theta' f(b)$. Now, define binary relations θ_1 on $\mathbf{A}/\chi$ and θ_2 on $\mathbf{A}/\tau$ in such a way that

$$a/\chi \theta_1 b/\chi \text{ iff } \langle a/\chi, a/\tau \rangle \, \theta' \, \langle b/\chi, b/\tau \rangle \, ;$$
$$a/\tau \theta_2 b/\tau \text{ iff } \langle a/\chi, a/\tau \rangle \, \theta' \, \langle b/\chi, b/\tau \rangle \, .$$

θ_1 and θ_2 are congruences, respectively, on $\mathbf{A}/\chi$ and $\mathbf{A}/\tau$. Moreover,

$$\begin{aligned} a\theta b \quad & \text{iff } \langle a/\chi, a/\tau\rangle\, \theta'\, \langle b/\chi, b/\tau\rangle \\ & \text{iff } a/\chi\theta_1 b/\chi \text{ and } a/\tau\theta_2 b/\tau \\ & \text{iff } \langle a/\chi, a/\tau\rangle\, \theta_1 \times \theta_2\, \langle b/\chi, b/\tau\rangle \end{aligned}$$

□

It is not hard to see that:

Lemma 43. *If* $\mathbf{A}$ *is a qMV algebra and* $\mathbf{B}$ *is a subalgebra of its, then* $\chi^{\mathbf{B}}$ *extends to* $\chi^{\mathbf{A}}$ *and* $\tau^{\mathbf{B}}$ *extends to* $\tau^{\mathbf{A}}$.

Having settled these preliminaries, we now proceed to prove that

Theorem 44. *The variety* $q\mathbb{MV}$ *has the CEP.*

Proof. Take a qMV algebra $\mathbf{A}$, a subalgebra $\mathbf{B}$ of its, and an arbitrary congruence θ on $\mathbf{B}$. By Lemma 42, there exist a congruence θ_1 on $\mathbf{B}/\chi^{\mathbf{B}}$ and a congruence θ_2 on $\mathbf{B}/\tau^{\mathbf{B}}$ such that, for any a, b in B, $a\theta b$ iff $\langle a/\chi^{\mathbf{B}}, a/\tau^{\mathbf{B}}\rangle\, \theta_1 \times \theta_2\, \langle b/\chi^{\mathbf{B}}, b/\tau^{\mathbf{B}}\rangle$. We recall that, since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, the sets $\{x/\chi^{\mathbf{A}} : x \in B\}$ and $\{x/\tau^{\mathbf{A}} : x \in B\}$ are subuniverses, respectively, of $\mathbf{A}/\chi^{\mathbf{A}}$ and $\mathbf{A}/\tau^{\mathbf{A}}$. Recall moreover that the CEP holds for MV algebras and, in the light of Lemma 41, also for flat qMV algebras. So, by Lemma 43 and the third isomorphism theorem, there exist congruences θ_1' on $\mathbf{A}/\chi^{\mathbf{A}}$ and θ_2' on $\mathbf{A}/\tau^{\mathbf{A}}$ s.t. $\theta_1 = \theta_1' \cap (B/\chi^{\mathbf{B}})^2$ and $\theta_2 = \theta_2' \cap (B/\tau^{\mathbf{B}})^2$. We have to show that $\theta_1 \times \theta_2 = (\theta_1' \times \theta_2') \cap (B/\chi^{\mathbf{B}} \times B/\tau^{\mathbf{B}})^2$. For the nontrivial direction, let $a, b \in B$, and $\langle a/\chi^{\mathbf{A}}, a/\tau^{\mathbf{A}}\rangle\, \theta_1' \times \theta_2'\, \langle b/\chi^{\mathbf{A}}, b/\tau^{\mathbf{A}}\rangle$. Then $a/\chi^{\mathbf{A}}\theta_1' b/\chi^{\mathbf{A}}$ and thus $a/\chi^{\mathbf{B}}\theta_1 b/\chi^{\mathbf{B}}$, while $a/\tau^{\mathbf{A}}\theta_2' b/\tau^{\mathbf{A}}$ and thus $a/\tau^{\mathbf{B}}\theta_2 b/\tau^{\mathbf{B}}$. So $\langle a/\chi^{\mathbf{B}}, a/\tau^{\mathbf{B}}\rangle\, \theta_1 \times \theta_2\, \langle b/\chi^{\mathbf{B}}, b/\tau^{\mathbf{B}}\rangle$. □

We now tackle the analogous problem for $\sqrt{'}q\mathbb{MV}$. The obvious option will consist in mimicking the previous strategy and exploiting the direct decomposition result of Theorem 30. There is an important difference, though: the role previously played by MV algebras - a subvariety for which the CEP is known to hold - is now played by the subquasivariety of cartesian algebras. Thus, we wish to investigate first whether *relative* congruences extend in this quasivariety. This is indeed the case:

Lemma 45. *The quasivariety* $\mathbb{C}$ *has the RCEP.*

Proof. By Theorem 32 and the observation that $\sqrt{'}q\mathbb{MV}-\mathbb{C}$ congruences are exactly the relative congruences in cartesian algebras, it suffices to prove that $\mathbb{C}$ has the ideal extension property, namely, that given any cartesian $\sqrt{'}$ qMV algebra $\mathbf{A}$, any subalgebra $\mathbf{B}$ of $\mathbf{A}$, and any ideal I of $\mathbf{B}$, there exists an ideal J of $\mathbf{A}$ s.t. $I = J \cap B$. In fact, let $(I]$ be the ideal of $\mathbf{A}$ generated by I. It is easy to see that

$$(I] = \{x \in A : x \leq a_1 \oplus ... \oplus a_n \text{ for some } a_1, ..., a_n \in I\}$$

and it is just as easy to show that $I = (I] \cap B$. □

The previous lemma turns out to be the key to the desired result:

Theorem 46. $\sqrt{'}q\mathbb{MV}$ *has the CEP.*

Proof. Lemma 42 can be easily adapted to cover the case of $\sqrt{'}q\mathbb{MV}$: if $\mathbf{A}$ is a $\sqrt{'}$qMV algebra and θ a congruence on $\mathbf{A}$, there exist a congruence θ_1 on $\mathcal{P}(\mathbf{R_A})$ and a congruence θ_2 on $\mathbf{F}(\mathbf{A}, k)$ such that, for any a, b in A,

$$a\theta b \text{ iff } \left\langle \left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle, \pi_2(h(a)) \right\rangle \theta_1 \times \theta_2 \left\langle \left\langle b \oplus 0, \sqrt{'}b \oplus 0 \right\rangle, \pi_2(h(b)) \right\rangle,$$

where $\pi_2(h(a))$ is the second component of the image of a under the injection of Theorem 30. By Lemma 45, if we can prove that θ_1 therein is necessarily a $\sqrt{'}q\mathbb{MV}-\mathbb{C}$ congruence, we are done, for we can resort to the strategy of Theorem 44. Carrying out the appropriate calculations, what must be shown is that $\langle a \oplus 0, k \rangle \, \theta_1 \, \langle b \oplus 0, k \rangle$ and $\left\langle \sqrt{'}a \oplus 0, k \right\rangle \theta_1 \left\langle \sqrt{'}b \oplus 0, k \right\rangle$ imply $\left\langle a \oplus 0, \sqrt{'}a \oplus 0 \right\rangle \theta_1 \left\langle b \oplus 0, \sqrt{'}b \oplus 0 \right\rangle$.

We distinguish several cases. First of all, remark that a and b must be either regular or coregular, because the first components of $h(a)$ and $h(b)$ must be, respectively, $\langle a \oplus 0, k \rangle$ and $\langle b \oplus 0, k \rangle$ for $\langle a \oplus 0, k \rangle \, \theta_1 \, \langle b \oplus 0, k \rangle$ to hold.

So, if a, b are both regular, then $\langle a, k \rangle \, \theta_1 \, \langle b, k \rangle$ and we are done. If a, b are both coregular, $\left\langle \sqrt{'}a, k \right\rangle \theta_1 \left\langle \sqrt{'}b, k \right\rangle$ and thus $\left\langle k, \sqrt{'}a' \right\rangle \theta_1 \left\langle k, \sqrt{'}b' \right\rangle$, whence $\left\langle k, \sqrt{'}a \right\rangle \theta_1 \left\langle k, \sqrt{'}b \right\rangle$. Finally, if w.l.g. a is regular and b is coregular, then $\langle a, k \rangle \, \theta_1 \, \langle k, k \rangle$ and $\langle k, k \rangle \, \theta_1 \left\langle \sqrt{'}b, k \right\rangle$, whereby $\langle k, k \rangle \, \theta_1 \left\langle k, \sqrt{'}b \right\rangle$ and by transitivity $\langle a, k \rangle \, \theta_1 \left\langle k, \sqrt{'}b \right\rangle$. □

6. QMV term reducts and term subreducts of $\sqrt{'}$ qMV algebras

Which qMV algebras are such that one can impose thereupon a square root of the inverse? And which qMV algebras are embeddable into the appropriate term reduct of a $\sqrt{'}$ qMV algebra? In this subsection we will on the one hand provide necessary and sufficient conditions for a qMV algebra to be a qMV term reduct of a $\sqrt{'}$ qMV algebra, and on the other hand show that qMV algebras are exactly the qMV term subreducts of $\sqrt{'}$ qMV algebras.

Definition 47. A qMV algebra $\mathbf{A}$ is called *extensible* iff it has the following three properties:

E1 it contains a regular element $k = k'$;

E2 the cloud of k contains a bijective copy $f(\mathcal{R}(\mathbf{A}))$ of $\mathcal{R}(\mathbf{A})$, with $f(k) = k$;

E3 *(Four-partitioning property)*: the set $NF_A = \{a \in A : a \neq a'\}$ can be partitioned into classes of cardinality 4, in such a way that every $a \in NF_A$ belongs to the same member of the partition as a'.

Extensibility is a necessary condition for being a qMV term reduct of a $\sqrt{'}$ qMV algebra:

Lemma 48. *Every qMV term reduct of a $\sqrt{'}$ qMV algebra is an extensible qMV algebra.*

Proof. Let $\mathbf{A} = \left\langle A, \oplus, \sqrt{'}, 0, 1, k \right\rangle$ be a $\sqrt{'}$ qMV algebra. By SQ1, $\langle A, \oplus, ', 0, 1\rangle$ is a qMV algebra; we now check one by one the requirements E1-E3 in order to show it is extensible.

(*ad* E1). k is regular and it is a fixpoint for the inverse operation, by Lemma 19(i)-(ii).

(*ad* E2). Let a be a regular element of A: by SQ3, $\sqrt{'}a$ is in the cloud of k, whence $f(a) = \sqrt{'}a$ maps $\mathcal{R}(\mathbf{A})$ to the cloud of k. It is also injective, since $\sqrt{'}a = \sqrt{'}b$ implies $a' = b'$ and thus $a = b$. Finally, by SQ2 $f(k) = k$.

(*ad* E3). Let θ be an equivalence on NF_A defined as follows: for every a in NF_A, $a/\theta = \left\{a, \sqrt{'}a, a', \sqrt{'}a'\right\}$. Since $a \neq a'$, by what we remarked above it has to be $\sqrt{'}a \neq \sqrt{'}a'$; also, $a \neq \sqrt{'}a$ (or else it would be $a = a'$) and similarly $a \neq \sqrt{'}a'$. Thus, each member of the given partition has the required cardinality. □

All that remains to be shown is that extensibility is a *sufficient* condition as well. In fact:

Theorem 49. *Every extensible qMV algebra is a qMV term reduct of a* $\sqrt{'}$ *qMV algebra.*

Proof. Let $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ be an extensible qMV algebra, on which we need to define an operation of square root of the inverse. By E1, there is in $\mathcal{R}(\mathbf{A})$ an element $k = k'$, and by E2 the cloud of k contains a bijective copy $f(\mathcal{R}(\mathbf{A}))$ of $\mathcal{R}(\mathbf{A})$, with $f(k) = k$. Moreover, if $F = \{a, b, a', b'\} \in NF_A$, the partition whose existence is guaranteed by E3, we define g in such a way that $g(a) = b, g(b) = a', g(a') = b'$ and $g(b') = a$ - or, alternatively, $g(a) = b', g(b') = a', g(a') = b$ and $g(b) = a$. Now, define the operation $\sqrt{'}$ as follows:

$$\sqrt{'}a = \begin{cases} f(a), \text{ if } a \in \mathcal{R}(\mathbf{A}); \\ \left(f^{-1}(a)\right)', \text{ if } a \in f(\mathcal{R}(\mathbf{A})) - \{k\}; \\ a, \text{ if } a \notin \mathcal{R}(\mathbf{A}) \cup f(\mathcal{R}(\mathbf{A})) \text{ and } a = a'; \\ g(a), \text{ if } a \notin \mathcal{R}(\mathbf{A}) \cup f(\mathcal{R}(\mathbf{A})) \text{ and } a \neq a'. \end{cases}$$

First of all, we have to show that this definition does not conflict with the behaviour of the inverse. In fact: i) if $a \in \mathcal{R}(\mathbf{A})$, $\sqrt{'}\sqrt{'}a = \sqrt{'}f(a) = a'$; ii) if $a \in f(\mathcal{R}(\mathbf{A})) - \{k\}$, $\sqrt{'}\sqrt{'}a = \sqrt{'}\left(f^{-1}(a)\right)' = f\left(f^{-1}(a)\right)' = a'$; (iii) if $a \notin \mathcal{R}(\mathbf{A}) \cup f(\mathcal{R}(\mathbf{A}))$ and $a = a'$, $\sqrt{'}\sqrt{'}a = a = a'$; finally, iv) if $a \notin \mathcal{R}(\mathbf{A}) \cup f(\mathcal{R}(\mathbf{A}))$ and $a \neq a'$, $\sqrt{'}\sqrt{'}a = \sqrt{'}g(a) = g(g(a)) = a'$.

Now, we check the remaining two axioms. SQ2 is satisfied because $\sqrt{'}k = f(k) = k$ by hypothesis; SQ3 holds because $a \oplus b$ is a regular element, whence $\sqrt{'}(a \oplus b) = f(a \oplus b)$ is in the cloud of k, which means that $\sqrt{'}(a \oplus b) \oplus 0 = k$. □

The conditions in Definition 47 are far from elementary. It would be nice to describe extensible qMV algebras using more manageable conditions from a model-theoretic viewpoint. Also, it would be interesting to characterize such algebras in terms of properties of their factors in the direct decomposition of Theorem 15.

We now turn to the issue of subreducts. As a preliminary move, we need to prove that every qMV algebra can be embedded into a qMV algebra where inverse has a fixpoint.

Lemma 50. *Every qMV algebra can be embedded into a qMV algebra with a regular element k s.t. $k = k'$.*

Proof. Let $\mathbf{Q}$ be a qMV algebra. By Theorem 15, there are an MV algebra $\mathbf{M}$ and a flat qMV algebra $\mathbf{F}$ s.t. $\mathbf{Q}$ is embeddable into $\mathbf{M} \times \mathbf{F}$. By Di Nola's representation theorem (cp. [6]), $\mathbf{M}$ is embeddable into an MV algebra $\mathbf{M}'$ where inverse has a fixpoint i, while $\mathbf{F}$ is (identically) embeddable into a flat qMV algebra $\mathbf{F}'$ where inverse has a fixpoint $0^{\mathbf{F}'}$. Let f, g, h be, respectively, such embeddings. Then the mapping

$$(g \times h)(\langle a, b \rangle) = \langle g(a), h(b) \rangle$$

is obviously an embedding of $\mathbf{M} \times \mathbf{F}$ into $\mathbf{M}' \times \mathbf{F}'$. Composing f with $g \times h$, we can embed $\mathbf{Q}$ into $\mathbf{M}' \times \mathbf{F}'$. It remains to be shown that $\mathbf{M}' \times \mathbf{F}'$ contains a regular element k s.t. $k = k'$: but it is immediate to check that the pair $\left\langle i, 0^{\mathbf{F}'} \right\rangle$ has the required properties. □

By definition, every qMV term reduct of a $\sqrt{'}$ qMV algebra is a qMV algebra, and so is every subalgebra of such since $q\mathbb{MV}$ is a variety. We now show the converse:

Theorem 51. *Every qMV algebra is a qMV term subreduct of a $\sqrt{'}$ qMV algebra.*

Proof. Let $\mathbf{A} = \left\langle A, \oplus^{\mathbf{A}}, '^{\mathbf{A}}, 0^{\mathbf{A}}, 1^{\mathbf{A}} \right\rangle$ be a qMV algebra. By Lemma 50, we can assume w.l.g. that it satisfies E1 in Definition 47. Consider the set

$$K_A = \left\{ \langle a, b \rangle \in A^2 : a = k \text{ or } b = k \right\},$$

and define the algebra $\mathbf{K} = \left\langle K_A, \oplus^{\mathbf{K}}, \sqrt{'}^{\mathbf{K}}, 0^{\mathbf{K}}, 1^{\mathbf{K}}, k^{\mathbf{K}} \right\rangle$, where:

- $\langle a, b \rangle \oplus^{\mathbf{K}} \langle c, d \rangle = \left\langle a \oplus^{\mathbf{A}} c, k \right\rangle$;
- $\sqrt{'}^{\mathbf{K}} \langle a, b \rangle = \left\langle b, a'^{\mathbf{A}} \right\rangle$;
- $0^{\mathbf{K}} = \left\langle 0^{\mathbf{A}}, k \right\rangle, 1^{\mathbf{K}} = \left\langle 1^{\mathbf{A}}, k \right\rangle, k^{\mathbf{K}} = \langle k, k \rangle$.

These operations are well-defined: the second component of a sum is always k, while the first or the second component of $\sqrt{\prime}^{\mathbf{K}}\langle a,b\rangle$ is k according as the second or the first component of $\langle a,b\rangle$ is k. Also, $\mathbf{K}$ is easily seen to be a $\sqrt{\prime}$ qMV algebra. We verify that the mapping $h(a)=\langle a,k\rangle$ embeds $\mathbf{A}$ into the qMV term reduct of $\mathbf{K}$. The mapping is clearly well-defined and injective; we now check that it preserves operations. Nullary operations are obviously preserved. As regards truncated sum,

$$\begin{aligned} h(a\oplus^{\mathbf{A}} b) &= \left\langle a\oplus^{\mathbf{A}} b,k\right\rangle \\ &= \langle a,k\rangle \oplus^{\mathbf{K}} \langle b,k\rangle \\ &= h(a)\oplus^{\mathbf{K}} h(b). \end{aligned}$$

Finally, we have preservation of inverses:

$$\begin{aligned} h(a'^{\mathbf{A}}) &= \left\langle a'^{\mathbf{A}},k\right\rangle \\ &= \langle a,k\rangle'^{\mathbf{K}} \\ &= h(a)'^{\mathbf{K}}. \end{aligned}$$

□

7. Semisimple algebras

One of the deepest results in the theory of MV algebras is the representation of semisimple MV algebras in terms of algebras of real-valued functions: [5], [1]. It is an important theorem since it allows to think of a semisimple MV algebra as an algebra of *fuzzy sets*, exactly like Boolean algebras (which are always semisimple) can be viewed as algebras of standard, "crisp" sets - and as such it acts as a proper fuzzy counterpart of Stone's representation theorem.

Can we get an analogous result in our framework? The aim of this section is to answer this question in the affirmative, although the theorem we will prove is, frankly speaking, far less elegant. We will confine our investigations to $\sqrt{\prime}q\mathbb{MV}$.

Lemma 52. *If* $\mathbf{R}_A$ *is a subalgebra of* $\mathbf{R}_B$*, the pair algebra* $\mathcal{P}(\mathbf{R}_A)$ *is a subalgebra of* $\mathcal{P}(\mathbf{R}_B)$*.*

Proof. Let f be an embedding of $\mathbf{R}_A$ into $\mathbf{R}_B$. Now define, for $a,b\in\mathcal{R}(\mathbf{A})$,

$$g(\langle a,b\rangle)=\langle f(a),f(b)\rangle .$$

This is a homomorphism from $\mathcal{P}(\mathbf{R}_A)$ to $\mathcal{P}(\mathbf{R}_B)$. To see that it is one-one, suppose that $g(\langle a,b\rangle) = g(\langle c,d\rangle)$, i.e. $\langle f(a), f(b)\rangle = \langle f(c), f(d)\rangle$. Then $f(a) = f(c)$ and $f(b) = f(d)$, whence by the injectivity of f, $a = c$ and $b = d$. Thus our conclusion follows. □

Before we address our main problem, we need to know something more about the structure of simple algebras - in particular, since every simple $\sqrt{'}$ qMV algebra is either cartesian or flat, we need additional information about simple cartesian algebras and simple flat algebras. The next three lemmas provide such information.

Lemma 53. *If* $\mathbf{A}$ *is a simple cartesian* $\sqrt{'}$ *qMV algebra, then* $\mathbf{A}$ *is strongly cartesian.*

Proof. Assume that $\mathbf{A}$ is cartesian yet not strongly cartesian and consider the congruence whose blocks are just $\mathcal{R}(\mathbf{A}) \cup \mathcal{COR}(\mathbf{A})$ and its set-theoretical complement w.r.t A. This congruence differs from ∇ as $\mathbf{A}$ is not strongly cartesian, and differs from Δ since $\mathbf{A}$ is cartesian. It follows that $\mathbf{A}$ is not simple. □

Lemma 54. *If* $\mathbf{A}$ *is a simple cartesian* $\sqrt{'}$ *qMV algebra, then* $\mathbf{R}_A$ *is a simple MV algebra.*

Proof. Lemma 53 entitles us to assume that $\mathbf{A}$ be strongly cartesian. Suppose $\theta \in \mathcal{C}(\mathbf{R}_A)$ is such that $\Delta \subset \theta \subset \nabla$. Let the partition induced by θ be

$$\{A_i\}_{i<\lambda} \cup \{k/\theta\},$$

where $\lambda \geq 1$ (since $\theta \subset \nabla$) and either k/θ or some A_i is not a singleton (since $\Delta \subset \theta$). Now, for $X \subseteq A$, let $\sqrt{'}X = \left\{\sqrt{'}x : x \in X\right\}$, and let φ be an equivalence on A such that

$$A/\varphi = \{A_i\}_{i<\lambda} \cup \left\{\sqrt{'}A_i\right\}_{i<\lambda} \cup \left\{k/\theta \cup \sqrt{'}k/\theta\right\}.$$

It is easy to see that φ is a congruence on $\mathbf{A}$. Also, $\Delta \subset \varphi \subset \nabla$ because of the above assumptions on θ. So $\mathbf{A}$ is not simple. □

Lemma 55. *The only simple flat* $\sqrt{'}$ *qMV algebra is* $\mathbf{F}_{100}$.

Proof. As we know from Example 26, $\mathbf{F}_{100}$ is simple. On the other hand, let $\mathbf{A}$ be a flat $\sqrt{'}$ qMV algebra with at least 3 elements. In the light of Lemma 40, the equivalence whose blocks are just $\{0\}$ and $A - \{0\}$ is a congruence, different from Δ ($\mathbf{A}$ has more than two elements) and from ∇ ($\mathbf{A}$ is nontrivial). Thus, $\mathbf{A}$ is not simple. □

We now prove an analogue of Hölder's theorem for cartesian $\sqrt{\prime}$ qMV algebras.

Theorem 56. *Every simple cartesian $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{S}_r$.*

Proof. By Theorem 29 $\mathbf{A}$ can be embedded into $\mathcal{P}(\mathbf{R}_A)$, and $\mathbf{R}_A$, according to Lemma 54, is a simple MV algebra, hence isomorphic to a subalgebra of $\mathbf{MV}_{[0,1]}$. Thus, by Lemma 52, $\mathcal{P}(\mathbf{R}_A)$ is isomorphic to a subalgebra of $\mathbf{S}_r$, whence our conclusion. □

Theorem 57. *Every semisimple $\sqrt{\prime}$ qMV algebra $\mathbf{A}$ is isomorphic to an algebra of functions which are either complex-valued or have values in $\mathbf{F}_{100}$.*

Proof. Let $\mathbf{A}$ be a semisimple $\sqrt{\prime}$ quasi-MV algebra. Thus, $\mathbf{A}$ can be represented as a subdirect product of simple $\sqrt{\prime}$ quasi-MV algebras, whence by Birkhoff's subdirect representation theorem there is a family $\{\theta_i\}_{i\in I}$ of congruences on $\mathbf{A}$ such that the $\mathbf{A}/\theta_i$'s are the simple factors in the subdirect representation of $\mathbf{A}$, and $\bigcap_{i\in I}\{\theta_i\} = \Delta$. Being simple, the $\mathbf{A}/\theta_i$'s are necessarily either cartesian or flat. So, by Lemma 55 and Theorem 56, for every $i \in I$ there is an isomorphism $f_i : \mathbf{A}/\theta_i \to \mathbf{B}_i$, where $\mathbf{B}_i$ is either $\mathbf{F}_{100}$ or a subalgebra of $\mathbf{S}_r$.

Now, for every $b \in \mathbf{A}$, let $\overline{b}$ be a function with domain $\{\theta_i\}_{i\in I}$ s.t. for every $i \in I$,

$$\overline{b}(\theta_i) = f_i\left(b/\theta_i\right).$$

Each $\overline{b}$, thus, is either a complex-valued function or an $\mathbf{F}_{100}$-valued function over $\{\theta_i\}_{i\in I}$. Let $F(A) = \left\{\overline{b} : b \in A\right\}$, and let $\mathbf{F_A}$ be the algebra of functions with universe $F(A)$ and pointwise defined operations. It is easy to check that the map $h : A \to F(A)$ defined by $h(b) = \overline{b}$ is a homomorphism from $\mathbf{A}$ onto $\mathbf{F_A}$. It remains to prove that it is injective. Thus, let $h(b) = h(c)$, i.e. $\overline{b}(\theta_i) = \overline{c}(\theta_i)$ for every $i \in I$. Then, for every $i \in I$, $f_i\left(b/\theta_i\right) = f_i\left(c/\theta_i\right)$ and, since f_i is bijective, $b/\theta_i = c/\theta_i$. Hence $\langle b, c\rangle \in \bigcap_{i\in I}\{\theta_i\} = \Delta$, i.e. $b = c$. □

Corollary 58. *Every semisimple cartesian* $\sqrt{'}$ *qMV algebra* $\mathbf{A}$ *is isomorphic to an algebra of complex-valued functions.*

8. Free algebras

A thorough and satisfactory description of free MV algebras over arbitrarily many generators is available as a consequence of *McNaughton's theorem* ([14], [16]): free MV algebras with κ many generators can be described as algebras of McNaughton functions from $[0,1]^{\kappa}$ to $[0,1]$. In this paper, for the sake of simplicity, we will also confine ourselves to free qMV and $\sqrt{'}$ qMV algebras with one generator, although the extension to an arbitrary number of generators does not seem to present, in principle, additional difficulties. We start by recalling the notion of McNaughton function in one variable.

Definition 59. A function $f : [0,1] \to [0,1]$ is called a *McNaughton function* iff it satisfies the following conditions:

- it is continuous;
- there are linear polynomials $p_1, ..., p_n$ in one variable and with integer coefficients s.t. for each $a \in [0,1]$ there is $1 \leq i \leq n$ s.t. $f(a) = p_i(a)$.

In other words, a McNaughton function in one variable is nothing but a continuous piecewise linear function over $[0,1]$.

Taking advantage of Theorem 16 and mimicking the proof of Proposition 3.1.4 in [6], it is not hard to see that the free qMV algebra over one generator is nothing but the algebra $\mathbf{Term}_1^{\mathbf{S}}$ of the qMV term functions over the standard qMV algebra $\mathbf{S}$, a notion that we now define more explicitly.

Definition 60. Consider the qMV algebra of functions $\mathbf{S}^{\mathbf{S}}$, with pointwise defined operations. The qMV algebra

$$\mathbf{Term}_1^{\mathbf{S}} = \left\langle Term_1^{\mathbf{S}}, \oplus^{\mathbf{Term}_1^{\mathbf{S}}}, '^{\mathbf{Term}_1^{\mathbf{S}}}, 0^{\mathbf{Term}_1^{\mathbf{S}}}, 1^{\mathbf{Term}_1^{\mathbf{S}}} \right\rangle$$

is its subalgebra whose universe is the set $Term_1^{\mathbf{S}}$, inductively defined as follows:

- the identity function X on $\mathbf{S}$ belongs to $Term_1^{\mathbf{S}}$;

- the constant functions $0^{\mathbf{S}}$ and $1^{\mathbf{S}}$, s.t. for every $a, b \in [0,1]$ it is $0^{\mathbf{S}} \langle a, b \rangle = \left\langle 0, \frac{1}{2} \right\rangle$ and $1^{\mathbf{S}} \langle a, b \rangle = \left\langle 1, \frac{1}{2} \right\rangle$, belong to $Term_1^{\mathbf{S}}$;
- If $\tau^{\mathbf{S}} \in Term_1^{\mathbf{S}}$, then $(\tau')^{\mathbf{S}} \in Term_1^{\mathbf{S}}$, where
$$\left(\tau'\right)^{\mathbf{S}} \langle a, b \rangle = \left\langle 1 - \pi_1(\tau^{\mathbf{S}} \langle a, b \rangle), 1 - \pi_2(\tau^{\mathbf{S}} \langle a, b \rangle) \right\rangle;$$
- If $\tau^{\mathbf{S}}, \sigma^{\mathbf{S}} \in Term_1^{\mathbf{S}}$, then $(\tau \oplus \sigma)^{\mathbf{S}} \in Term_1^{\mathbf{S}}$, where
$$(\tau \oplus \sigma)^{\mathbf{S}} \langle a, b \rangle = \left\langle \min(1, \pi_1(\tau^{\mathbf{S}} \langle a, b \rangle) + \pi_1(\sigma^{\mathbf{S}} \langle a, b \rangle)), \tfrac{1}{2} \right\rangle.$$

This algebra can be easily described with reference to the characterization of free MV algebras with one generator in terms of McNaughton functions. In fact:

Theorem 61. *A function $f \in ([0,1] \times [0,1])^{[0,1] \times [0,1]}$ belongs to $Term_1^{\mathbf{S}}$ iff for every $a, b \in [0,1]$ the following conditions are satisfied:*

- *$\pi_2(f \langle a, b \rangle)$ is either b or $1 - b$ or $\frac{1}{2}$;*
- *if $\pi_2(f \langle a, b \rangle) = b$, then $\pi_1(f \langle a, b \rangle) = a$;*
- *if $\pi_2(f \langle a, b \rangle) = 1 - b$, then $\pi_1(f \langle a, b \rangle) = 1 - a$;*
- *if $\pi_2(f \langle a, b \rangle) = \frac{1}{2}$, then $\pi_1(f \langle a, b \rangle)$ is a McNaughton function in one variable.*

Proof. A function in $Term_1^{\mathbf{S}}$, in fact, is either the identity function, or a constant function ($0^{\mathbf{S}}$ or $1^{\mathbf{S}}$), or a function f s.t. $f \langle a, b \rangle = \langle 1 - a, 1 - b \rangle$, or a function f s.t. $f \langle a, b \rangle = \left\langle f'(a), \frac{1}{2} \right\rangle$, where $f' \in Term_1^{[0,1]}$. Our claim, then, is a direct consequence of McNaughton's theorem for free MV algebras. □

The description of free $\sqrt{'}$ qMV algebras is similar, albeit slightly more convoluted. Once again, taking advantage of Theorem 37 and mimicking the proof of Proposition 3.1.4 in [6], it is not hard to see that the free $\sqrt{'}$qMV algebra over one generator is nothing but the algebra $\mathbf{Term}_1^{\mathbf{S}_r}$ of the qMV term functions over the standard $\sqrt{'}$qMV algebra $\mathbf{S}_r$, a notion that we now define in full.

Definition 62. Consider the $\sqrt{'}$qMV algebra of functions $\mathbf{S}_r^{\mathbf{S}_r}$, with pointwise defined operations. The $\sqrt{'}$qMV algebra

$$\mathbf{Term}_1^{\mathbf{S}_r} = \left\langle Term_1^{\mathbf{S}_r}, \oplus^{\mathbf{Term}_1^{\mathbf{S}_r}}, \sqrt{'}^{\mathbf{Term}_1^{\mathbf{S}_r}}, 0^{\mathbf{Term}_1^{\mathbf{S}_r}}, 1^{\mathbf{Term}_1^{\mathbf{S}_r}}, k^{\mathbf{Term}_1^{\mathbf{S}_r}} \right\rangle$$

is its subalgebra whose universe is the set $Term_1^{\mathbf{S}_r}$, obtained by adding to the definition of $Term_1^{\mathbf{S}}$ the following two inductive clauses:

- the constant function $k^{\mathbf{S}_r}$, s.t. for every $a, b \in [0,1]$ it is $k^{\mathbf{S}_r} \langle a, b \rangle = \left\langle \frac{1}{2}, \frac{1}{2} \right\rangle$, belongs to $Term_1^{\mathbf{S}_r}$;
- If $\tau^{\mathbf{S}_r} \in Term_1^{\mathbf{S}_r}$, then $\left(\sqrt{'}\tau\right)^{\mathbf{S}_r} \in Term_1^{\mathbf{S}_r}$, where

$$\left(\sqrt{'}\tau\right)^{\mathbf{S}_r} \langle a, b \rangle = \left\langle \pi_2(\tau^{\mathbf{S}_r} \langle a, b \rangle), 1 - \pi_1(\tau^{\mathbf{S}_r} \langle a, b \rangle) \right\rangle .$$

(The clause concerning $(\tau')^{\mathbf{S}}$ in Definition 60 becomes obviously redundant.) This algebra can also be described with reference to the characterization of free MV algebras with one generator in terms of McNaughton functions. Omitting details, we just state the result:

Theorem 63. *A function $f \in \mathbf{S}_r^{\mathbf{S}_r}$ belongs to $Term_1^{\mathbf{S}_r}$ iff for every $a, b \in [0,1]$ one of the following conditions is satisfied:*

$f \langle a, b \rangle = \langle a, b \rangle$ *or*
$f \langle a, b \rangle = \langle 1 - a, 1 - b \rangle$
$f \langle a, b \rangle = \langle b, 1 - a \rangle$ *or*
$f \langle a, b \rangle = \langle 1 - a, b \rangle$ *or*
$f \langle a, b \rangle = \left\langle f'(a), \frac{1}{2} \right\rangle$ *(* $\left\langle f'(b), \frac{1}{2} \right\rangle$ *) where f' is a McNaughton function, or*
$f \langle a, b \rangle = \left\langle \frac{1}{2}, f'(b) \right\rangle$ *(* $\left\langle \frac{1}{2}, f'(a) \right\rangle$ *) where f' is a McNaughton function.*

Acknowledgement. We are grateful to Felix Bou, Maria Luisa Dalla Chiara, Francesc Esteva, Josep Font, Lluis Godo, Franco Montagna, Carles Noguera, Ramon Jansana, Giuseppe Sergioli and Matthew Spinks for the insightul conversations about the topics covered in this paper. We also thank two anonymous referees for their precious remarks which led us to considerably improve the presentation of our paper.

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