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ADDITIVITY OF THE COMMUTATOR AND RESIDUATION

A b s t r a c t. The notion of a commutator lattice is investigated. It is shown that the class of commutator lattices is coextensive with the class of lattices with residuation.

The work is concerned with the study of commutator lattices, a new notion that originates from commutator theory, a branch of universal algebra. The notion of a commutator lattice embodies, in the abstract and condensed form, the most characteristic features of the commutator. The paper contains a number of observations about commutator lattices and related notions. In particular, an emphasis is put on the residuation operation corresponding to the commutator. It is proved that every commutator defines in a canonical way the residuation operation in the underlying complete lattice. And conversely, every complete lattice endowed with a residuation, satisfying some natural conditions, determines the commutator operation in the lattice. This observation enables one to isolate the notion of a complete lattice with a commutator residuation. The notions of a commutator lattice and of a lattice with the commutator residuation are

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coextensive and mutually interdefinable. An array of examples is presented and some open problems pertinent to commutator lattices are discussed.

1. Commutator lattices.

Definition 1.1. A *proto-commutator lattice* is an algebra $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ of type $(2, 2, 2)$ such that

(1) the reduct $(A, \wedge, \vee)$ is a complete lattice,
(2) the operation $\bullet$ is commutative,
(3) $a \bullet b \leq a \wedge b$ for all $a, b \in A$,
(4) the operation $\bullet$ is monotone, i.e., for all $a, b, c \in A$, $a \leq b$ implies $a \bullet c \leq b \bullet c$.

A *commutator lattice* is an algebra $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ of type $(2, 2, 2)$ satisfying conditions (1) - (3) and the following condition:

(5) $a \bullet \bigvee B = \bigvee\{a \bullet b : b \in B\}$ for any $a \in A$ and $B \subseteq A$.

(Here $\leq$ is the lattice order of $(A, \wedge, \vee)$ and $\vee X$ stands for the supremum of X in the sense of $\leq$, where $X \subseteq A$.) □

Comments to Definition 1.1.

$\bullet$ is called the *commutator operation in* $\boldsymbol{A}$ and the value $a \bullet b$ is called the commutator of a and b. (The commutator operation is usually denoted by [,]. In the context of the present work the use of the symbol $\bullet$ seems to be more transparent.) The commutator operation is *not* assumed to be associative. Furthermore, it *is not* assumed that $\mathbf{1}$, the top element of the lattice $(A, \wedge, \vee)$ is neutral with respect to $\bullet$, i.e., the equation $x \bullet \mathbf{1} \approx x$ need not hold in $\boldsymbol{A}$. This means that the reduct $(A, \bullet, \mathbf{1})$ need not be a monoid.

Condition (5) is called the (*total*) *additivity* property of the commutator. If B is empty, then $\bigvee B = \mathbf{0}$, the least element of the lattice. Consequently, (5) implies that $a \bullet \mathbf{0} = \mathbf{0}$ for all $a \in A$. But this equality also trivially follows from (3).

The restriction of condition (5) to *finite* subsets B is called *finite additivity.* It is clear that finite additivity is equivalent to the equation:

$$a \bullet (b_1 \vee b_2) = (a \bullet b_1) \vee (a \bullet b_2),$$

for all $a, b_1, b_2 \in A$.

Proto-commutator lattice in which $\bullet$ is finitely additive are called *finitely additive commutator lattices*. This definition is correct but we must remember that finitely additive commutator lattices need not be commutator lattices in the sense of the above definition. Of course, if $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ is a finitely additive commutator lattice and the underlying set A is finite, then $\boldsymbol{A}$ is a commutator lattice.

Finite additivity of the operation $\bullet$ trivially implies its *monotonicity*. Indeed, if $a \leq b$, then $b = a \vee b$. It follows that $b \bullet c = (a \vee b) \bullet c = (a \bullet c) \vee (b \bullet c)$ which means that $a \bullet c \leq b \bullet c$. Hence every commutator lattice is a proto-commutator lattice. □

Theorem 1.2. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be an algebra of type* $(2, 2, 2)$. *Suppose* $\boldsymbol{A}$ *satisfies* (1), (2) *and* (3). *Then* $\boldsymbol{A}$ *satisfies* (5) *if and only if, for every* $a \in A$ *and any set* $B \subseteq A$,

(5)* *if* $a \leq \bigvee B$, *then* $a \bullet c \leq \bigvee\{b \bullet c : b \in B\}$, *for all* $c \in A$.

Proof. ($\Rightarrow$). Assume $\boldsymbol{A}$ satisfies (5). Then $\bullet$ is monotone. Suppose $a \leq \bigvee B$. Then $a \bullet c \leq \bigvee B \bullet c$, by the monotonicity of $\bullet$. As (5) holds, we also have: $\bigvee B \bullet c = \bigvee\{b \bullet c : b \in B\}$. So $a \bullet c \leq \bigvee\{b \bullet c : b \in B\}$.

($\Leftarrow$). We assume $\boldsymbol{A}$ validates (5)*. We first note that (5)* trivially implies the monotonicity of the commutator operation. Indeed, let $a \leq b$ for some $a, b \in A$. Putting $B := \{b\}$, we have that for any $c \in A$, $a \bullet c \leq \bigvee\{y \bullet c : y \in B\} = b \bullet c$.

To show (5), suppose $a \in A$ and $B \subseteq A$. In view of monotonicity, $a \bullet \bigvee B \geq a \bullet b$ for all $b \in B$. Consequently,

(a) $a \bullet \bigvee B \geq \bigvee\{a \bullet b : b \in B\}$.

In the other direction, putting $b_0 := \bigvee B$, we trivially have that $b_0 \leq \bigvee B$, Whence, by (5)*, $a \bullet b_0 \leq \bigvee\{a \bullet b : b \in B\}$, i.e.,

(b) $a \bullet \bigvee B \leq \bigvee\{a \bullet b : b \in B\}$.

In virtue of (a) and (b), condition (5) holds. □

Examples. Since in every proto-commutator lattice the commutator $\bullet$ satisfies the double inequality: $\mathbf{0} \leq a \bullet b \leq a \wedge b$ for any $a, b \in A$, it is then natural to consider two extreme possibilities.

Case 1. *The zero commutator.*

This is the commutator which satisfies $a \bullet b = \mathbf{0}$, for all $a, b \in A$. Equivalently, $\bullet$ is the zero commutator if and only if $\mathbf{1} \bullet \mathbf{1} = \mathbf{0}$.

It is clear that the zero commutator satisfies the condition of total additivity. We thus see that every complete lattice endowed with the zero commutator becomes the commutator lattice. In this case the properties of the commutator have no impact on the structure properties of the lattice on which the zero commutator is defined.

Case 2. *The full commutator.*

This is the commutator which satisfies $a \bullet b = a \wedge b$, for all $a, b \in A$.

Every complete lattice equipped with the full commutator becomes a proto-commutator lattice. This commutator need not be additive. It is easy to see that the full commutator, defined in a complete lattice $\boldsymbol{A}$, is totally additive if and only if the lattice $\boldsymbol{A}$ is distributive in the infinite sense, i.e., $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ for any $a \in A$ and any set $B \subseteq A$. In this case the commutator is stricly interconnected with the structure of the underlying lattice.

In the above context it is then natural to ask about the existence of plausible lattice-theoretic conditions, which, when imposed on a complete lattice, would yield the existence of the totally additive commutator but different from the above two extreme cases.

One may argue that the above definition of a commutator lattice is *ad hoc* since it is too general and not well anchored in the existing literature on commutator theory, an important part of universal algebra. The natural environment for commutator theory is that part of universal algebra that studies congruence-modular equational classes of algebras, see [3], or, more generally, relatively congruence-modular quasivarieties [7]. The notion of the commutator is there inherently linked with congruence lattices of algebras belonging to such classes and has a genuinely algebraic flavour. Much more general approach to the commutator, based on the theory of n-dimensional protoalgebraic systems has been proposed in [1]. □

The greatest commutator in a lattice.

Given a complete lattice $\boldsymbol{A} = (A, \wedge, \vee)$, we let

$$Comm(\boldsymbol{A})$$

denote the set of all commutator operations defined in A. Thus, for every $\bullet$ in $Comm(\boldsymbol{A})$, the algebra $(A, \wedge, \vee, \bullet)$ is a commutator lattice. The set $Comm(\boldsymbol{A})$ is non-empty because the zero commutator is in it. $Comm(\boldsymbol{A})$ is furnished with the following binary relation $\leq_c$:

$$\bullet_1 \leq_c \bullet_2 \text{ iff for all } x, y \in A,\ x \bullet_1 y \leq x \bullet_2 y,$$

where $\bullet_1$ and $\bullet_2$ are in $Comm(\boldsymbol{A})$.

Theorem 1.3. *The pair* $(Comm(\boldsymbol{A}), \leq_c)$ *is a complete lattice.*

Proof. It is easy to see that $\leq_c$ is a partial order and the zero commutator is the least element of the above poset.

Let C be a subset of $Comm(\boldsymbol{A})$. Let the mapping $\bullet_C : A \times A \to A$ be defined as follows: for any $x, y \in A$,

$$x \bullet_C y := \sup\{x \bullet y : \bullet \in C\},$$

where the supremum is taken in the lattice $\boldsymbol{A}$. (We use here the commutator notation earlier adopted in this work.)

The theorem is an obvious consequence of the following lemma.

Lemma 1.4. *For every set* $C \subseteq Comm(\boldsymbol{A})$*, the operation* $\bullet_C$ *is a commutator in the lattice* $\boldsymbol{A}$*.*

Moreover $\bullet_C$ *is the supremum of the set* C *in the poset* $(Comm(\boldsymbol{A}), \leq_c)$*.*

Proof of the lemma. We shall check that $\bullet_C$ is totally additive. (Verification of other conditions is easy.) Let $a \in A$ and let B be a subset of A. Then:

$$a \bullet_C \bigvee B = \sup\{a \bullet \bigvee B : \bullet \in C\} = \sup\{\bigvee\{a \bullet b : b \in B\} : \bullet \in C\} =$$
$$\sup\{\sup\{a \bullet b : b \in B\} : \bullet \in C\} = \sup\{\sup\{a \bullet b : \bullet \in C\} : b \in B\} =$$
$$\sup\{a \bullet_C b : b \in B\}. \qquad \square$$

The following observation immediately follows from Theorem 4.1:

Corollary 1.5. *In every complete lattice* $\boldsymbol{A} = (A, \wedge, \vee)$ *there exists the largest commutator in the sense of the order* $\leq_c$*. It is denoted by* $\bullet_\Omega$*.* $\square$

The above proof of the above theorem and of Corollary 1.5 is not constructive in the sense that it does not provide an explict uniform definition of $\bullet_\Omega$ (formulated in the second-order language of complete lattices). The known exceptions are of course distributive lattices in which $\bullet_\Omega$ coincides

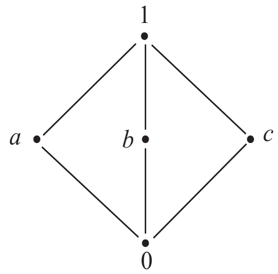

Figure 1: $\boldsymbol{N_5}$

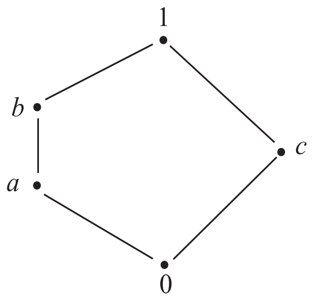

Figure 2: $\boldsymbol{M_5}$

with the meet operation. A characterization of the operation $\bullet_\Omega$ in modular algebraic lattices is an open and challenging problem.

Let $\boldsymbol{N_5}$ denote the non-distributive lattice shown in Figure 1.

We shall show that the greatest commutator of $\boldsymbol{N_5}$ coincides with the zero commutator. It suffices to prove that the zero commutator is the only commutator operation defined in $\boldsymbol{N_5}$. For let $\bullet$ be a commutator in $\boldsymbol{N_5}$. It suffices to show that $\mathbf{1} \bullet \mathbf{1} = \mathbf{0}$. Evidently, $a \bullet b = b \bullet c = a \bullet c = \mathbf{0}$, because $a \wedge b = b \wedge c = a \wedge c = \mathbf{0}$. Then, by additivity,

$$a \bullet \mathbf{1} = a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c) = \mathbf{0} \vee \mathbf{0} = \mathbf{0},$$

$$b \bullet \mathbf{1} = b \bullet (a \vee c) = (b \bullet a) \vee (b \bullet c) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}.$$

Consequently, $\mathbf{1} \bullet \mathbf{1} = \mathbf{1} \bullet (a \vee b) = (\mathbf{1} \bullet a) \vee (\mathbf{1} \bullet b) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$.

$\boldsymbol{M_5}$ denotes the pentagon lattice, the non-modular lattice given by the diagram shown in Figure 2.

The greatest commutator $\bullet_\Omega$ in $\boldsymbol{M_5}$ is non-zero. It is given by the following table:

$\bullet_\Omega$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	0	a
b	0	a	b	0	a
c	0	0	0	c	c
1	0	a	a	c	1

As $\mathbf{1} \bullet_\Omega b = a$, $\bullet_\Omega$ does not coincide with the meet operation in $\boldsymbol{M_5}$.

Dual commutators.

Every commutator operation $\bullet$ in a complete lattice may be called a *meet- commutator*, $\wedge$*-commutator* for short, because it is majorized by the meet operation of the lattice. One may also introduce the notion dual to the commutator, namely the notion of a *join-commutator*, $\vee$*-commutator* for short, which is defined by the conditions dual to those which define commutators.

Definition 1.6. An algebra $\boldsymbol{A} = (A, \wedge, \vee, \bullet^d)$ of type (2,2,2) is called a *dual proto-commutator lattice* if

$(1)^d$ the reduct $(A, \wedge, \vee)$ is a complete lattice,

$(2)^d$ the operation $\bullet^d$ is commutative,

$(3)^d$ $a \bullet^d b \geq a \vee b$ for all $a, b \in A$,

$(4)^d$ the operation $\bullet^d$ is monotone, i.e., for all $a, b, c \in A$, $a \leq b$ implies $a \bullet^d c \leq b \bullet^d c$.

If moreover $\bullet^d$ satisfies the condition:

$(5)^d$ $a \bullet^d \bigwedge B = \bigwedge\{a \bullet^d b : b \in B\}$ for any $a \in A$ and $B \subseteq A$,

the operation $\bullet^d$ is called a *dual-commutator*, or a *join-commutator*. □

We note that $(4)^d$ already follows from $(5)^d$.

One may also consider extreme, limit cases of dual commutators, paralel to those for the usual commutator.

The dual commutator satisfying the condition: $a \bullet^d b = \mathbf{1}$, for all $a, b \in A$, is called the *unit* dual commutator. Equivalently, this is the dual commutator such that $\mathbf{0} \bullet^d \mathbf{0} = \mathbf{1}$. In turn, if the join operation itself is a dual commutator in a complete lattice $(A, \wedge, \vee)$, then $\vee$ is called the *full dual* commutator. In this case, $(5)^d$ entails the following infinite distributivity law:

$$a \vee \bigwedge B = \bigwedge\{a \vee b : b \in B\} \text{ for any } a \in A \text{ and } B \subseteq A,$$

which must hold in the lattice $(A, \wedge, \vee)$.

Given a complete lattice $\boldsymbol{A} = (A, \wedge, \vee)$, we let

$$DComm\,(\boldsymbol{A})$$

denote the set of all dual commutator operations defined in A. Thus, for every $\bullet^d$ in $DComm\,(\boldsymbol{A})$, the algebra $(A, \wedge, \vee, \bullet^d)$ is a dual commutator lattice. The set $DComm\,(\boldsymbol{A})$ is non-empty because the unit dual commutator is in it. $DComm\,(A)$ is furnished with the following binary relation $\leq_{dc}$:

$$\bullet^d_1 \leq_{dc} \bullet^d_2 \text{ if and only if, for all } x, y \in A,\ x \bullet^d_1 y \leq x \bullet^d_2 y,$$

where $\bullet^d_1$ and $\bullet^d_2$ are in $DComm\,(\boldsymbol{A})$.

Theorem 1.7. *The pair* $(DComm\,(\boldsymbol{A}), \leq_{dc})$ *is a complete lattice.*

The proof is analogous to that of Theorem 1.3. $\leq_{dc}$ is a partial order with the unit dual commutator being the greatest element.

Let D be a subset of $DComm\,(\boldsymbol{A})$. Let the mapping $\bullet^d_D : A \times A \to A$ be defined as follows: for any $x, y \in A$,

$$x \bullet^d_D y := \inf\{x \bullet^d y : \bullet^d \in D\},$$

where the infimum is taken in the lattice $\boldsymbol{A}$.

The following lemma is immediate:

Lemma 1.8. *For every set* $D \subseteq DComm\,(\boldsymbol{A})$*, the operation* $\bullet^d_D$ *is a dual commutator in the lattice* $\boldsymbol{A}$*. Moreover* $\bullet^d_D$ *is the infimum of the set* D *in the poset* $(DComm\,(\boldsymbol{A}), \leq_{dc})$*.*

Proof of the lemma. We shall check that $\bullet^d_D$ is totally multiplicative. (Verification of other conditions is easy.) Let $a \in A$ and let B be a subset of A. Then:

$$a \bullet^d_D \bigwedge B = \inf\{a \bullet^d \bigwedge B : \bullet^d \in D\} = \inf\{\bigwedge\{a \bullet^d b : b \in B\} : \bullet^d \in D\}$$
$$= \inf\{\inf\{a \bullet^d b : b \in B\} : \bullet^d \in D\} = \inf\{\inf\{a \bullet^d b : \bullet^d \in D\} : b \in B\}$$
$$= \inf\{a \bullet^d_D b : b \in B\}. \qquad \square$$

Theorem 1.7 follows from the above lemma. $\square$

Corollary 1.9. *In every complete lattice* $\boldsymbol{A} = (A, \wedge, \vee)$ *there exists the least dual commutator in the sense of the order* $\leq_{dc}$. *It is denoted by* $\bullet^d_{A}$. ☐

(The subscript “A” above stands for the Greek capital “alpha” .)

The lattice $\boldsymbol{N_5}$ is equipped with only one dual commutator it is the unit dual commutator. The proof of this fact is obtained by a straightforward “dualization” of the argument that the zero commutator is the greatest commutator in $\boldsymbol{N_5}$.

In an analogous way one may describe the least dual commutator $\bullet^d_A$ in the lattice $\boldsymbol{M_5}$. $\bullet^d_A$ differs from the unit dual commutator.

2. Residuations in proto-commutator lattices.

Definition 2.1. Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a complete lattice endowed with a binary operation $\bullet$. A binary operation $\rightarrow$ defined on the set A is called a *residuation for* $\bullet$ in $\boldsymbol{A}$ if for every triple $a, b, c \in A$ the following equivalence holds:

(6) $\quad a \bullet c \leq b$ if and only if $c \leq a \rightarrow b$.

The equivalence (6) is called the *residuation property* of the mapping $\rightarrow$. If (6) holds, we also say that $\bullet$ is *residuated by* $\rightarrow$. ☐

Given a complete lattice $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ equipped with a binary operation $\bullet$, we define, for each pair $a, b \in A$:

(Res) $\quad a \rightarrow b := \bigvee\{x \in A : a \bullet x \leq b\}$.

$\rightarrow$ is thus a well-defined binary operation in the set A.

Every commutator lattice is endowed with a residuation operation defined in a certain canonical way.

Theorem 2.2. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a commutator lattice. Define the binary operation* $\rightarrow$ *in* A *according to formula* (Res). *Then* $\rightarrow$ *is a residuation for* $\bullet$ *in* $\boldsymbol{A}$.

Proof. ($\Rightarrow$). Assume $a \bullet c \leq b$. This means that c belongs to the set $\{x \in A : a \bullet x \leq b\}$. Hence the definition of $a \rightarrow b$ implies that $c \leq a \rightarrow b$.

($\Leftarrow$). Assume $c \leq a \rightarrow b$. It follows that $a \bullet c \leq a \bullet (a \rightarrow b) = a \bullet \bigvee\{x \in A : a \bullet x \leq b\} =$ (by (5)) $\bigvee\{a \bullet x : x \in A, a \bullet x \leq b\} \leq b$. Hence $a \bullet c \leq b$. ☐

We shall later show that the canonical residuation in commutator lattices possess some other, inherently infinitistic properties, which unambiguously characterize this operation see Lemma 2.8 below.

The following observation says that commutator lattices possess exactly one residuation operation, viz. the canonical residuation defined by the condition (Res) above. This follows from the following general fact.

Theorem 2.3. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a proto-commutator lattice. Suppose a binary operation* $\rightarrow^*$ *defined in* A *is a residuation for* $\bullet$*, i.e., for all* $a, b, c \in A$*,* $a \bullet c \leq b$ *if and only if* $c \leq a \rightarrow^* b$*. Then* $a \rightarrow^* b = \bigvee\{x \in A : a \bullet x \leq b\}$*, for all* $a, b \in A$*.*

Proof. Let $a, b \in A$. Since $\rightarrow^*$ is a residuation, it is clear that $a \rightarrow {}^*b \geq \bigvee\{x \in A : a \bullet x \leq b\}$. Let $X := \{x \in A : a \bullet x \leq b\}$. Then $X = \{x \in A : x \leq a \rightarrow^* b\}$ and trivially $a \rightarrow^* b$ belongs to X. Consequently, $a \rightarrow^* b \leq \bigvee\{x \in A : a \bullet x \leq b\}$. □

We underlie here that fact that a proto-commutator lattice $\boldsymbol{A}$ need not possesses a residuation at all. The above theorem thus states that if $\boldsymbol{A}$ has a residuation, then this operation is unique and defined by the formula (Res). Consequently, in this context, we may say about *the* residuation operation for $\bullet$.

The next observation says that in proto-commutator lattices, the total additivity of the operation $\bullet$ is equivalent to the fact that the operation $\rightarrow$ defined by (Res) is indeed a residuation.

Theorem 2.4. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a proto-commutator lattice. Let the operation* $\rightarrow$ *in* A *be defined according to* (Res) *above. Then* $\rightarrow$ *satisfies* (6) *if and only if the operation* $\bullet$ *is totally additive, i.e.,* (5) *holds.*

The above theorem thus states that the class of commutator lattices coincides with the class of proto-commutator lattices in which the operation $\rightarrow$ defined by (Res) is a residuation.

Proof. ($\Leftarrow$). Assume (5) holds for $\boldsymbol{A}$. Hence $\boldsymbol{A}$ is a commutator lattice. Applying Theorem 2.2, we get that $\rightarrow$ is a residuation.

($\Rightarrow$). Suppose $\rightarrow$ satisfies (6). Let $a \in A$ and $B \subseteq A$. By the monotonicity of $\bullet$, $a \bullet \bigvee B \geq a \bullet b$, for all $b \in B$. Consequently, $a \bullet \bigvee B \geq \bigvee\{a \bullet b : b \in B\}$. This means that $a \bullet \bigvee B$ is an upper bound of the set $\{a \bullet b : b \in B\}$. We claim that $a \bullet \bigvee B$ is the least upper bound of this set. Suppose y is an upper bound of $\{a \bullet b : b \in B\}$, i.e., $a \bullet b \leq y$ for all $b \in B$. Hence $b \leq a \rightarrow y$ for all $b \in B$, by residuation (6). This implies $\bigvee B \leq a \rightarrow y$. Applying again (6), we get that $a \bullet \bigvee B \leq y$. This proves that $a \bullet \bigvee B = \bigvee\{a \bullet b : b \in B\}$. □

Note. If $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ is a commutator lattice, we put for any $a, b \in A$:

$$a \backslash b := a \rightarrow b \text{ and } a/b := b \rightarrow a.$$

Trivially

(*) $\quad a/b = b \backslash a,$

and

$$a \bullet c \leq b \text{ iff } c \leq a \backslash b \text{ iff } a \leq b/c,$$

for all $a, b, c \in A$.

Thus every commutator lattice is residuated lattice ordered commutative grupoid in the sense known from the literature, see e.g. [4], GJKO, for short. But strictly speaking, commutator lattices $\boldsymbol{A}$ are *not* residuated lattices in the sense of GJKO, because their reducts $(A, \bullet, \mathbf{1})$ need not be monoids (i.e., $\bullet$ is asssociative, with unit element $\mathbf{1}$). (The condition that $(A, \bullet, \mathbf{1})$ be a monoid is assumed in the definition of a residuated lattice.) Moreover, (*) says that the operation \ is the reverse of /, and hence / is superfluous as a primitive operation in the signature of commutator lattices.

Following the common notation adopted in logic and in the theory of commutative residuated lattices, we shall rather use the symbol $\rightarrow$ and express $a \backslash b$ and a/b as $a \rightarrow b$ and $b \rightarrow a$, respectively. □

Examples. Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a commutator lattice with the zero commutator. It is easy to see that the residuation $\rightarrow$ in $\boldsymbol{A}$ defined by means of (Res) satisfies the following condition: $a \rightarrow b = \mathbf{1}$, for all $a, b \in A$.

In turn, if $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ is equipped with the full commutator, then for all $a, b \in A$, $a \rightarrow b = \bigvee\{x \in A : a \wedge x \leq b\}$. The operation $\rightarrow$ has the properties on intuitionistic implication and the algebra $(A, \wedge, \vee, \rightarrow, \mathbf{0})$ becomes a complete Heyting algebra. (Negation $\neg$ is defined by $\neg a := a \rightarrow \mathbf{0}$, for all $a \in A$.) □

The following result characterizes the residuation operation in commutator lattices:

Theorem 2.5. *Let* $(A, \wedge, \vee, \bullet)$ *be a commutator lattice. Let* $\rightarrow$ *be the binary operation defined in* A *according to the formula* (Res)*. Then, for all* $a, b, c \in A$ *and all subsets* $B \subseteq A$*, the following conditions hold*:

(a) $a \leq b$ *implies* $c \rightarrow a \leq c \rightarrow b$,

(b) $a \rightarrow \bigwedge B = \bigwedge\{a \rightarrow b : b \in B\}$,

(c) $b \leq a \rightarrow (a \wedge b)$,

(d) $a \leq b \rightarrow c$ *if and only if* $b \leq a \rightarrow c$,

(e) $a \leq b$ *implies* $b \rightarrow c \leq a \rightarrow c$.

Proof. We note that the above conditions do not involve occurrences of $\bullet$ whatsoever.

The theorem follows from the following lemmas which seem to be interesting in their own right:

Lemma 2.6. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a commutator lattice. Then for any* $a, b, c \in A$:

(7) $a \bullet (a \rightarrow b) \leq a \wedge b$.

(8) $b \leq a \rightarrow (a \bullet b)$.

(9) $a \leq b$ *implies* $a \rightarrow b = \mathbf{1}$.

(10) $a \rightarrow b = \mathbf{1}$ *if and only if* $a \bullet \mathbf{1} \leq b$.

(11) $a \rightarrow a = \mathbf{1}$.

Proof.

(7). As $a \bullet (a \wedge b) \leq a \wedge (a \wedge b) = a \wedge b \leq b$, and hence $a \bullet (a \wedge b) \leq b$, the residuation gives that $a \bullet (a \rightarrow b) \leq a \wedge b$.

(8). As $a \bullet b \leq a \bullet b$, residuation gives that $b \leq a \rightarrow a \bullet b$.

(9). Suppose $a \leq b$. Then $\{x \in A : a \bullet x \leq b\} \supseteq \{x \in A : a \wedge x \leq b\} = A$. Consequently, $a \rightarrow b = \bigvee\{x \in A : a \bullet x \leq b\} = \bigvee A = \mathbf{1}$.

(10). Assume $a \rightarrow b = \mathbf{1}$. Hence for every $x \in A$ it is the case that $x \leq a \rightarrow b$. This by residuation, gives that $\{x \in A : a \bullet x \leq b\} = A$. Putting $x = \mathbf{1}$, we obtain that $a \bullet \mathbf{1} \leq b$.

Conversely, let $a \bullet \mathbf{1} \leq b$. Then, by residuation, $\mathbf{1} \leq a \rightarrow b$. Hence $a \rightarrow b = \mathbf{1}$.

(11) follows from (10). □

Lemma 2.7. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a commutator lattice. Then, for any* $a, b, c \in A$, $\boldsymbol{A}$ *satisfies* (a), (c), (d) *and* (e).

Proof.

(a) and (e). Suppose $a \leq b$. It follows from (7) that $c \bullet (c \to a) \leq a$. Hence $c \bullet (c \to a) \leq b$. This gives that $c \to a \leq c \to b$, by residuation. So (a) holds. To prove (e), we observe that $\{x \in A : b \bullet x \leq c\} \subseteq \{x \in A : a \bullet x \leq c\}$, because $a \leq b$. Consequently, $b \to c = \bigvee\{x \in A : b \bullet x \leq c\} \leq \bigvee\{x \in A : a \bullet x \leq c\} = a \to c$. So (e) holds.

(c). As $a \bullet b \leq a \wedge b$, residuation implies that $b \leq a \to a \wedge b$.

(d). By the residuation property and the commutativity of $\bullet$, $a \leq b \to c$ iff $b \bullet a \leq c$ iff $a \bullet b \leq c$ iff $b \leq a \to c$. □

Lemma 2.8. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a commutator lattice. Then for any* $a \in A$ *and any set* $B \subseteq A$:

(A) $\quad a \to \bigwedge B = \bigwedge\{a \to b : b \in B\}$.

(B) $\quad \bigvee B \to a = \bigwedge\{b \to a : b \in B\}$.

Proof. Note that (A) is the same as (b) in Theorem 2.5. Let $a \in A$ and $B \subseteq A$.

(A). Evidently, $\bigwedge B \leq b$, for all $b \in B$. As, by Lemma 2.7, $\boldsymbol{A}$ satisfies (a), we have that, $a \to \bigwedge B \leq a \to b$, for all $b \in B$. Consequently $a \to \bigwedge B \leq \bigwedge\{a \to b : b \in B\}$. We claim that the element $a \to \bigwedge B$ is the greatest lower bound of the set $X := \{a \to b : b \in B\}$. Suppose x is a lower bound of X. As $x \leq a \to b$, for all $b \in B$, the residuation property gives that $a \bullet x \leq b$, for all $b \in B$. Consequently, $a \bullet x \leq \bigwedge B$. Then, by residuation, $x \leq a \to \bigwedge B$. So (A) holds.

(B). As $b \leq \bigvee B$, for all $b \in B$, the fact that $\boldsymbol{A}$ satisfies (d) (by Lemma 2.7) implies that $\bigvee B \to a \leq b \to a$, for all $b \in B$. Hence $\bigvee B \to a \leq \bigwedge\{b \to a : b \in B\}$. We claim that $\bigvee B \to a$ is the greatest lower bound of the set $X := \{b \to a : b \in B\}$. Suppose x is a lower bound of X. Since $x \leq b \to a$, for all $b \in B$, the residuation property gives that $b \bullet x \leq a$, for all $b \in B$. Hence $\bigvee\{b \bullet x : b \in B\} \leq a$. Consequently, $x \bullet \bigvee B \leq a$ by additivity. It follows that $x \leq \bigvee B \to$ a by the residuation property. So (B) holds. □

The theorem follows from Lemmas 2.7 2.8. □

The next observation concerns definability of the commutator operation in terms of residuation and the lattice order.

Theorem 2.9. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a commutator lattice. Let* $\rightarrow$ *be the residuation for* $\bullet$*. Then for any* $a, b \in A$*,*

$$a \bullet b = \bigwedge\{x \in A : b \leq a \rightarrow x\}.$$

Proof. Let $X := \{x \in A : b \leq a \rightarrow x\}$. As $b \leq a \rightarrow x$, for all $x \in X$, the residuation property gives that $a \bullet b \leq x$, for all $x \in X$. Hence $a \bullet b$ is a lower bound of X. Consequently, $a \bullet b \leq \bigwedge\{x \in A : b \leq a \rightarrow x\}$. But Lemma 2.6.(8) implies that $a \bullet b \in X$, because $b \leq a \rightarrow (a \bullet b)$. Hence $\bigwedge\{x \in A : b \leq a \rightarrow x\} \leq a \bullet b$. This proves the theorem. □

The next simple observation provides equational characterization of residuation operations in proto-commutator lattices.

Theorem 2.10. *Let* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *be a proto-commutator lattice. Let* $\rightarrow$ *be a binary operation defined in* A*. Then* $\rightarrow$ *is the residuation for* $\bullet$ *if and only if the following two conditions hold*:

(a) $x \bullet (x \rightarrow y) \leq y$,

(b) $y \leq x \rightarrow ((x \bullet y) \vee z)$.

Proof. ($\Rightarrow$). Assume $\rightarrow$ satisfies (6). As the inequality $x \rightarrow y \leq x \rightarrow y$ trivially holds in $\boldsymbol{A}$, the residuation property gives that $x \bullet (x \rightarrow y) \leq y$. So (a) holds.

As $x \bullet y \leq (x \bullet y) \vee z$, residuation gives that $y \leq x \rightarrow ((x \bullet y) \vee z)$. So (b) holds.

($\Leftarrow$). Suppose that $\rightarrow$ satisfies (a) and (b). We claim (6) holds in $\boldsymbol{A}$. Let $a, b, c \in A$. Let us first assume that $a \bullet c \leq b$. Then $(a \bullet c) \vee b = b$. According to (b) we have that $c \leq a \rightarrow ((a \bullet c) \vee b) = a \rightarrow b$. Hence $c \leq a \rightarrow b$.

Conversely, suppose that $c \leq a \rightarrow b$. Then, by the monotonicity of $\bullet$ and (a), $a \bullet c \leq a \bullet (a \rightarrow b) \leq b$. So $a \bullet c \leq b$.

This proves that $\rightarrow$ saisfies (6). □

Corollary 2.11. *Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a proto-commutator lattice. Suppose $\rightarrow$ is a binary operation defined in A satisfying conditions* (a) *and* (b) *of the above theorem. Then $\boldsymbol{A}$ is a commutator lattice.*

Proof. In view of Theorem 2.10, $\rightarrow$ is a residuation in $\boldsymbol{A}$. Theorem 2.3 implies that $a \rightarrow b = \bigvee\{x \in A : a \bullet x \leq b\}$, for all $a, b \in A$. This means that the residuation $\rightarrow$ is equal to the operation defined by (Res). Then applying Theorem 2.4, we get that $\bullet$ is totally additive, which means that $\boldsymbol{A}$ is a commutator lattice. $\square$

Corollary 2.12. *The class of commutator lattices coincides with the class of proto-commutator lattices endowed with the residuation operation for the commutator.* $\square$

It should be noted that if a binary operation $\rightarrow$ in a proto-commutator lattice $\boldsymbol{A}$ is defined according to formula (Res), then $\rightarrow$ already satisfies condition (b) of Theorem 2.10. Indeed, let $a, b, c \in A$. Then $b \leq a \rightarrow ((a \bullet c) \vee c)$ means that $b \leq \bigvee\{x \in A : a \bullet x \leq (a \bullet b) \vee c\}$. But the last inequality trivially holds because b is a member of the set $\{x \in A : a \bullet x \leq (a \bullet b) \vee c\}$.

The above remark yields the following corollary which supplements Theorem 2.10:

Corollary 2.13. *Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a proto-commutator lattice. Let the operation $\rightarrow$ in A be defined according to* (Res) *above. Then $\rightarrow$ is the residuation for $\bullet$ in $\boldsymbol{A}$ if and only if $a \bullet (a \rightarrow b) \leq b$ for all $a, b \in A$, which means that*

$$a \bullet \bigvee\{x \in A : a \bullet x \leq b\} \leq b. \qquad \square$$

Combining the above corollary with Corollary 2.11, we obtain:

Corollary 2.14. *Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a proto-commutator lattice. Then $\boldsymbol{A}$ is a commutator lattice if and only if, for every pair $a, b \in A$, $a \bullet \bigvee\{x \in A : a \bullet x \leq b\} \leq b$.* $\square$

The above corollary is thus a paraphrase of Theorem 1.2.

3. Residuations and the commutator.

Let $\boldsymbol{A} = (A, \wedge, \vee)$ be a complete lattice. Suppose $\bullet_1$ and $\bullet_2$ are binary operations defined in A that possess the same residuation $\rightarrow$, i.e., for all $a, b, c \in A$ it is the case that

(+) $a \bullet_1 c \le b$ iff $c \le a \to b$ iff $a \bullet_2 c \le b$.

Then it trivially follows from (+) that the operations $\bullet_1$ and $\bullet_2$ coincide.

In other words, if a binary mapping $\bullet$ possesses a residuation, then $\to$ unambiguously determines $\bullet$. This simple observation gives rise to the following question:

Problem. Let $\boldsymbol{A} = (A, \wedge, \vee)$ be a complete lattice. Suppose A is endowed with a binary operation $\to$. Find necessary and sufficient conditions imposed on $\to$ ensuring the existence of a binary operation $\bullet$ on A under which $\to$ becomes the residuation for $\bullet$. Moreover, formulate additional conditions under which $\to$ is the residuation for the additive commutator.

The following observation, which is the converse of Theorem 2.5, entirely solves the above problem.

Theorem 3.1. *Let* $(A, \wedge, \vee)$ *be a complete lattice. Suppose* $\to$ *is a binary operation defined in* A *such that, for all* $a, b, c \in A$ *and all subsets* $B \subseteq A$*, it satisfies conditions* (a) *and* (b) *of Theorem* 2.5*, i.e.,*

(a) *if* $a \le b$ *then* $c \to a \le c \to b$,

(b) $a \to \bigwedge B = \bigwedge\{a \to b : b \in B\}$.

We define the binary operation $\bullet$ *in* A *according to the formula*:

(Com) $a \bullet b := \bigwedge\{x \in A : b \le a \to x\}$,

for all $a, b \in A$. (*cf. Theorem* 2.9)*. Then* $\to$ *is a residuation for* $\bullet$ *in the above lattice.*

Moreover, if for all $a, b, c \in A$*, the operation* $\to$ *satisfies conditions* (c)*,* (d) *and* (e) *of Theorem* 2.5*, i.e.,*

(c) $b \le a \to (a \wedge b)$,

(d) $a \le b \to c$ *if and only if* $b \le a \to c$,

(e) $a \le b$ *implies* $b \to c \le a \to c$,

for all $a, b, c \in A$*, then* $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ *is a commutator lattice. Furthermore*

$$a \to b = \bigvee\{x \in A : a \bullet x \le b\}, \text{ for all } a, b \in A.$$

Proof. $\bullet$ is well-defined. We claim that $\to$ is the residuation for $\bullet$. Let $a, b \in A$. It immediately follows from the definition of $\bullet$ that for any $y \in A$,

$$b \leq a \to y \text{ implies } a \bullet b \leq y.$$

Let $X := \{x \in A : b \leq a \to x\}$. To prove the reverse implication, suppose $a \bullet b \leq y$ for some $y \in A$. This means that $\bigwedge X \leq y$. The last inequality together with (a) imply that $a \to \bigwedge X \leq a \to y$. But by (b), $a \to \bigwedge X = \bigwedge\{a \to x : x \in X\}$. Hence $\bigwedge\{a \to x : x \in X\} \leq a \to y$. But $b \leq a \to x$, for all $x \in X$, by the definition of X. Consequently, $b \leq \bigwedge\{a \to x : x \in X\} \leq a \to y$. Thus $b \leq a \to y$.

To prove the second part of the theorem, assume $a, b, c \in A$. As by (c), $b \leq a \to (a \wedge b)$, the just proved residuation property gives that $a \bullet b \leq a \wedge b$. In turn, (d) implies that $\{x \in A : b \leq a \to x\} = \{x \in A : a \leq b \to x\}$. Hence $a \bullet b = \bigwedge\{x \in A : b \leq a \to x\} = \bigwedge\{x \in A : a \leq b \to x\} = b \bullet a$. This shows that $\bullet$ is commutative.

We shall check that $\bullet$ is monotone. Suppose $a \leq b$. (e) then implies that $b \to x \leq a \to x$, for all $x \in A$. It follows that $\{x \in A : c \leq b \to x\} \subseteq \{x \in A : c \leq a \to x\}$ and, consequently,

$$a \bullet c = \bigwedge\{x \in A : c \leq a \to x\} \leq \bigwedge\{x \in A : c \leq b \to x\} = b \bullet c.$$

To conclude the proof, we show that $\bullet$ is totally additive. We argue as in the proof of Theorem 2.4. Let B be a subset of A. By the monotonicity of $\bullet$ we have that $a \bullet \bigvee B \geq a \bullet b$, for all $b \in B$. Consequently, $a \bullet \bigvee B \geq \bigvee\{a \bullet b : b \in B\}$. This means that $a \bullet \bigvee B$ is an upper bound of the set $\{a \bullet b : b \in B\}$. We claim that $a \bullet \bigvee B$ is the least upper bound of this set. Suppose y is an upper bound of $\{a \bullet b : b \in B\}$. As $a \bullet b \leq y$ for all $b \in B$, we get $b \leq a \to y$ for all $b \in B$, by residuation. This implies $\bigvee B \leq a \to y$. Again applying residuation, we get that $a \bullet \bigvee B \leq y$. This proves that $a \bullet \bigvee B = \bigvee\{a \bullet b : b \in B\}$.

The last equality follows from Theorem 2.3 and the just proved facts stating that $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ is a commutator lattice and that $\to$ is the residuation for $\bullet$ in $\boldsymbol{A}$. □

Theorems 2.5 and 3.1 taken together state that the theory of commutator lattices is coextensive with the theory of complete lattices endowed with a binary operation $\to$ satisfying the above conditions (a) (e). To put the above theorems in a more transparent context, we introduce one more definition.

Definition 3.2. Let $B = (B, \wedge, \vee, \to)$ be a complete lattice endowed with a binary operation $\to$. If for all $a, b, c \in A$ and all $B \subseteq A$, $\to$ satisfies conditions (a) (e), then the operation $\to$ is called a *commutator residuation*, and the algebra $(B, \wedge, \vee, \to)$ is called a *lattice with a commutator residuation.* □

The above theorems yield the following

Corollary 3.3.

(α) *Let $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ be a commutator lattice. Let $\to$ be the binary operation defined in A according to the formula* (Res). *Then the algebra $\boldsymbol{A}^{\to} := (A, \wedge, \vee, \to)$ is a lattice with a commutator residuation.*

(β) *Let $\boldsymbol{B} = (B, \wedge, \vee, \to)$ be a lattice with a commutator residuation. Let $\bullet$ be the binary operation in B defined according to the formula* (Com) *above. Then the algebra $B^{\bullet} = (B, \wedge, \vee, \bullet)$ is a commutator lattice.*

(γ) *The algebras $\boldsymbol{A}^{\to\bullet}$ and $\boldsymbol{A}$ coincide.*

(δ) *The algebras $\boldsymbol{B}^{\bullet\to}$ and $\boldsymbol{B}$ coincide.* □

4. Distributivity and commutator lattices.

In light of the above examples it is natural to ask whether a completely distributive lattice possesses a totally additive commutator different from the zero and full commutators.

Given a distributive commutator lattice $\boldsymbol{A} = (A, \wedge, \vee, \bullet)$ and and element $a \in A$, we define the following binary operation $\bullet_a$ in A: for $x, y \in A$.

$(1)_a$ $\quad x \bullet_a y := a \wedge (x \bullet y).$

It follows from the above definition that $x \bullet_a y \leq x \bullet y$, for all $x, y \in A$. Moreover $\bullet_1$ coincides with $\bullet$ and $\bullet_0$ is the zero commutator in A. Here $\mathbf{1}$ is the unit and $\mathbf{0}$ is the zero of the lattice $(A, \wedge, \vee)$.

Proposition 4.1. *For every $a \in A$, the algebra $\boldsymbol{A}_a = (A, \wedge, \vee, \bullet_a)$ is a commutator lattice.*

Proof. We shall check additivity of $\bullet_a$. The other commutator properties are immediate. Suppose $x \in A$ and $B \subseteq A$. Then, by the total distributivity of $(A, \wedge, \vee)$ and the additivity of $\bullet$,

$$x \bullet_a \bigvee B = a \wedge (x \bullet \bigvee B) = a \wedge (\bigvee\{x \bullet b : b \in B\}) =$$
$$\bigvee\{a \wedge (x \bullet b) : b \in B\} = \bigvee\{x \bullet_a b : b \in B\}.$$ □

Proposition 4.2. *Let* $B \subseteq A$ *and let* $a := \bigvee B$. *Then* $x \bullet_a y = \bigvee\{x \bullet_b y : b \in B\}$, *for all* $x, y \in A$.

Proof. $x \bullet_a y = a \wedge (x \bullet y) = \bigvee B \wedge (x \bullet y) =$ (by distributivity) $\bigvee\{b \wedge (x \bullet y) : b \in B\} = \bigvee\{x \bullet_b y : b \in B\}$. □

As the class of all finitely additive commutator lattices is equationally definable, it is closed under the operations of forming direct products, homomorphic images and subalgebras. In particular, this class is closed under the formation of subdirect products.

We let $\mathbf{2}^\wedge$ and $\mathbf{2}^0$ denote the two-element distributive lattice equipped with the full and the zero commutator operation, respectively. $\mathbf{2}^\wedge$ is thus the well-known truth-table for disjunction and conjunction. Closing the set $\{\mathbf{2}^\wedge, \mathbf{2}^0\}$ under the above algebraic operations yields an interesting class of distributive finitely additive commutator lattices. Finite algebras in this class are commutator lattices. E.g. taking the direct product $\mathbf{2}^\wedge \times \mathbf{2}^0$, we get a distributive commutator lattice in which the operation $\bullet$ is different from the full and zero commutators. Further examples of distributive commutator lattices with the above property of the commutator can be easily produced. This observation does not contradict the well-known theorem that in the commutator lattices investigated in universal algebra, viz. congruence lattices, the fact that the lattice is distributive implies that the commutator trivializes, i.e., it coincides with the meet operation. The reason is that the commutator in congruence lattices is defined in a certain canonical way, much depending on the properties of congruences, see [3], or [7]. For a general account of commutator theory see [1]. From the universal algebraic perspective, it is natural to regard the largest commutator $\bullet_\Omega$, defined in an arbitrary complete lattice, as the proper lattice-theoretic counterpart of the commutators studied in congruence lattices because the latter coincide with the former in modular congruence-modular lattices, see [3].

5. Final remarks. Commutators in congruence lattices.

The notion of the commutator known from universal algebra, has a global character in the sense that it is not separately defined for each individual congruence lattice, but uniformly defined for classes of lattices, viz. for the classes of congruence lattices associated with varieties of quasivarieties of algebras. We argue that the notion of a commutator in universal algebra is genuinely *logical* and not lattice-theoretic notion in the sense that it is inherently linked with the equational systems corresponding to varieties (or, more generally, quasivarieties) of algebras. One of a few conceivable definitions of the commutator can be expressed in terms of commutator equations. But all the known definitions of the commutator turn out to be equivalent for *congruence-modular* varieties (or quasi-varieties) of algebras.

Let τ be a signature of algebras. Te_τ stands for the set of of terms of signature τ. Te_τ is defined in the usual recursive way from a countably infinite set of individual variables.

Let

$$\alpha(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) := \alpha(x_1,\dots,x_m,y_1,\dots,y_m,z_1,\dots,z_n,w_1,\dots,w_n,u_1,\dots,u_k)$$

and

$$\beta(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) := \beta(x_1,\dots,x_m,y_1,\dots,y_m,z_1,\dots,z_n,w_1,\dots,w_n,u_1,\dots,u_k)$$

be terms in Te_τ built up with at most the variables $\underline{x} = x_1,\dots,x_m$, $\underline{y} = y_1,\dots,y_m$, $\underline{z} = z_1,\dots,z_n$, $\underline{w} = w_1,\dots,w_n$, and $\underline{u} = u_1,\dots,u_k$. Note that the lengths of the strings $\underline{x}$ and $\underline{y}$ are equal, $|\underline{x}| = |\underline{y}| = m$ and, similarly, $|\underline{z}| = |\underline{w}| = n$.

Definition 5.1. Let $\mathbf{K}$ be a class of algebras.

$$\alpha(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) \approx \beta(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u})$$

is called a *commutator equation for* $\mathbf{K}$ in the variables $\underline{x},\underline{y}$ and $\underline{z},\underline{w}$ (with parameters $\underline{u}$) if the following quasi-equations are valid in $\mathbf{K}$:

$$x_1 \approx y_1 \wedge \dots \wedge x_m \approx y_m \Rightarrow \alpha(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) \approx \beta(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u})$$

$$z_1 \approx w_1 \wedge \dots \wedge z_m \approx w_n \Rightarrow \alpha(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) \approx \beta(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}).$$

Equivalently, $\alpha(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u}) \approx \beta(\underline{x},\underline{y},\underline{z},\underline{w},\underline{u})$ is a commutator equation for $\mathbf{K}$ in the variables $\underline{x},\underline{y}$ and $\underline{z},\underline{w}$ if and only if the equations

$$\alpha(\underline{x},\underline{x},\underline{z},\underline{w},\underline{u}) \approx \beta(\underline{x},\underline{x},\underline{z},\underline{w},\underline{u}) and \alpha(\underline{x},\underline{y},\underline{z},\underline{z},\underline{u}) \approx \beta(\underline{x},\underline{y},\underline{z},\underline{z},\underline{u})$$

are valid in $\mathbf{K}$.

A *quaternary commutator equation* for $\mathbf{K}$ (with parameters) is a commutator equation $\alpha(x,y,z,w,u) \approx \beta(x,y,z,w,u)$ for $\mathbf{K}$ in the variables x,y and z,w. □

Note. It follows from the above definition that

$$\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u})$$

is a commutator equation for $\mathbf{K}$ (in the variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$) if and only if it is a commutator equation (in the variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$) for the variety $\boldsymbol{Va}(\mathbf{K})$ generated by $\mathbf{K}$. Consequently, the classes $\mathbf{K}$ and $\boldsymbol{Va}(\mathbf{K})$ possess the same commutator equations. □

We note the following straightforward consequences of the above definition:

Lemma 5.2. (1). *Let $\alpha := \alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u})$ be any term. Then $\alpha \approx \alpha$ is a commutator equation for $\mathbf{K}$ in $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$.*

(2). *More generally, if $\alpha \approx \beta$ is an identity of $\mathbf{K}$, then it is a commutator equation for $\mathbf{K}$ (in whatever variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$).*

(3). *If $\alpha \approx \beta$ is a commutator equation for $\mathbf{K}$ in $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$, then so is $\beta \approx \alpha$.* □

Lemma 5.3. *Let $\alpha_i(\underline{x}_i, \underline{y}_i, \underline{z}_i, \underline{w}_i, \underline{u}_i,) \approx \beta_i(\underline{x}_i, \underline{y}_i, \underline{z}_i, \underline{w}_i, \underline{u}_i,)$ be a commutator equation for $\mathbf{K}$ in $\underline{x}_i, \underline{y}_i$ and $\underline{z}_i, \underline{w}_i$, for $i = 1, \dots, k$. Let f be a k-ary operation symbol of τ and let $\underline{x}$:= the union of $\underline{x}_1, \dots, \underline{x}_k$, $\underline{y}$:= the union of $y_1, \dots, \underline{y}_k$, $\underline{z}$:= the union of $z_1, \dots, \underline{z}_k$, $\underline{w}$:= the union of $\underline{w}_1, \dots, \underline{w}_k$, $\underline{u}$:= the union of $\underline{u}_1, \dots, \underline{u}_k$.* [The sets $\underline{x}, \underline{y}, \underline{z}, \underline{w}$ and $\underline{u}$ are assumed to be pairwise disjoint.] *Let $\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) := f(\alpha_1, \dots, \alpha_k), \beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) := f(\beta_1, \dots, \beta_k)$. Then*

$$\alpha(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u}) \approx \beta(\underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{u})$$

is a commutator equation for $\mathbf{K}$ in $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$.

The above lemma states that the set of commutator equations for $\mathbf{K}$ has the substitution property.

If Φ is a congruence of an algebra $\boldsymbol{A}$ and $a = \langle a_1, \dots, a_m \rangle, \underline{b} = \langle b_1, \dots, b_m \rangle$ are sequences of elements of A of the same length, we write $\underline{a} \equiv \underline{b}\,(\Phi)$ to indicate that $a_i \equiv b_i\,(\Phi)$ for $i = 1, \dots, m$.

Let $\mathbf{K}$ be a class of algebras. *Coeq* $(\mathbf{K})$ denotes the set of all commutator equations for $\mathbf{K}$ and *Coeq*$^{(4)}(\mathbf{K})$ is the set of all *quaternary* commutator equations for $\mathbf{K}$.

Given an algebra $\boldsymbol{A} \in \boldsymbol{Va}(\mathbf{K})$ and congruences Φ, Ψ on $\boldsymbol{A}$, we define

$$Com_{\mathbf{K}}(\Phi,\Psi) := \{\langle \alpha(\underline{a},\underline{b},\underline{c},\underline{d},\underline{e}), \beta(\underline{a},\underline{b},\underline{c},\underline{d},\underline{e})\rangle : \alpha \approx \beta \in Coeq(\mathbf{K}),$$
$$\underline{a},\underline{b},\underline{c},\underline{d},\underline{e} \text{ are strings of elements of } A \text{such that } \underline{a} \equiv \underline{b}\,(\Phi), \underline{c} \equiv \underline{d}(\Psi)\}.$$

Lemma 5.4. *The relation $Com_{\mathbf{K}}(\Phi,\Psi) \subseteq A\times A$ is reflexive, symmetric on A and has the substitution property.*

Proof. Reflexivity of $Com_{\mathbf{K}}(\Phi,\Psi)$ follows from the fact that $\underline{x} \approx \underline{x}$ is a commutator equation (in whatever variables $\underline{x}, \underline{y}$ and $\underline{z}, \underline{w}$). The remaining properties are consequences of Lemmas 5.2 5.3. □

$Com^{(4)}_{\mathbf{K}}(\Phi,\Psi)$ is a subrelation of $Com_{\mathbf{K}}(\Phi,\Psi)$ determined by quaternary commutator equations for $\mathbf{K}$. Thus

$$Com^{(4)}_{\mathbf{K}}(\Phi,\Psi) := \{\langle \alpha(a,b,c,d,\underline{e}), \beta(a,b,c,d,\underline{e})\rangle : \alpha \approx \beta \in Coeq(4)(\mathbf{K}),$$
$$a,b,c,d \in A, \underline{e} \in A^{<\omega} \text{ and } a \equiv b\,(\Phi), c \equiv d(\Psi)\}.$$

Definition 5.5. Let $\mathbf{K}$ be a class of algebras. Given an algebra $\boldsymbol{A} \in \boldsymbol{Va}(\mathbf{K})$ and congruences Φ, Ψ on $\boldsymbol{A}$, we define

$\Phi \bullet \Psi :=$ the congruence of $\boldsymbol{A}$ generated by the relation $Com_{\mathbf{K}}(\Phi,\Psi)$.

$\Phi \bullet \Psi$ is called the *commutator* of the congruences Φ and Ψ in the algebra $\boldsymbol{A}$. (In the literature this congruence is denoted by $[\Phi,\Psi]$.) □

It is easy to see that for every algebra $\boldsymbol{A} \in \mathbf{V}$, the lattice $Con(\boldsymbol{A})$ of congruences of $\boldsymbol{A}$ endowed with the operation $\bullet$ becomes a proto-commutator lattice.

In this section, by the *commutator* we shall mean the binary operation $\bullet$ on congruence algebras of $\mathbf{V}$ defined as in Definition 5.5.

The above definition of the commutator differs from that one can encounter e.g. in [3]. But for many varieties of algebas, viz. for congruence modular varieties, the above definition is equivalent with the "standard" one and it indeed yields commutator lattices. This fact was established in [1].

We are concerned with the additivity of $\bullet$. In the context of congruence lattices, the additivity property is denoted by (C1). Thus the commutator $\bullet$ is additive if, for any algebra $\boldsymbol{A} \in \mathbf{V}$, for any set $\{\Phi_i : i \in I\}$ of congruences of $\boldsymbol{A}$ and any $\Psi \in Con(\boldsymbol{A})$:

(C1) $\bigvee\{\Phi_i : i \in I\} \bullet_A \Psi = \bigvee\{\Phi_i \bullet_A \Psi : i \in I\}$

in the lattice Con (A).

We need one more property of the commutator $\bullet$:

(C2) If $h : \boldsymbol{A} \to B$ is a surjective homomorphism between $\mathbf{V}$-algebras and $\Phi, \Psi \in Con(\boldsymbol{A})$, then $ker(h) + (\Phi \bullet_A \Psi) = h^{-1}(h\Phi \bullet_A h\Psi)$.

The above conditions are extensively applied in the commutator theory, see [3] or [7].

The following observations concerning the commutator defined in the sense of Definition 5.5 are proved in [1]:

Proposition 5.6. *For any variety* $\mathbf{V}$, (C1) *implies* (C2). □

The following result characterizes additivity of the commutator in terms of quaternary commutator equations. Thus it provides a necessary and sufficient condition for proto-commutator lattices $Con(\boldsymbol{A})$ to be commutator lattices, for all algebras $\boldsymbol{A}$ belonging to a variety.

Theorem 5.7. *Let* $\mathbf{V}$ *be a variety of algebras. The following conditions are equivalent*:

(1) *The commutator satisfies* (C1) *in any algebra* $\boldsymbol{A} \in \mathbf{V}$;

(2) *There exists a set* $\Delta_0(x, y, z, w, \underline{u})$ *of quaternary commutator equations for* $\mathbf{V}$ *in* x, y *and* z, w (*with parameters*) *such that for any algebra* $\boldsymbol{A} \in \mathbf{V}$ *and any sets* $X, Y \subseteq A^2$

$$\Theta^A(X) \bullet_A \Theta^A(Y) = \Theta^A(\cup\{\langle \alpha(a, b, c, d, \underline{e}), \beta(a, b, c, d, \underline{e})\rangle : \alpha \approx \beta \in \Delta_0, \langle a, b\rangle \in X, \langle c, d\rangle \in Y, \underline{e} \in A^{<w}). \qquad \square$$

The above theorem states that $\bullet$ is additive in the algebras of $\mathbf{V}$ if it is generated by a set of quaternary commutator equations, which is uniform for all algebras of $\mathbf{V}$. In fact, the above theorem can be generalized for quasivarieties $\mathbf{Q}$ of algebras; in this case one works rather with the lattice $Con_{\mathbf{Q}}(\boldsymbol{A})$ of $\mathbf{Q}$-congruences than the lattice $Con(\boldsymbol{A})$, for all $A \in \mathbf{Q}$.

If $\mathbf{V}$ is a congruence-modular variety, the above definition of the commutator coincides with the one studied in the algebraic literature. Moreover, in this case the set $\Delta_0(x, y, z, w, \underline{u})$ of quaternary commutator equations for $\mathbf{V}$ is supplied by Day terms for $\mathbf{V}$ (see [2], [5], [6]).

The following theorem states that if the above commutator $\bullet$ is additive, then it is the largest one among all additive commutator operations defined on the lattices $Con(\boldsymbol{A}), \boldsymbol{A} \in \mathbf{V}$, and satisfying (C2):

Theorem 5.8. *Let* $\mathbf{V}$ *be a variety of algebras. Suppose that* $\star$ *is a binary operator defined in the lattices* $Con(\boldsymbol{A})$*, for all* $\boldsymbol{A} \in \mathbf{V}$ *such that for every algebra* $\boldsymbol{A} \in \mathbf{V}, (Con(\boldsymbol{A}), \star_A)$ *is a commutator lattice. If, furthermore,* $\star$ *satisfies* (C2)*, i.e.,*

(C2) *If* $h : \boldsymbol{A} \to B$ *is a surjective homomorphism between* $\mathbf{V}$*-algebras and* $\Phi, \Psi \in Con(\boldsymbol{A})$*, then* $ker(h) + (\Phi \star_A \Psi) = h^{-1}(h\Phi \star_A h\Psi)$*,*

then the commutator $\star$ *is included in* $\bullet$*, i.e.,* $\Phi \star_A \Psi \subseteq \Phi \bullet_A \Psi$*, for all* $\boldsymbol{A} \in \mathbf{V}$ *and all* $\Phi, \Psi \in Con(\boldsymbol{A})$*.* □

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