

Mai GEHRKE and Hilary A. PRIESTLEY

## CANONICAL EXTENSIONS AND COMPLETIONS OF POSETS AND LATTICES

**A b s t r a c t.** The purpose of this note is to expose a new way of viewing the canonical extension of posets and bounded lattices. Specifically, we seek to expose categorical features of this completion and to reveal its relationship to other completion processes.

### 1. Introduction

The theory of canonical extensions is introduced by Jónsson and Tarski [15, 16] for Boolean algebras with operators. Their approach was based on a complete-lattice theoretic characterisation of the dual space of a Boolean algebra and provides access to the benefits of Stone duality in a uniform way for a class of varieties of algebras having a Boolean reduct. The most

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significant contribution made by this pioneering work of Jónsson and Tarski is to provide a framework for transporting the benefits of Boolean duality theory to a wider setting. It is chiefly in this respect that the *raison d'être* of the theory of canonical extensions lay.

In [9] and [10] Jónsson and Gehrke showed that topological duality for distributive lattices can be captured in a similarly algebraic way, and that results and tools may be developed for a much broader class of additional operations than those considered by Jónsson and Tarski. In the Boolean and distributive settings the *existence* of the canonical extension of the lattice reduct is provided by topological duality. Thus finding the complete-lattice theoretic characterisation of the dual space is all that needs to be done before addressing the main issue: the treatment of other algebraic operations and their properties.

In [8] the first steps were taken in generalising the theory beyond the distributive setting. Here the requirements changed somewhat in that existence also is an issue. In the setting of non-distributive lattices a number of candidate completions were available including the particular completion introduced by Harding in [13] as a candidate for a canonical extension for lattices as well as the completions obtainable via the existing dualities for lattices [20, 4, 14]. While the canonical extension characterised in [8] may be seen to be the one obtained by all the above mentioned candidate methods, a simple and direct construction of the canonical extensions using a Galois connection between filters and ideals of a lattice was used to prove existence instead. This construction has the advantage of being constructive. Furthermore, as we indicate below, it shifts the focus towards viewing the canonical extension as one completion among a whole hierarchy of completions of a lattice.

There is a significant difference between the character of canonical extension theory in the distributive and non-distributive settings. In the distributive (and Boolean) settings the theory simply rephrases the duality theoretic methods in an algebraic form and thus facilitates the treatment of additional operations. By contrast, it has turned out that in the non-distributive setting the canonical extension theory works as in the distributive setting. It is therefore *much simpler* than the available duality theories. Thus it has proved to be a method for transporting insight, results and methodology from *distributive and Boolean* duality theory to canonical extensions of lattice-based and even poset-based algebras and,

from there, also *to* the duality theory for these structures [3, 7].

Canonical extensions for partially ordered algebras were introduced in [3] and the existence issue had to be treated again. While a construction similar to the one given in [8] works in this setting as well, an alternate description was outlined. This description clearly identifies the canonical extension as obtained in two steps. The first consists in making an amalgam of the directed join and meet completions of the original poset. This so-called *intermediate structure* is quite important; it was first exploited by Ghilardi and Meloni in [12]. At the second step of the construction one forms the MacNeille completion of the intermediate structure. Viewing the canonical extension as a two-step process in this way makes clear the close relationship of canonical extension to these classical forms of completion. In this paper we take this idea further: we consider a hierarchy of completions for posets and identify the position and significance of the canonical extension within this hierarchy.

The traditional focus of canonical extension theory is additional operations. We will not consider these here. We will however concern ourselves with finding a direct and two-sided treatment of the extension of homomorphisms. So far, in the theory of canonical extensions homomorphisms have been treated within the framework of operators and dual operators. In fact, in [3] the dual object and the extensions of arbitrary lattice homomorphisms are easily derived precisely because one can split the persona of a lattice homomorphism into two parts: a box which is meet-preserving and a diamond which is join-preserving. The additional (Sahlqvist) inequalities  $\square \leq \diamond$  and  $\diamond \leq \square$  tell us that these two personae are actually equal. It is clear that this approach is useful for obtaining a dual correspondent for the notion of homomorphism. However, it seems natural from a categorical point of view to ask whether there is a more direct and intrinsic way of extending morphisms than through a detour to the much more general operators and dual operators. We show here that there is indeed such a way.

We begin the paper by considering completions in general. Posets and order preserving maps do not allow for a free extension mechanism to the category of complete lattices and complete lattice homomorphisms. They do however allow for free one-sided completions. The directed join completion of a poset is the unique directed-join-compact dcpo generated by the poset and may be obtained as the lattice of all ideals of the poset. Dually,

posets of course also have free directed meet completions and both have the nicest categorical properties one could wish for. By alternating these two completions we obtain a hierarchy of extensions of a poset akin to the hierarchy of Borel sets in analysis or the hierarchies for quantifier complexity for sentences in first order logic. The directed join completion of a poset  $P$  is its  $\Sigma_0$  completion, the directed join completion of the directed meet completion of  $P$  is the  $\Sigma_1$  completion, and so on. In this setting it is then natural to ask for  $\Delta_n$  completions to be two-sided objects that are the greatest common lower bounds of the  $\Sigma_n$  and  $\Pi_n$  completions.

In this paper we identify the  $\Delta_1$  object. We show that for a poset  $P$  it contains both the  $\Sigma_0$  completion (that is, the directed join completion) and the  $\Pi_0$  completion (that is, the directed meet completion) of  $P$  and that the order structure on the union of these in this  $\Delta_1$  object is exactly the order structure of the intermediate structure from [3]. While this order was chosen ad hoc in [3] the findings of this paper identify its naturality. Furthermore we show that, for lattices, the  $\Delta_1$  object is exactly the MacNeille completion of the intermediate structure. As defined in [3], this MacNeille completion is exactly the canonical extension. Thus, in the lattice case, our analysis identifies the canonical extension as a very natural object in the completion hierarchy. Outside the lattice case, the  $\Delta_1$  object is in general not a bi-dcpo which makes it less desirable. We show that it is reasonable to adopt the canonical extension as identified in [3] as the  $\Delta_1$ -like object in general. For this reason we introduce here and propose the future change in notation by which the canonical extension of a poset is denoted  $P^\delta$  rather than the traditional  $P^\sigma$ .

As stated above our second purpose is to understand directly how and why lattice homomorphisms extend to complete lattice homomorphisms under canonical extension. MacNeille completion, the last step in constructing the canonical extension, is well known not to be well-behaved on maps. However, in [6] Ern  shows that the maps between posets that extend to complete lattice homomorphisms of the MacNeille completions are those that are *cut-stable*. Cut-stability is nice in that it has a first-order description. On the other hand, a major drawback is that not all lattice homomorphism (even between Boolean algebras for example) are cut-stable. Interestingly and significantly, cut-stability becomes a much more reasonable requirement for intermediate structures. In particular we show that the unique extension of a lattice homomorphism to the inter-

mediate structures is always cut-stable thus explaining the functoriality of canonical extension intrinsically.

## 2. Free join- and meet-completions

This section collects together in the form we require it material on free join- and meet-completions of bounded lattices, and of the extensions of the ideas to posets. Sources for this are [17], Section 6, and [11], Sections I-4 and IV-1 and [19]; for an outline of the constructions in the poset case see [2], Exercise 9.6.

Let  $Q$  be a poset. Then we write  $\bigsqcup D$  for the supremum of a directed subset  $D$  of  $Q$ , when this supremum exists. A dcpo (directedly complete partial order) is a poset  $Q$  in which  $\bigsqcup D$  exists for every directed subset  $D$  of  $Q$ . Given a subset  $R$  of a dcpo  $Q$  we say that  $Q$  is  $\bigsqcup$ -generated by  $R$  if every element of  $Q$  is the directed join of elements from  $R$ . An element  $k$  of a dcpo  $Q$  is said to be *compact* (*alias finite*) if  $k \leq \bigsqcup D$  implies that  $k \leq d$  for some  $d \in D$ . The set of compact elements of  $Q$  will be denoted by  $F(Q)$ . In the terminology of [11], a dcpo  $Q$  is said to be an *algebraic domain* if  $Q$  is  $\bigsqcup$ -generated by  $F(Q)$ . A particular instance of an algebraic domain is the family  $\text{Id}(P)$  of all ideals (= directed down-sets) of a poset  $P$ , ordered by inclusion; in this context directed joins are given by union. We have  $F(\text{Id}(P)) = \{ \downarrow p \mid p \in P \}$  and  $\downarrow : p \mapsto \downarrow p$  is an order-isomorphism from  $P$  onto  $F(\text{Id}(P))$ .

Let  $P$  be a poset. Then a  $\bigsqcup$ -completion of  $P$  consists of a pair  $(\alpha, Q)$  where  $Q$  is a dcpo and  $\alpha : P \hookrightarrow Q$  is an order-embedding such that  $Q$  is  $\bigsqcup$ -generated by  $\alpha(P)$ . Such a completion is said to be *compact* if

$$\alpha(p) \leq \bigsqcup D \implies \exists d \in D (\alpha(p) \leq d)$$

(equivalently,  $\alpha(p) \leq \bigsqcup \alpha(S)$ , where  $\alpha(S)$  is a directed subset of  $\text{Im } \alpha$ , then  $p \leq s$  for some  $s \in S$ ).

**Proposition 2.1.** *Assume that  $(\alpha, Q)$  is a  $\bigsqcup$ -completion of a poset  $P$ . Then the following are equivalent:*

- (1)  $(\alpha, Q)$  is a compact completion;

- (2) whenever  $Q'$  is a dcpo and  $f : P \rightarrow Q'$  is an order-preserving map, then there exists a unique map  $\bar{f} : Q \rightarrow Q'$  preserving directed joins and such that  $\bar{f} \circ \alpha = f$ , given by

$$\bar{f}(x) := \bigsqcup \{ f(p) \mid x \geq \alpha(p) \};$$

- (3) there exists an isomorphism  $\eta : Q \cong \text{Id}(P)$  with  $\eta(\alpha(p)) = \downarrow(p)$  for all  $p \in P$ .

The proposition shows that a poset  $P$  has one, and up to isomorphism only one, compact  $\bigsqcup$ -completion, and that this has the universal mapping property given by (2). For this reason we shall refer to this completion as the *free  $\bigsqcup$ -completion* of  $P$ . We denote it by  $\mathbb{F}_{\bigsqcup}(P)$ , and the embedding of  $P$  into  $\mathbb{F}_{\bigsqcup}(P)$  by  $\alpha_P$ .

**Lemma 2.2.** *Let  $P$  be a poset. Then  $\alpha_P : P \rightarrow \mathbb{F}_{\bigsqcup}(P)$  has the following properties.*

- (i)  $\alpha_P$  preserves all existing meets;
- (ii)  $\alpha_P$  preserves all existing finite joins.

**Proof.** Consider (i). Let  $S \subseteq P$  and assume that  $\bigwedge_P S = s_0$  exists in  $P$ . Clearly  $s_0$  is a lower bound for  $S$  in  $\mathbb{F}_{\bigsqcup}(P)$ . Consider  $y \in \mathbb{F}_{\bigsqcup}(P)$  for which  $y \leq s$  for all  $s \in S$ . Then  $P \ni p \leq y$  implies  $p \leq s$  for all  $s \in S$ , so that  $p \leq s_0$ . Since  $y = \bigsqcup \{ p \in P \mid p \leq y \}$  we deduce that  $y \leq s_0$ . Therefore  $s_0 = \bigwedge_{\mathbb{F}_{\bigsqcup}(P)} S$ , as required.

We now prove (ii). Certainly the empty join is preserved if it exists in  $P$ , since this is equivalent to  $P$  having a bottom element, which is mapped by  $\alpha_P$  to the bottom element of  $\mathbb{F}_{\bigsqcup}(P)$ . Now assume that  $p_1$  and  $p_2$  in  $P$  are such that  $p_1 \vee p_2$  exists in  $P$ . Since  $\alpha_P$  is an order-embedding,  $p_i \leq p_1 \vee p_2$  ( $i = 1, 2$ ) in  $\mathbb{F}_{\bigsqcup}(P)$  as well. Suppose  $y$  is an upper bound for  $p_1$  and  $p_2$  in  $\mathbb{F}_{\bigsqcup}(P)$ . Then, since  $y = \bigsqcup \{ p \in P \mid p \leq y \}$ , there exists  $p' \in P$  with  $p_i \leq p' \leq y$ , for  $i = 1, 2$ . Therefore  $p_1 \vee p_2 \leq y$ .  $\square$

Notice that the compactness property tells us that  $\alpha_P$  must destroy any inherently infinite directed joins which happen to exist in  $P$ .

The following interaction with the existence of  $\wedge$  is worthy of note.

**Proposition 2.3.** *If  $P$  is a  $\wedge$ -semilattice then  $\mathbb{F}_{\bigsqcup}(P)$  is also a  $\wedge$ -semilattice.*

$$\begin{array}{ccc}
 \mathbb{F}_{\sqcup}(P) & \xrightarrow{\mathbb{F}_{\sqcup}(f)} & \mathbb{F}_{\sqcup}(P') \\
 \alpha_P \uparrow & & \alpha_{P'} \uparrow \\
 P & \xrightarrow{f} & P'
 \end{array}$$

Figure 1:

**Proof.** It is easy to see that if  $I$  and  $J$  are up-directed down-sets of a  $\wedge$ -semilattice then  $I \cap J$  is also an up-directed downset.  $\square$

Given two posets  $P$  and  $P'$ , and an order-preserving map  $f : P \rightarrow P'$  then we have a unique directed join preserving map  $\mathbb{F}_{\sqcup}(f) : \mathbb{F}_{\sqcup}(P) \rightarrow \mathbb{F}_{\sqcup}(P')$  such that the diagram in Figure 1 commutes.

We can, and sometimes will, suppress the embedding of a poset into its free  $\sqcup$ -completion and regard  $P$  as a subposet of  $\mathbb{F}_{\sqcup}(P)$ . When this is done, the formula for  $\mathbb{F}_{\sqcup}(f)$  takes the form

$$\mathbb{F}_{\sqcup}(f)(y) = \sqcup \{ f(p) \mid y \geq p \} \quad \text{for } y \in \mathbb{F}_{\sqcup}(P).$$

It is immediate that this map  $\mathbb{F}_{\sqcup}(f)$  is order-preserving and extends  $f$ .

The following lemma records elementary properties of the lifting of a map  $f$ . Parts (i)(a) and (b) are essentially well known (cf. [11], [19], [2], 9.11).

**Lemma 2.4.** *Let  $P$  and  $P'$  be posets and let  $f : P \rightarrow P'$  be an order-preserving map.*

- (i) (a)  $\mathbb{F}_{\sqcup}(f)$  is the unique extension of  $f$  that preserves  $\sqcup$ .
- (b) Assume  $P$  and  $P'$  are  $\vee$ -semilattices and that  $f$  preserves  $\vee$  and 0. Then  $\mathbb{F}_{\sqcup}(f)$  preserves  $\vee$ .
- (c) Let  $P$  and  $P'$  be  $\wedge$ -semilattices and assume that  $f$  preserves  $\wedge$ . Then  $\mathbb{F}_{\sqcup}(f)$  preserves  $\wedge$ .
- (ii) Assume that  $f$  is an order-embedding. Then  $\mathbb{F}_{\sqcup}(f)$  is an order-embedding.

**Proof.** To prove (i)(a), notice that since  $\mathbb{F}_{\sqcup}(P)$  is  $\sqcup$ -generated by  $P$ , there is at most one  $\sqcup$ -preserving extension of  $f$  to  $\mathbb{F}_{\sqcup}(P)$ . We show that

the map  $\mathbb{F}_{\sqcup}(f)$  as specified above does indeed preserve directed joins. To this end let  $D$  be a directed set in  $\mathbb{F}_{\sqcup}(P)$ . Then

$$\begin{aligned}\mathbb{F}_{\sqcup}(f)(\sqcup D) &= \sqcup \{ f(p) \mid \sqcup D \geq p \} \\ &= \sqcup \{ f(p) \mid \exists d \in D (d \geq p) \} \\ &= \sqcup_{d \in D} \sqcup \{ f(p) \mid d \geq p \}.\end{aligned}$$

To prove (b), simply replace, in the above calculation, directed joins by arbitrary non-empty joins and exploit the fact that  $f$  preserves finite joins. The empty join is preserved since  $f$  preserves 0.

Now consider (c). Let  $y = y_1 \wedge y_2$  in  $\mathbb{F}_{\sqcup}(P)$ . Then  $\mathbb{F}_{\sqcup}(f)(y) \leq \mathbb{F}_{\sqcup}(f)(y_1) \wedge \mathbb{F}_{\sqcup}(f)(y_2)$  because  $\mathbb{F}_{\sqcup}(f)$  is order-preserving. Also

$$\mathbb{F}_{\sqcup}(f)(y_1) \wedge \mathbb{F}_{\sqcup}(f)(y_2) = \bigvee \{ p' \in P' \mid p' \leq \mathbb{F}_{\sqcup}(f)(y_i) \ (i = 1, 2) \}.$$

By compactness, there exist  $p_1$  and  $p_2$  in  $P$  such that  $p' \leq f(p_i)$  and  $p_i \leq y_i$  ( $i = 1, 2$ ). Then  $p' \leq f(p_1) \wedge f(p_2) = f(p_1 \wedge p_2)$ . Hence  $p' \leq \mathbb{F}_{\sqcup}(y)$ . The required result follows.

Finally we prove (ii). Take  $y_1 \not\leq y_2$  in  $\mathbb{F}_{\sqcup}(P)$ . Then there exists  $p \in P$  such that  $p \leq y_1$  but  $p \not\leq y_2$ . Then  $f(p) \leq f(y_1)$ . We claim that  $f(p) \not\leq f(y_2)$ . If this were not so we would have

$$f(p) \leq \sqcup \{ f(q) \mid y_2 \geq q \}.$$

By compactness, there exists  $q \leq y_2$  such that  $f(p) \leq f(q)$ . Since  $f$  is an order-embedding we have  $p \leq q \leq y_2$ , contrary to hypothesis.  $\square$

We have, of course, order dual versions of all of the above constructions and properties. We adopt the notation  $\mathbb{F}_{\sqcap}(P)$  for the free  $\sqcap$ -completion of a poset  $P$ . We write  $\mathbb{F}_{\sqcap}(f)$  for the directed meet preserving extension of an order-preserving map  $f : P \rightarrow P'$ . It is given by

$$\mathbb{F}_{\sqcap}(f)(y) = \sqcap \{ f(p) \mid y \leq p \in P \}.$$

We can of course iterate the above constructions, to form  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P))$  and  $\mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(P))$  and so on. This enables us to form an infinite hierarchy of completions by alternating up-directed and down-directed completions. In the next section we investigate the bottom end of this hierarchy.

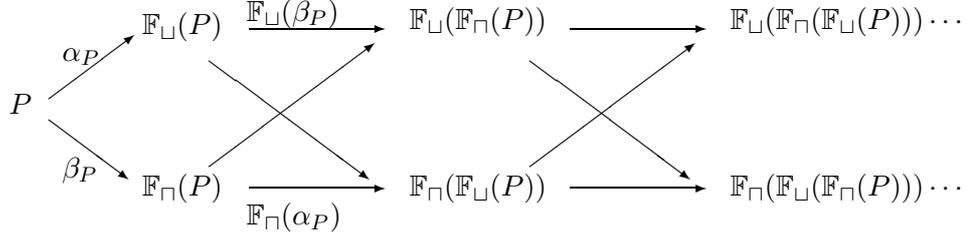


Figure 2:

### 3. The intermediate structure and bi-directed completions

Figure 2 shows the first stages of a hierarchy of completions of a poset  $P$ , in which directed completions and dually directed completions alternate. As in first-order logic and set theory, one can think of this as a complexity hierarchy. From this perspective  $\mathbb{F}_{\sqcup}(P)$  is the  $\Sigma_0$  directed completion of  $P$  while, for example,  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\triangle}(P))$  is the  $\Sigma_1$  directed completion of  $P$  and  $\mathbb{F}_{\triangle}(\mathbb{F}_{\sqcup}(P))$  is the  $\Pi_1$  directed completion of  $P$ . With this idea in mind, it is natural to ask for two-sided objects of type  $\Delta_0, \Delta_1$  etc. Here we shall consider this question on level 1 as we are ultimately interested in elucidating the canonical extension construction.

Essentially, what we want to do is to identify the 'common part' of the  $\Sigma_1$  and the  $\Pi_1$  directed completion of  $P$ . If  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\triangle}(P))$  and  $\mathbb{F}_{\triangle}(\mathbb{F}_{\sqcup}(P))$  were subsets of some common set  $X$ , then we would simply take the common part to be their intersection. In the absence of such an  $X$ , a natural substitute is to ask for a greatest interpolant  $Q$  in the diagram in Figure 3. Specifically, in the diagram we want all the maps to be embeddings and the compositions along the upward and downward diagonals to be  $\alpha_{\mathbb{F}_{\triangle}(P)}$  and  $\beta_{\mathbb{F}_{\sqcup}(P)}$ , respectively and we want the diagram to commute.

We now embark on showing how to construct the required interpolant  $Q$ . First we note that both  $\mathbb{F}_{\sqcup}(P)$  and  $\mathbb{F}_{\triangle}(P)$  sit inside both of  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\triangle}(P))$  and  $\mathbb{F}_{\triangle}(\mathbb{F}_{\sqcup}(P))$ , for example, in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\triangle}(P))$  we have a copy of  $\mathbb{F}_{\sqcup}(P)$  in the guise of  $\text{Im}(\mathbb{F}_{\sqcup}(\beta_P))$  and a copy of  $\mathbb{F}_{\triangle}(P)$  in the guise of  $\text{Im}(\alpha_{\mathbb{F}_{\triangle}(P)})$ .

**Theorem 3.1.** *For a poset  $P$ , the order induced on*

$$\text{Int}(P) = \mathbb{F}_{\sqcup}(P) \cup \mathbb{F}_{\triangle}(P)$$

*in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\triangle}(P))$  is the same as the one induced in  $\mathbb{F}_{\triangle}(\mathbb{F}_{\sqcup}(P))$ . For  $y \in \mathbb{F}_{\sqcup}(P)$*

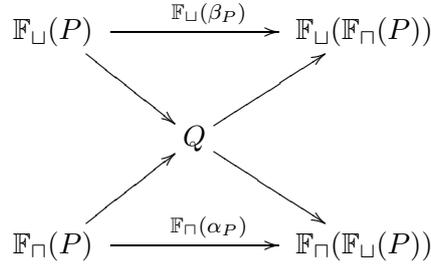


Figure 3:

and  $x \in \mathbb{F}_{\sqcap}(P)$  it is given by

$$\begin{aligned}
y \leq x &\iff \forall p, q \in P \text{ } ([\alpha_P(p) \leq y \text{ and } x \leq \beta_P(q)] \implies p \leq q) \\
x \leq y &\iff \exists p \in P \text{ } (x \leq \beta_P(p) \text{ and } \alpha_P(p) \leq y).
\end{aligned}$$

In particular, for  $y \in \mathbb{F}_{\sqcup}(P)$  and  $x \in \mathbb{F}_{\sqcap}(P)$  we have  $y = x$  if and only if there is a  $p \in P$  with  $\alpha_P(p) = y$  and  $\beta_P(p) = x$ .

**Proof.** Let  $y \in \mathbb{F}_{\sqcup}(P)$  and  $x \in \mathbb{F}_{\sqcap}(P)$  and consider their embeddings in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P))$ , namely  $\mathbb{F}_{\sqcup}(\beta_P)(y)$  and  $\alpha_{\mathbb{F}_{\sqcap}(P)}(x)$ . Since maps in the image of  $\mathbb{F}_{\sqcup}$  preserve  $\sqcup$  (Lemma 2.4) and the  $\alpha$ -embeddings preserve  $\wedge$  (Lemma 2.2), it follows that

$$\mathbb{F}_{\sqcup}(\beta_P)(y) = \sqcup \{ \mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) \mid \alpha_P(p) \leq y, p \in P \}$$

and

$$\alpha_{\mathbb{F}_{\sqcap}(P)}(x) = \bigwedge \{ \alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(q)) \mid x \leq \beta_P(q), q \in P \}.$$

Thus we see that  $\mathbb{F}_{\sqcup}(\beta_P)(y) \leq \alpha_{\mathbb{F}_{\sqcap}(P)}(x)$  if and only if

$$\forall p, q \in P \text{ } [(\alpha_P(p) \leq y \text{ and } x \leq \beta_P(q)) \implies \mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) \leq \alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(q))].$$

Now, since the diagram in Figure 1 with  $f = \beta_P$  commutes we have  $\mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) = \alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(p))$  and thus the inequality  $\mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) \leq \alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(q))$  is equivalent to  $\alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(p)) \leq \alpha_{\mathbb{F}_{\sqcap}(P)}(\beta_P(q))$ . In turn, since both  $\alpha_{\mathbb{F}_{\sqcap}(P)}$  and  $\beta_P$  are embeddings, this inequality is equivalent to  $p \leq q$  and we have proved that  $\mathbb{F}_{\sqcup}(\beta_P)(y) \leq \alpha_{\mathbb{F}_{\sqcap}(P)}(x)$  if and only if

$$\forall p, q \in P \text{ } [(\alpha_P(p) \leq y \text{ and } x \leq \beta_P(q)) \implies p \leq q].$$

That is, we have shown that, in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\cap}(P))$ , the condition for  $y \leq x$  is the one claimed.

Now we consider

$$\begin{aligned} \alpha_{\mathbb{F}_{\cap}(P)}(x) &\leq \mathbb{F}_{\sqcup}(\beta_P)(y) \\ &= \bigsqcup \{ \mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) \mid \alpha_P(p) \leq y, p \in P \}. \end{aligned}$$

As the image of  $\alpha_{\mathbb{F}_{\cap}(P)}$  is  $\bigsqcup$ -compact in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\cap}(P))$ , it follows that this inequality is equivalent to

$$\exists p \in P (\alpha_P(p) \leq y \text{ and } \alpha_{\mathbb{F}_{\cap}(P)}(x) \leq \mathbb{F}_{\sqcup}(\beta_P)(\alpha_P(p)) = \alpha_{\mathbb{F}_{\cap}(P)}(\beta_P(p)).$$

Now as  $\alpha_{\mathbb{F}_{\cap}(P)}$  is an embedding we have  $\alpha_{\mathbb{F}_{\cap}(P)}(x) \leq \alpha_{\mathbb{F}_{\cap}(P)}(\beta_P(p))$  if and only if  $x \leq \beta_P(p)$ . Putting these things together we have that  $\alpha_{\mathbb{F}_{\cap}(P)}(x) \leq \mathbb{F}_{\sqcup}(\beta_P)(y)$  if and only if

$$\exists p \in P (\alpha_P(p) \leq y \text{ and } x \leq \beta_P(p)).$$

Checking that the conditions for  $\beta_{\mathbb{F}_{\sqcup}(P)}(y) \leq \mathbb{F}_{\cap}(\alpha_P)(x)$  and  $\mathbb{F}_{\cap}(\alpha_P)(x) \leq \beta_{\mathbb{F}_{\sqcup}(P)}(y)$  in  $\mathbb{F}_{\cap}(\mathbb{F}_{\sqcup}(P))$  are the same can be done similarly and is left to the reader.  $\square$

According to the final statement in Theorem 3.1  $\alpha_P(p)$  and  $\beta_P(p)$  are identified for each  $p \in P$  and we regard  $\text{Int}(P)$  as being partially ordered, and as containing  $P$  as a subposet. We refer to this ordered set as the *intermediate structure* of  $P$ . This structure was first exploited by Ghilardi and Meloni in [12]. It was identified as being the amalgamation of  $\mathbb{F}_{\sqcup}(P)$  and  $\mathbb{F}_{\cap}(P)$  given above in [3]; see Theorem 2.5 and the remarks following it. New to this paper is the point that this is the order induced on the union in the  $\Sigma_1$ - and  $\Pi_1$ -completions of  $P$ .

**Remark 3.2.** One of the advantages of the canonical extension point of view is that lattice ordered algebras and their duals both are encoded in one and the same structure making it easier to talk about the relationship between the two. However, in the traditional approach to duality and completions the duals are given set theoretically. In that setting one may want to think of the elements of  $\mathbb{F}_{\sqcup}(P)$  as the ideals of  $P$  ordered by inclusion while the elements of  $\mathbb{F}_{\cap}(P)$  are the filters of  $P$  ordered by the opposite

order to inclusion. As an aside we note that in this incarnation the order on  $\text{Int}(P)$  is given below. For  $F, F' \in \mathbb{F}_{\sqcap}(P)$  and  $I, I' \in \mathbb{F}_{\sqcup}(P)$

$$\begin{aligned} I \leq F &\iff \forall x, y ([x \in I \text{ and } y \in F] \implies x \leq y) \\ F \leq I &\iff F \cap I \neq \emptyset; \\ F \leq F' &\iff F \supseteq F'; \\ I \leq I' &\iff I \subseteq I'. \end{aligned}$$

In addition,  $F \leq I$  and  $I \leq F$  if and only if there exists  $p \in P$  such that  $F = \uparrow p$  and  $I = \downarrow p$ .

Theorem 3.1 together with the results of the preceding section yield the following proposition.

**Proposition 3.3.** *Let  $P$  be a poset. The following statements hold:*

- (i)  $\mathbb{F}_{\sqcup}(P)$  is meet-dense in  $\text{Int}(P)$  and  $\mathbb{F}_{\sqcap}(P)$  is join-dense in  $\text{Int}(P)$ .
- (ii) In  $\text{Int}(P)$  up-directed joins exist for elements drawn from  $\mathbb{F}_{\sqcup}(P)$ , and may be regarded as being calculated either in  $\mathbb{F}_{\sqcup}(P)$  or in  $\text{Int}(P)$ . Dual assertions hold for down-directed meets of elements drawn from  $\mathbb{F}_{\sqcap}(P)$ .
- (iii) Let  $L$  be a bounded lattice. Then in  $\text{Int}(L)$ , arbitrary joins and finite meets exist for elements drawn from  $\mathbb{F}_{\sqcup}(L)$ , and may be regarded as being calculated either in  $\mathbb{F}_{\sqcup}(L)$  or in  $\text{Int}(L)$ . Likewise for arbitrary meets and finite joins of elements drawn from  $\mathbb{F}_{\sqcap}(L)$ .

Now we have established that  $Q = \text{Int}(P)$  works as an interpolant but we would like to find the largest such  $Q$ . Suppose  $Q$  is an interpolant as in Figure 3 and let  $q \in Q$ . Since  $Q$  embeds in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P))$  we have that  $\downarrow q \cap \mathbb{F}_{\sqcap}(P)$  is a directed set whose join is  $q$  in  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P))$  and thus certainly also in the subposet  $Q$ . By symmetry we also have the dual property that

$\uparrow q \cap \mathbb{F}_{\sqcup}(P)$  is a down-directed set whose meet in  $Q$  is  $q$ . In particular, this means that  $\text{Int}(P)$  is both  $\vee$ -dense and  $\wedge$ -dense in  $Q$ , and thus  $Q$  may be seen as a subposet of the MacNeille completion  $\mathcal{N}(\text{Int}(P))$ . For a discussion of the basic properties of MacNeille completion see for example [2]; cf. also [1] and [18].

**Definition 3.4.** For a poset  $P$ , define the bi-directional interpolant,  $P^\delta$ , by

$$P^\delta = \{ u \in \mathcal{N}(\text{Int}(P)) : \uparrow u \cap \mathbb{F}_\sqcup(P) \text{ is down-directed and} \\ \downarrow u \cap \mathbb{F}_\sqcap(P) \text{ is up-directed} \}.$$

The next proposition follows immediately from the discussion above.

**Proposition 3.5.** For a poset  $P$ , the bi-directional interpolant,  $P^\delta$ , is the greatest possible interpolant  $Q$  in Figure 3.

We can easily prove that in the lattice case the greatest interpolant coincides with the canonical extension.

**Proposition 3.6.** Let  $L$  be a lattice. Then  $L^\delta = \mathcal{N}(\text{Int}(L))$  and is thus the canonical extension of  $L$ .

**Proof.** As noted above, we know from the treatment in [3] that the canonical extension of a poset  $P$  is  $\mathcal{N}(\text{Int}(P))$ .

As we saw in Proposition 2.3, if  $L$  is a lattice then  $\mathbb{F}_\sqcup(L)$  is a  $\wedge$ -semilattice and, dually,  $\mathbb{F}_\sqcap(L)$  is a  $\vee$ -semilattice. Therefore  $\uparrow u \cap \mathbb{F}_\sqcup(L)$  is down-directed and  $\downarrow u \cap \mathbb{F}_\sqcap(L)$  is up-directed, for each  $u \in \mathcal{N}(\text{Int}(P))$ . This shows that  $L^\delta = \mathcal{N}(\text{Int}(P))$ .  $\square$

We must now ask whether, for an arbitrary poset  $P$ , the greatest interpolant  $P^\delta$  has the completeness properties we would want of a bi-directional completion. It does not:  $P^\delta$  need not be a bi-dcpo (that is, a dcpo and a dual dcpo).

Consider the poset depicted in Figure 4. This has an infinite decreasing sequence of ideals, obtained by starting from the improper ideal  $P$  and discarding the uppermost north-east diagonal ray of points repeatedly; this sequence has  $\{r, s\}$  as its set of common lower bounds in  $\text{Int}(P)$ , but  $\{r, s\}$  is not an up-directed set.

Notwithstanding the bad behaviour exhibited by our example, there is however a sizeable class of posets for which the interpolant is a bi-dcpo. This class of course includes the finite posets and it includes all lattices. Further exploration of this class would certainly be of interest but will not be pursued here.

Our analysis above points towards  $\mathcal{N}(\text{Int}(P))$  as another reasonable  $\Delta_1$ -like object of study. We pursue this in the next section. Before concluding

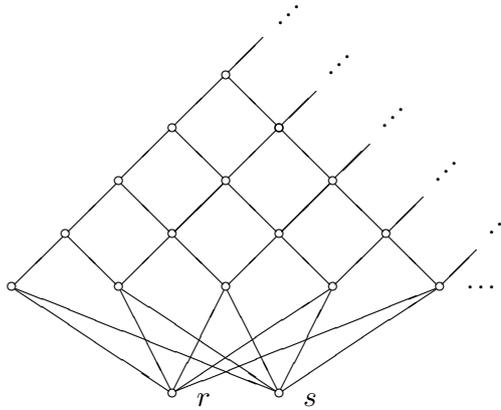


Figure 4:

$$\begin{array}{ccc}
 \mathbb{F}_{\sqcap}(P) & \xrightarrow{\mathbb{F}_{\sqcap}(f)} & \mathbb{F}_{\sqcap}(P') \\
 \beta_P \uparrow & & \beta_{P'} \uparrow \\
 P & \xrightarrow{f} & P'
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P)) & \xrightarrow{\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(f))} & \mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(P')) \\
 \mathbb{F}_{\sqcup}(\beta_P) \uparrow & & \mathbb{F}_{\sqcup}(\beta_{P'}) \uparrow \\
 \mathbb{F}_{\sqcup}(P) & \xrightarrow{\mathbb{F}_{\sqcup}(f)} & \mathbb{F}_{\sqcup}(P').
 \end{array}$$

Figure 5:

this section we shall show that order-preserving maps have an unambiguous extension to the intermediate structure.

Let  $P$  and  $P'$  be posets and let  $f : P \rightarrow P'$  be an order-preserving map. The basic commutative diagram for the free directed meet extension and the result of applying the functor  $\mathbb{F}_{\sqcup}$  to it yield the diagram shown in Figure 5. On the other hand, applying first the functor  $\mathbb{F}_{\sqcup}$  and then the functor  $\mathbb{F}_{\sqcap}$  to the basic commutative diagram for the free directed join extension yields the diagram shown in Figure 6. These two figures, Figure 5 and Figure 6, show that in both the lifting of  $f$  to the  $\Pi_1$ -completion,  $\mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(f))$ , and the lifting of  $f$  to the  $\Sigma_1$ -completion,  $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(f))$ , the restriction to  $\mathbb{F}_{\sqcup}(P)$  acts exactly as  $\mathbb{F}_{\sqcup}(f)$  and is thus unambiguously determined. Dually we get that the action of the lifting of  $f$  on  $\mathbb{F}_{\sqcap}(P)$  is unambiguously determined and is that of  $\mathbb{F}_{\sqcap}(f)$ .

For a more compact notation we shall henceforth adopt the customary terminology and refer to the elements of  $\mathbb{F}_{\sqcup}(P)$  and  $\mathbb{F}_{\sqcap}(P)$  as *open* and *closed*, respectively, and denote these by  $O$  and  $K$ . Whenever  $P$  is

$$\begin{array}{ccc}
 \mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(P)) & \xrightarrow{\mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(f))} & \mathbb{F}_{\sqcap}(\mathbb{F}_{\sqcup}(P')) \\
 \beta_{\mathbb{F}_{\sqcup}(P)} \uparrow & & \beta_{\mathbb{F}_{\sqcup}(P')} \uparrow \\
 \mathbb{F}_{\sqcup}(P) & \xrightarrow{\mathbb{F}_{\sqcup}(f)} & \mathbb{F}_{\sqcup}(P') \\
 \alpha_P \uparrow & & \alpha_{P'} \uparrow \\
 P & \xrightarrow{f} & P'.
 \end{array}$$

Figure 6:

clear from the context, we suppress all reference to it when dealing with open and closed elements. Usually in the canonical extension literature  $O$  has been denoted  $O(P^\sigma)$  whenever it was desirable to make the poset  $P$  explicit. Actually the collection of 'opens', as the  $\vee$ -closure of the original poset in the extension, depends on both the poset and the extension and should really be named  $O(P, P^\sigma)$ . In fact, as we have seen here, the opens are fully determined by  $P$ , being the free directed completion of  $P$ , and we will have occasion to consider this poset as sitting inside various extensions. Therefore we elect not to make  $P^\sigma$  overt when dealing with open elements. Instead, when we need to make clear which poset we are talking about, we will write  $K_P$  and  $O_P$ .

We thus have the following explicit description of the extension of maps to the intermediate structure

$$\text{Int}(f)(x) = \begin{cases} \sqcup\{f(p) : P \ni p \leq x\} & (x \in O_P), \\ \sqcap\{f(p) : P \ni p \geq x\} & (x \in K_P). \end{cases}$$

#### 4. The MacNeille completion of the intermediate structure

In this section we focus on the canonical extension viewed as the MacNeille completion of the intermediate structure. We recall again that the basic properties of the MacNeille completion are discussed for example in [2]; cf. also [1] and [18].

In [3], the canonical extension of a poset  $P$ , as defined there was shown to be the MacNeille completion of  $\text{Int}(P)$  up to isomorphism. As we have seen above  $P^\delta = \mathcal{N}(\text{Int}(P))$  for lattices. Even beyond the setting of lattices

$\mathcal{N}(\text{Int}(P))$  has several merits. For one, it is a lattice completion and thus has much better closure properties than  $P^\delta$  does in general. Secondly, and maybe even more importantly, as was exploited in [3] and [7], this completion lands us within a category that is part of a discrete duality and thus allows us to apply/develop duality theory.

Our concern here is with extension of maps. In particular, we want to show that lattice homomorphisms extend to complete lattice homomorphisms without developing the more general theory of operators and dual operators. It is well known that the MacNeille completion has nice categorical properties with respect to embeddings [1], but that it does not behave well with respect to maps in general.

For complete lattices the complete lattice homomorphisms are the most natural. This is particularly true in the context of a two-sided completion such as the MacNeille completion. To this end the results of M. Ern e in [6] are relevant. There he shows that in order for the class of posets (and then in fact quosets will do) to be a reflective subcategory of the category of complete lattices with complete lattice homomorphisms, the notion of map must be extremely restricted (to what he calls cut-stable maps).

In [6] Ern e identifies the cut-stable maps as those maps  $f : P \rightarrow P'$  between posets such that

$$\forall p', q' \in P' [p' \not\leq q' \implies \exists p, q \in P (p \not\leq q \text{ and } [f^{-1}(\uparrow p') \subseteq \uparrow p] \text{ and } [f^{-1}(\downarrow q') \subseteq \downarrow q])].$$

We now adapt this condition to the special setting of maps  $\text{Int}(f)$ .

Let  $P$  be a poset. Then  $\text{Int}(P)$  is such that the open elements,  $O_P$ , and the closed elements,  $K_P$ , are respectively  $\wedge$ - and  $\vee$ -dense in  $\text{Int}(P)$ . We shall adopt the notation, adopted in [6], of superscript arrows to denote common bounds, but, as in [3] and [7], we restrict the bounds under consideration to be open or closed elements. To be precise, given a poset  $P$  and a subset  $S$  of  $\text{Int}(P)$  we define

$$S^\uparrow = \{ y \in O_P \mid \forall s \in S (y \geq s) \},$$

$$S^\downarrow = \{ x \in K_P \mid \forall s \in S (x \leq s) \}.$$

The set  $S$  is a lower cut if  $S = S^{\uparrow\downarrow}$  and the associated cut is the pair  $(S, S^\uparrow)$ . The pair  $(\uparrow, \downarrow)$  forms a Galois connection, and the associated complete lattice of Galois-closed sets or cuts is the MacNeille completion  $\mathcal{N}(\text{Int}(P))$ .

**Lemma 4.1.** *Let  $f : P \rightarrow P'$  be order preserving and  $\text{Int}(f) : \text{Int}(P) \rightarrow \text{Int}(P')$  the extension of  $f$  to the intermediate structure. Then  $\text{Int}(f)$  is cut stable if and only if the following condition holds:*

*For all  $x' \in K_{P'}$  and  $y' \in O_{P'}$  with  $x' \not\leq y'$  there exist  $x \in K_P$  and  $y \in O_P$  with  $x \not\leq y$  such that:*

$$\begin{aligned} \forall p \in P (x' \leq f(p) &\implies x \leq p), \\ \forall p \in P (f(p) \leq y' &\implies p \leq y). \end{aligned}$$

**Proof.** If  $\text{Int}(f)$  is cut-stable, then, as  $x', y' \in \text{Int}(P')$  and  $x' \not\leq y'$ , there are  $u, v \in \text{Int}(P)$  with  $u \not\leq v$  and  $\text{Int}(f)^{-1}(\uparrow x') \subseteq \uparrow u$  and  $\text{Int}(f)^{-1}(\downarrow y') \subseteq \downarrow v$ . Since  $K_P$  is  $\vee$ -dense in  $\text{Int}(P)$  and  $O_P$  is  $\wedge$ -dense in  $\text{Int}(P)$ , there are  $x \in K_P$  and  $y \in O_P$  with  $x \leq u$ ,  $v \leq y$  and  $x \not\leq y$ . Now let  $p \in P$  with  $x' \leq f(p) = \text{Int}(f)(p)$ . Then  $p \in \text{Int}(f)^{-1}(\uparrow x')$  and thus  $u \leq p$  which implies  $x \leq p$ . Dually, if  $p \in P$  with  $y' \geq f(p) = \text{Int}(f)(p)$ , then  $p \in \text{Int}(f)^{-1}(\downarrow y')$  and thus  $p \leq v$  which implies  $p \leq y$ . That is,  $x, y$  is the pair of elements required for the above condition.

Conversely, suppose the condition of the lemma holds, and let  $u', v' \in \text{Int}(P')$  with  $u' \not\leq v'$ . Again, as  $K_{P'}$  is  $\vee$ -dense in  $\text{Int}(P')$  and  $O_{P'}$  is  $\wedge$ -dense in  $\text{Int}(P')$ , there are  $x' \in K_{P'}$  and  $y' \in O_{P'}$  with  $x' \leq u'$ ,  $v' \leq y'$  and  $x' \not\leq y'$ . Now by the condition of the lemma, we have  $x \in K_P$  and  $y \in O_P$  with  $x \not\leq y$  such that for all  $p \in P$  we have  $(x' \leq f(p) \implies x \leq p)$  and  $(f(p) \leq y' \implies p \leq y)$ .

We first show that it follows that for all  $z \in \text{Int}(P)$  we have  $(x' \leq \text{Int}(f)(z) \implies x \leq z)$  and  $(\text{Int}(f)(z) \leq y' \implies z \leq y)$ . Let  $z \in K_P$  with  $x' \leq \text{Int}(f)(z) = \bigcap \{f(p) \mid z \leq p \in P\}$ . Then for each  $p \in P$  with  $z \leq p$  we have  $x' \leq f(p)$  and thus  $x \leq p$ . But then we have  $x \leq \bigcap \{p \mid z \leq p \in P\} = z$  as desired. Now let  $z \in O_P$  with  $x' \leq \text{Int}(f)(z) = \bigcup \{f(p) \mid z \geq p \in P\}$ . By the  $\bigcup$ -compactness of  $\mathbb{F}_{\bigcap}(P') = K_{P'}$  in  $\mathbb{F}_{\bigcup}(\mathbb{F}_{\bigcap}(P'))$  and thus in  $\text{Int}(P')$ , it follows that there is  $p \in P$  with  $z \geq p$  and  $x' \leq f(p)$ . Consequently  $x \leq p$  and thus  $x \leq z$ .

The proof that for all  $z \in \text{Int}(P)$  we have  $(\text{Int}(f)(z) \leq y' \implies z \leq y)$  is order dual.

Finally suppose  $u' \leq \text{Int}(f)(z)$  for  $z \in \text{Int}(P)$ . Then  $x' \leq \text{Int}(f)(z)$  and thus  $x \leq z$ . Dually if  $\text{Int}(f)(z) \leq v'$  for  $z \in \text{Int}(P)$ , then  $\text{Int}(f)(z) \leq y'$  and thus  $z \leq y$ . So the pair  $x, y$  shows that the cut-stability condition holds for the pair  $u', v'$ .  $\square$

Combining this lemma with Ern e's result from [6] that the cut-stable maps are exactly the ones that extend to complete lattice homomorphisms when taking the MacNeille completion, we obtain the following theorem.

**Theorem 4.2.** *Let  $f : P \rightarrow P'$  be an order preserving map between posets. Then  $\text{Int}(f)$  has a (necessarily unique) extension*

$$\mathcal{N}(\text{Int}(f)) : \mathcal{N}(\text{Int}(L)) \rightarrow \mathcal{N}(\text{Int}(M))$$

to a complete lattice homomorphism if and only if we have:

For all  $x' \in K_{P'}$  and  $y' \in O_{P'}$  with  $x' \not\leq y'$  there exist  $x \in K_P$  and  $y \in O_P$  with  $x \not\leq y$  such that:

$$\begin{aligned} \forall p \in P \ (x' \leq f(p) \implies x \leq p), \\ \forall p \in P \ (f(p) \leq y' \implies p \leq y) \end{aligned}$$

Translating the condition of the lemma and the theorem into properties involving filters and ideals we obtain the condition for what we will call  $\delta$ -morphisms.

**Definition 4.3.** *Let  $P$  and  $P'$  be posets and  $f : P \rightarrow P'$  an order preserving map. Then we say that  $f$  is a  $\delta$ -morphism if and only if, for each pair  $F', I'$  consisting of a filter and an ideal of  $P'$  that are disjoint, there is a filter  $F$  and an ideal  $I$ , both of  $P$ , that are also disjoint and so that  $f^{-1}(F') \subseteq F$  and  $f^{-1}(I') \subseteq I$ .*

**Corollary 4.4.** *An order preserving map  $f : P \rightarrow P'$  extends to a complete homomorphism  $f^\delta : \mathcal{N}(\text{Int}(P)) \rightarrow \mathcal{N}(\text{Int}(P'))$  if and only if  $f$  is a  $\delta$ -homomorphism.*

**Proposition 4.5.** *Let  $P$  and  $Q$  be posets and assume that  $f : P \rightarrow Q$  is an order-preserving map which possesses an upper and a lower adjoint, then  $f$  is a  $\delta$ -morphism.*

**Proof.** Suppose  $f$  has an upper and a lower adjoint and let  $F', I'$  be a disjoint pair consisting of a filter and an ideal of  $P'$ . Let  $F = \uparrow f^\flat(F')$  and  $I = \downarrow f^\sharp(I')$ . We show that  $F$  is a filter of  $P$ . Clearly it is an upset. Now let  $r, s \in F$ , then there are  $r', s' \in F'$  with  $f^\flat(r') \leq r$  and  $f^\flat(s') \leq s$ . Consequently we have  $r' \leq f(r)$  and  $s' \leq f(s)$ . Now pick  $t' \in F'$  with  $t' \leq r'$  then  $t' \leq f(r)$  and thus  $f^\flat(t') \leq r$ . Similarly  $f^\flat(t') \leq s$  and thus  $F$

is down-directed. The dual argument shows that  $I$  is an ideal. Finally we show that  $I \cap F = \emptyset$ . If  $p \in I \cap F$  then there exists  $p' \in F'$  and  $q' \in I'$  with  $f^b(p') \leq p \leq f^\sharp(q')$ . Now  $f^b(p') \leq f^\sharp(q')$  implies  $p' \leq f(f^\sharp(q')) \leq q'$ , which is a contradiction of  $F' \cap I' = \emptyset$ .  $\square$

**Theorem 4.6.** *Assume that  $L$  and  $M$  are bounded lattices and that  $h : L \rightarrow M$  is a lattice homomorphism. Then  $h$  is a  $\delta$ -morphism.*

**Proof.** Let  $F'$  be a filter of  $L'$ ,  $I'$  and ideal of  $L'$ ,  $F = h^{-1}(F')$ , and  $I = h^{-1}(I')$ . Then it is easy to check that  $F$  and  $I$  have the required properties.  $\square$

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IMAPP, Radboud Universiteit,  
PO Box 9010, 6500 GL Nijmegen, Netherlands  
Mathematical Institute, University of Oxford,  
24/29 St Giles, Oxford OX1 3LB, United Kingdom  
M.Gehrke@math.ru.nl    hap@maths.ox.ac.uk