REPORTS ON MATHEMATICAL LOGIC
43 (2008), 73–83

Wojciech DZIK

UNIFICATION IN SOME SUBSTRUCTURAL LOGICS OF BL-ALGEBRAS AND HOOPS

A b s t r a c t. Abstract. It is shown that substructural logics of k-potent BL-algebras and k-potent hoops have unitary unification (in fact, transparent unifiers) while Basic Fuzzy Logic, BL (the logic of BL-algebras), and ∞-valued Łukasiewicz logic (the logic of MV-algebras) do not have unitary unification. It follows that every k-potent substructural logic containing BL is structurally complete in the restricted sense, but Basic Logic itself is not.

Given an equational theory E, equational unification or E-*unification* is concerned with finding a substitution σ of individual variables that makes two given terms t_1, t_2 equal, or *unified*, modulo the theory E, i.e. $\vdash_E \sigma t_1 = \sigma t_2$. Such a substitution is called a *unifier* for t_1 and t_2; if such a unifier exists then t_1 and t_2 are called *unifiable*. Given two unifiers of τ and σ, we say that σ is *more general then* τ, in symbols $\tau \preceq \sigma$, if τ is an instance of σ, i.e. $\vdash_E \delta(\sigma(x)) = \tau(x)$, for some substitution δ. A unifier σ for t_1 and t_2 is a *most general unifier, a mgu*, if it is more general then any other unifier for t_1 and t_2.

Received 7 May 2007

Unification type of a theory E, equivalently, of a variety $\mathcal{V}_E$, can be unitary, finitary, infinitary or nullary, according to a number of $\preceq$-maximal E-unifiers of „the worst case" of t_1 and t_2, see [1], [8].
Unification is basic to Resolution Theorem and Term Rewriting Systems.
We consider logics of partially ordered algebras such that among the algebraic operations there is a pair $(\cdot, \Rightarrow)$ called an *adjoint* or *residuated pair*, which satisfies the following condition of residuation:

$$c \leq a \Rightarrow b \text{ iff } a \cdot c \leq b \tag{1}$$

in other words, $\Rightarrow$ is a residuum of $\cdot$. $(\cdot, \Rightarrow)$ are represented by the connectives $(\&, \rightarrow)$ of fusion and implication in the corresponding logics. In particular we consider the following classes of algebras which contain the residuated pair $(\cdot, \Rightarrow)$ as a reduct: hoops, basic hoops, Wajsberg (or Łukasiewicz) hoops, BL-algebras, Gödel algebras, and corresponding to them, substructural logics which are related to fuzzy logics, see [7], [10], [6].

All of the above algebras belong to the class of FL_{ew}-algebras, i.e. commutative integral residuated lattices, corresponding to the substructural logics over FL_{ew}, Full Lambek calculus with weakening and exchange but without contraction, introduced by Ono, see [7], [13], [14].

Hoops originated in a manuscript by Büchi and Owens in the 70's. Later, they were considered by Blok and Pigozzi [3], Blok, Ferreirim [2] and others. Hoops capture a common $\{\&, \rightarrow\}$ fragment of all fuzzy logics.

BL-algebras or Basic Logic algebras have been introduced by P.Hájek [10] as an algebraic counterpart of Basic Fuzzy Logic, BL, which is a common generalization of the three main fuzzy logics: Gödel logic, Łukasiewicz logic and Product logic. Gödel algebras, Wajsberg algebras and product algebras are the algebraic counterparts of these logics, respectively. It was proved that the variety of BL-algebras is generated by all algebras $([0,1], \star, \Rightarrow, 0, 1)$, where $\star$ is a continuous t-norm on the real interval $[0,1]$ and $\Rightarrow$ its residuum.

In case of algebras with 1 (*unit*) and with $\Rightarrow, \cdot$ considered here, unification problem $t_1 =^? t_2$ in a class $\mathcal{V}$ of algebras is equivalent to $t =^? 1$, for $t = (t_1 \Rightarrow t_2) \cdot (t_1 \Rightarrow t_2)$ or, in logical terms, to finding a substitution σ such that $\vdash_L \sigma\phi$, where L is a logic determined by $\mathcal{V}$. The subsumption preorder $\preceq$ is related to $\vdash_L$, provability in L.

The set of all formulas F_Δ is built up from variables $x_1, x_2, \ldots$ by means of the connectives from a set Δ, where $\{\&, \rightarrow\} \subseteq \Delta \subseteq \{\&, \rightarrow, \wedge, \vee, \bot\}$, ($\bot$ = constant falsity). Further connectives are defined: $\top$ (constant truth) is $x \rightarrow x$, a negation $\neg\phi$ is $\phi \rightarrow \bot$ and $\phi \equiv \psi$ is $(\phi \rightarrow \psi)\&(\psi \rightarrow \phi)$.
By a logic we mean a set of formulas closed on Modus Ponens and substitution. $\Gamma \vdash_L \phi$ means that a formula ϕ is derivable from a set of formulas Γ and from L by Modus Ponens. $\vdash_L \phi$ means that ϕ is provable in L.
Let $\mathbf{A} = (A, \cdot, \Rightarrow, \wedge, \vee, 1, 0)$ be a residuated lattice. A map $\epsilon : \{x_i\} \rightarrow A$ can be uniquely extended to a homomorphism (denoted by the same letter) $\epsilon : F \rightarrow A$, called a valuation, in such a way that: $\epsilon(A\&B) = \epsilon(A) \cdot \epsilon(B)$, $\epsilon(A \rightarrow B) = \epsilon(A) \Rightarrow \epsilon(B)$, $\epsilon(A \vee B) = \epsilon(A) \vee \epsilon(B)$, $\epsilon(A \wedge B) = \epsilon(A) \wedge \epsilon(B)$, $\epsilon(\bot) = 0$. By restriction of ϵ to $\{\&, \rightarrow\}$ we get a valuation in a hoop $(A, \cdot, \Rightarrow)$.

The *logic of* $\mathbf{A}$ (i.e. a logic determined by $\mathbf{A}$), denoted by $L(\mathbf{A})$, is a set of all formulas valid in $\mathbf{A}$, i.e. $\phi \in L(\mathbf{A})$ iff $\epsilon(\phi) = 1$, for every valuation $\epsilon : F \rightarrow A$; that is $L(\mathbf{A})$ consists of all $\mathbf{A}$-tautologies. The *logic of* a class of algebras $\mathcal{V}$, is a set of all formulas ϕ such that $\phi \in L(\mathbf{A})$, for all $\mathbf{A} \in \mathcal{V}$.

We denote by $\underline{x}$ a finite set of variables $x_1, \ldots, x_n$. We write $\phi(\underline{x})$ to express that a formula ϕ contains variables only from $\underline{x}$. $F(\underline{x})$ denotes the set of all formulas of the form $\phi(\underline{x})$. A substitution $\sigma : \underline{x} \rightarrow F$ is a *unifier* for $\phi(\underline{x})$ in L, if $\vdash_L \sigma\phi$; then ϕ is called *unifiable*. The most general unifier, a mgu, for ϕ in L, is a unifier σ for ϕ such that $\sigma \preceq \tau$ for any unifier τ for ϕ. A unifier of the type $\sigma : \underline{x} \rightarrow \{\bot, \top\}$ is called *ground*.
Unification in L depends on the collection of connectives used in L. Hence we write ,,a logic L in Δ", where $\{\&, \rightarrow\} \subseteq \Delta \subseteq \{\&, \rightarrow, \wedge, \vee, \neg\}$, to mean that $L \subseteq F_\Delta$, i.e. all formulas in L contain only connectives from Δ.

We begin with logics of hoops in $\{\&, \rightarrow\}$ and expand to logics of residuated lattices in $\{\&, \wedge, \vee, \rightarrow, (\neg)\}$, in two parts: I. without negation and II. with negation, or constant $\bot$.

Hoops. A **hoop** is an algebra $\mathbf{A} = (A, \cdot, \Rightarrow, 1)$ of type $(2, 2, 0)$ such that $(A, \cdot, 1)$ is a commutative *monoid* with the unit 1, (i.e. satisfying: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x \cdot y = y \cdot x$, $x \cdot 1 = 1$) which satisfies the axioms:

(H1) $x \Rightarrow x = 1$

(H2) $x \Rightarrow (y \Rightarrow z) = (x \cdot y) \Rightarrow z$

(H3) $(x \Rightarrow y) \cdot x = (y \Rightarrow x) \cdot y$.

A relation $\leq$ on any hoop $(A, \cdot, \Rightarrow, 1)$ defined as follows: $x \leq y$ iff $x \Rightarrow y = 1$ is a partial order; then $(A, \leq)$ is a $\wedge$-semilattice. In fact, for any $x, y \in A$:

$$x \wedge y = x \cdot (x \Rightarrow y), \tag{2}$$

which is called *divisibility*. Hence the class of hoops form a variety.
A hoop $(A, \cdot, \Rightarrow, 1)$ is *basic* if it is *prelinear*, i.e. for $x, y, z \in A$:

$$(x \Rightarrow y) \Rightarrow z \ \leq \ ((y \Rightarrow x) \Rightarrow z) \Rightarrow z. \tag{3}$$

In basic hoops we have $x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge ((y \Rightarrow x) \Rightarrow x)$, i.e. the join is definable. Hence any basic hoop $(A, \cdot, \Rightarrow, 1)$ is definitionally equivalent to a commutative integral divisible residuated lattice, with the greatest element 1, $(A, \wedge, \vee, \cdot, \Rightarrow, 1)$, cf.[12]. Basic hoops are also called *generalized BL-algebras* since they are subalgebras of 0-free reducts of BL-algebras cf [6].
A hoop $(A, \cdot, \Rightarrow, 1)$ is a *Wajsberg hoop* or a *Łukasiewicz hoop* if, for $x, y \in A$: $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$. Every Wajsberg hoops is basic.
A *bounded hoop* is an algebra $(A, \cdot, \Rightarrow, 0, 1)$ such that $(A, \cdot, \Rightarrow, 1)$ is a hoop and $0 \leq x$, for all $x \in A$. In such hoops negation is defined by $\neg x = x \Rightarrow 0$. Bounded Wajsberg hoops are equivalent to Wajsberg algebras and to MV-algebras, which are algebraic models of Łukasiewicz many-valued logics.

An $\mathbf{FL_{ew}}$**-algebra** is an algebra $(A, \cdot, \Rightarrow, \wedge, \vee, 0, 1)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded (i.e. with 0 and 1) lattice, $(A, \cdot, \Rightarrow, 1)$ is a commutative monoid and the condition of residuation (1) holds.

A **BL-algebra** (a Basic Logic algebra) is a $\mathrm{FL_{ew}}$-algebra satisfying divisibility (2) and

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1 \quad (\textit{prelinearity}). \tag{4}$$

Note: (4) is equivalent to (3), and (2) implies distributivity in $(A, \wedge, \vee)$.

A hoop or a BL-algebra A is *k-potent*, if, for every $x \in A$,

$$(E_k) \qquad\qquad x^{k+1} = x^k. \tag{5}$$

An element $x \in A$, is *idempotent*, if $x^2 = x$; similarly, a formula ϕ is *idempotent* in L, if $\vdash_L \phi^2 \equiv \phi$. A logic L is *k-potent* if $\vdash_L \phi^k \rightarrow \phi^{k+1}$ for any ϕ. If L is *k-potent*, then $\vdash_L (\phi^k)^2 \equiv \phi^k$, for any ϕ. By Blok and Pigozzi

[3], a variety of hoops (BL-algebras) has Equationally Definable Principal Congruences, EDPC, iff it is k-potent, for some k.

Transparent unifiers were defined by Wroński [19] (see also [18]), for quasivarieties of equivalence algebras as follows. Let $\mathbf{K}$ be a class of algebras of the same type, F be the set of all terms ($F(\underline{x})$ terms built in $\underline{x}$ respectively) of $\mathbf{K}$, $p(\underline{x}), q(\underline{x})$ terms with variables in $\underline{x} = x_1, \ldots, x_k$. A unifier σ for p, q is *transparent* if $\models_{\mathbf{K}} (p = q \Rightarrow \sigma(x) = x)$, for every x. In view of the completeness theorem: $\models_{\mathbf{K}} \; = \; \vdash_L$ we arrive at the following definition.
Let $\phi(\underline{x})$ be unifiable in L. A substitution $\sigma : \underline{x} \to F$ is a *transparent* unifier for ϕ in L if σ is a unifier for ϕ in L, $\vdash_L \sigma\phi$, and, for $x \in \underline{x}$

$$\phi \vdash_L (\sigma(x) \equiv x). \tag{6}$$

The formula ϕ in (6) should be in the form that allows to generate a congruence (e.g. idempotent), see [5], Sect. 2.4. We say that a *logic L has transparent unifiers* if every unifiable formula in L has a transparent unifier. Transparent unifiers have an advantage (here): they are preserved under extensions. Unitary unification is not preserved ,,upwords": the logic KC of weak excluded middle has unitary unification [9] but some its extensions have nullary unification.

1. Substructural logics without negation ¬.

In the following theorem (i) is due to Blok and Pigozzi [3] and (ii) was proved in [17] with the assistance of the automated reasoning program Otter.

Theorem 1. *Let $(A, \cdot, \Rightarrow, 1)$ be a hoop and e an idempotent element of A. Then the following equations hold for $x, y \in A$*

$$\begin{array}{lrcl} (i) & e \Rightarrow (x \Rightarrow y) & = & (e \Rightarrow x) \to (e \Rightarrow y) \\ (ii) & e \Rightarrow (x \cdot y) & = & (e \Rightarrow x) \cdot (e \Rightarrow y) \end{array}$$

Hence the map $f(x) = e \Rightarrow x$, for $x \in A$, is an endomorphism of $(A, \cdot, \Rightarrow, 1)$. Moreover, f is a retraction, $f \circ f = f$.

Corollary 2. *For any logic L of hoops, and an idempotent formula $\phi \in F(\underline{x})$, $\sigma(x) = \phi \rightarrow x$, for $x \in \underline{x}$, is a transparent unifier (hence a mgu).*[1]

Note that σ is ,,weakly transparent", i.e., for any unifier τ for ϕ, we have $\vdash_L \tau(\sigma(x)) \equiv \tau(x)$, for $x \in \underline{x}$; hence σ is a mgu for ϕ.

Corollary 3. *Every logic in $\{\&, \rightarrow\}$ or $\{\&, \wedge, \rightarrow\}$ determined by any class of k-potent hoops, $k \geq 1$, has transparent unifiers; hence it has unitary unification.*

Corollary 4. *Every idempotent formula of any logic in $\{\&, \wedge, \vee, \Rightarrow\}$ of basic hoops has a transparent unifier.*

Corollary 5. *Any logic in $\{\&, \wedge, \vee, \Rightarrow\}$ of k-potent basic hoops (in particular, of k-potent generalized BL-algebra [12]) has unitary unification. The transparent unifier for ϕ has the form $\sigma(x) = \phi^k \rightarrow x$ for $x \in \underline{x}$.*

Remark. If the assumption of prelinearity is dropped then unitary unification is lost; example: the $\{\wedge, \vee, \rightarrow\}$-reduct of intuitionistic logic.
A (Hilber style) rule $\phi_1, \ldots, \phi_n/\psi$ is *admissible in L* if, for every substitution τ, $\vdash_L \tau\phi_1, \cdots \vdash_L \tau\phi_n$ implies $\vdash_L \tau\psi$; it is *derivable in L*, if $\phi_1, \ldots, \phi_n \vdash_L \psi$. L is *structurally complete* if every structural (i.e. closed under substitution) and admissible rule in L is derivable in L, see [15].

Corollary 6. *Any logic L in $\{\cdot, \wedge, \rightarrow\}$ (in $\{\cdot, \wedge, \vee, \rightarrow\}$) of k-potent (basic) hoops is hereditarily Structurally Complete; in terms of algebra: every subquasivariety of k-potent (basic) hoops is a variety, i.e. it is deductive, [11].*

2. Substructural Logics with negation $\neg$

. In this part we consider logics with negation $\neg$ (or $\bot$) and the algebras with the least element 0. Recall that $\neg x = x \Rightarrow 0$ and $\neg\phi \equiv \phi \rightarrow \bot$. Allowing negation often changes unification type to ,,worse".

Example: Consider reducts of intuitionistic logic in $\{\wedge, \vee, \rightarrow, \neg\}$. Unification is: unitary in $\{\rightarrow\}$-reduct (Prucnal 1972); **not** unitary in $\{\rightarrow, \neg\}$-

[1] Prucnal [16] used such a substitution in implicational fragment of intuitionistic logic.

reduct (Wroński [18]), unitary in $\{\to, \wedge\}$ and $\{\to, \wedge, \neg\}$-reducts (Wroński [18]); note that it is finitary (not unitary) in full $\{\to, \wedge, \vee, \neg\}$, (Ghilardi [9]).

In any BL-algebra the set $\{0, 1\}$ is closed under $\cdot, \Rightarrow, \wedge, \vee, \neg$, hence every formula unifiable in BL has a ground unifier.

Lemma 7. *Let* $\mathbf{A}$ *be a BL-algebra and* e *an idempotent element in* $\mathbf{A}$. *Then for every* $a, b \in A$, *the following equations hold*

(i) $e \cdot a = e \wedge a$,

(ii) $\neg\neg e = (\neg\neg e)^2$,

(iii) $\neg\neg e \cdot [(e \Rightarrow a) \Rightarrow (e \Rightarrow b)] = [(e \Rightarrow a) \Rightarrow \neg\neg e \cdot (e \Rightarrow b)]$,

(iv) $[(e \Rightarrow a) \Rightarrow (e \Rightarrow b)] = [\neg\neg e \cdot (e \Rightarrow a) \Rightarrow \neg\neg e \cdot (e \Rightarrow b)] =$
$[\neg\neg e \cdot (e \Rightarrow a) \Rightarrow (e \Rightarrow b)]$.

The proof is omitted; for useful equations see [4],[6],[10].

Theorem 8. *Let* $(A, \cdot, \Rightarrow, \wedge, \vee, 0, 1)$ *be a BL-algebra,* e *an idempotent element of* A *and* $\tau : A \to \{0, 1\}$ *a homomorphism,* $\tau(e) = 1$. *Then the map*

$$f(x) = (e \Rightarrow x) \cdot (\neg e \Rightarrow \tau(x)), \ \textit{for } x \in A, \tag{7}$$

is an endomorphism of $(A, \cdot, \Rightarrow, 1)$. *Moreover,* f *is a retraction,* $f \circ f = f$.

Proof. Let e be an idempotent element in A and $\tau : A \to \{0, 1\}$ a homomorphism such that $\tau(e) = 1$. By induction on the length of a term t in variables $\underline{x}$ we show that:

$$(\star) \qquad f(t) = \begin{cases} e \Rightarrow t, & \text{if} \quad \tau(t) = 1, \\ (e \Rightarrow t) \cdot \neg\neg e, & \text{if} \quad \tau(t) = 0. \end{cases}$$

Assume that $(\star)$ holds for t_1 and t_2. Let $t = t_1 \Rightarrow t_2$. We have the cases: 1) $\tau t_1 = 1$, $\tau t_2 = 0$, i.e. $\tau t = 0$, then, by Theorem 1 and by Lemma 7, (iii), $(\star)$ holds ; 2) $\tau t_1 = 0$, $\tau t_2 = 0$ and 3) $\tau t_1 = 0$, $\tau t_2 = 1$ follows similarly by Lemma 7, (iv); and 4) $\tau t_1 = 1$, $\tau t_2 = 1$, by Theorem 1, (i).
For $t = t_1 \cdot t_2$, $(\star)$ holds, by Theorem 1, (ii) and Lemma 7. For $t = 0$ we have $f(0) = \neg\neg t \cdot \neg t = 0$. The other operations are definable. □

Theorem 9. *For every logic L containing Basic Logic* BL*, if $\phi \in F(\underline{x})$ is a formula idempotent and unifiable in L, with a ground unifier τ, than it has the transparent of the form:*

$$\sigma_\phi(x) = (\phi \to x)\&(\neg\phi \to \tau(x)), \quad \textit{for } x \in \underline{x}.$$

In particular, for every k-potent logic L containing Basic Logic BL*, if $\phi \in F(\underline{x})$ is unifiable in L, than it has the transparent unifier of the form:*

$$\sigma_\phi(x) = (\phi^k \to x)\&(\neg\phi^k \to \tau(x)), \quad \textit{for } x \in \underline{x},$$

where τ is a ground unifier for ϕ. Hence unification in L is unitary.

Proof. Let ϕ be a unifiable and idempotent formula in L and τ a ground unifier for ϕ. By induction on the length of $\psi \in F(\underline{x})$ we show that:

$$(\star) \qquad \sigma_\phi(\psi) \equiv \begin{cases} \phi \to \psi, & \text{if} \quad \tau(\psi) \equiv \top, \\ (\phi \to \psi)\&\neg\neg\phi, & \text{if} \quad \tau(\psi) \equiv \bot\,. \end{cases}$$

In particular, $\sigma_\phi(\phi) \equiv \phi \to \phi$, σ_ϕ is a unifier for ϕ. Moreover, since ϕ is idempotent, σ_ϕ is transparent, i.e., $\vdash_L \phi \to [(\sigma_\phi(x) \to x)\&(x \to \sigma_\phi(x))]$. □

Remark. It follows that unification is unitary in Gödel logic (Wroński's result, cf. [18]) and in finite-valued Łukasiewicz logics.
If k-potency, or EDPC, is dropped, then unitary unification is lost.

Lemma 10. *In Łukasiewicz logic $Ł_\infty$ the following holds:*

$$(Dis) \qquad \textit{If } \vdash_{Ł_\infty} \phi \vee \neg\phi, \textit{ then } \vdash_{Ł_\infty} \phi \textit{ or } \vdash_{Ł_\infty} \neg\phi.$$

Proof. Recall that Łukasiewicz logic $Ł_\infty$ is characterized by the Wajsberg algebra $\mathfrak{W}_{[0,1]} = ([0,1], \ \cdot, \to, 0, 1)$, where: $\neg x = 1 - x$ and $x \to y = min\{1, 1 - x + y\}$ are continuous real functions; their compositions (corresponding to formulas) are continuous functions $[0,1]^k \to [0,1]$. Now observe that, by the Darboux property of continuous functions, for every formula ϕ:
$\phi \vee \neg\phi$ is valid in $\mathfrak{W}_{[0,1]}$ iff ϕ is valid in $\mathfrak{W}_{[0,1]}$ or $\neg\phi$ is valid in $\mathfrak{W}_{[0,1]}$. □

Corollary 11. *Unification in Łukasiewicz logic $Ł_\infty$ is not unitary.*

Proof. The formula $x \vee \neg x$ has two unifiers $\sigma_0(x) = \bot$ and $\sigma_1(x) = \top$ in $Ł_\infty$. Assume to the contrary, that $Ł_\infty$ has unitary unification; let σ be a mgu for $x \vee \neg x$. i.e. $\vdash_{Ł_\infty} \sigma(x) \vee \neg\sigma(x)$. By the property (Dis) we have $\vdash_{Ł_\infty} \sigma(x)$ or $\vdash_{Ł_\infty} \neg\sigma(x)$, i.e. σ is equivalent either to σ_1 or to σ_0, a contradiction. □

Conjecture. Unification in Łukasiewicz logic $Ł_\infty$ is infinitary or nullary.

Theorem 12 (Glivenko property)**.** *(Cignoli, Torrens [4])*

(i) $\vdash_{\mathrm{BL}} \neg\neg\phi$ *iff* $\vdash_{Ł_\infty} \phi$

(ii) $\vdash_{\mathrm{BL}} \neg\phi$ *iff* $\vdash_{Ł_\infty} \neg\phi$

Lemma 13. *In Basic Logic* BL *the following holds:*

(Dis) *If* $\vdash_{\mathrm{BL}} \neg\phi \vee \neg\neg\phi$, *then* $\vdash_{\mathrm{BL}} \neg\phi$ *or* $\vdash_{\mathrm{BL}} \neg\neg\phi$.

Corollary 14. *If* $\mathrm{BL} \subseteq L \subseteq Ł_\infty$, *then unification in* L *is not unitary.*

Recall that every structural rule, in Hilbert style, can be represented as a join of rules with basic sequents, i.e. such that there is a sequent $\phi_1(\underline{x}), \ldots, \phi_n(\underline{x})/\psi(\underline{x})$ such that any application of the rule is an instance of $\phi_1(\underline{x}), \ldots, \phi_n(\underline{x})/\psi(\underline{x})$. A rule with the basic sequent $\phi_1(\underline{x}), \ldots, \phi_n(\underline{x})/\psi(\underline{x})$ has *unifiable premises*, if $\phi_1(\underline{x})\& \ldots \&\phi_n(\underline{x})$ is unifiable; e.g. for Modus Ponens: $x_1 \to x_2, x_1/x_2$, the formula $(x_1 \to x_2)\&x_1$ is unifiable. Applying the transparent unifier for $\phi_1(\underline{x})\& \ldots \&\phi_n(\underline{x})$ we get:

Corollary 15. *Every k-potent logic L containing Basic Logic* BL *is structurally complete in the following restricted sense: every structural and admissible rule with unifiable premises in L is derivable in L.*

Remarks. In Łukasiewicz logics and in Basic Logic not all consistent formulas are unifiable (the converse is true). Example: $\psi(x_1) \to \neg\psi(x_1)$, where $\psi(x_1)$ is any formula such that for any valuation v in a linear Wajsberg algebra we have $v(\psi(x_1)) = 1$, for $v(x_1) \in \{0, 1\}$ and $v(\psi(x_1)) = 0$, otherwise.

Corollary 16. *The logics* $Ł_\infty$ *and* BL *are not structurally complete, even in the restricted sense.*

Proof. Consider the following rule with the basic sequent:

$$\neg\neg[(x \vee \neg x)^2] \vee \neg[(x \vee \neg x)^2] \ \ / \ \ \neg\neg[(x \vee \neg x)^2].$$

The premise is unifiable. By the property (Dis) the rule is admissible, but it is not derivable in $Ł_\infty$ (and in BL). □

References

[1] F. Baader, W. Snyder, *Unification Theory.* In: A. Robinson, A. Voronkov, (eds.) Handbook of Automated Reasoning, Elsevier Science Publisher, 2001, Ch. 8, pp. 445–533.

[2] W. Blok, I. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** (2000), pp. 233–257.

[3] W. Blok, D. Pigozzi, *On the structure of varieties with equationally definable principal congruences III*, Algebra Universalis **32** (1994), pp. 545–608.

[4] R. Cignoli, A. Torrens, *Hajek basic fuzzy logic and Łukasiewicz infinite-valued logics*, Arch. Math. Logic, **42**, (2003), pp. 361–370.

[5] W. Dzik, Unification Types in Logic, Wydawnictwo Uniwersytetu Śląskiego, Katowice 2007.

[6] F. Esteva, L. Godo, P. Hajek, F. Montagna, *Hoops and Fuzzy Logic*, J. of Logic and Computation **13**, N.4 (2003), pp. 531–555.

[7] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices. An Algebraic Glimpse At Substructural Logics Elsevier, Netherlands 2007.

[8] S. Ghilardi, *Unification through Projectivity*, J. of Symbolic Computation **7** (1997), pp. 733–752.

[9] S. Ghilardi, *Unification in Intuitionistic Logic,* J. of Symbolic Logic **64** (1999), pp. 859–880.

[10] P. Hájek, Metamathematics of fuzzy logic, Kluwer Academic Press, Dordrecht 1998.

[11] L. Hogben, C. Bergman, *Deductive varieties of modules and universal algebras*, Trans. of the Amer. Math. Soc. **289** (1985), 303–320.

[12] P. Jipsen, C. Tsinakis *A survey of residuated lattices*, Ordered Algebraic Structures (J.Martinez, ed.), Kluwer Academic Press, Dordrecht, 1998.

[13] H. Ono, *Logics without contraction rule and residuated lattices I.* To appear in the book for the Festschrift of Prof. R. K. Meyer.

[14] H. Ono, *Substructural logics and residuated lattices an introduction.* In F.V. Hendricks, J. Malinowski (eds.): Trends in Logic: 50 Years of Studia Logica, 2003, pp. 177–212.

[15] Pogorzelski, W.A., *Structural Completeness of the Propositional Calculus*, Bulletin de l'Academie Polonaise des Sciences, Ser. Sci. Math., Astron., Phys., **19** (1971), pp. 349–351.

[16] Prucnal, T., *Structural Completeness of some Pure Implicational Propositional Calculi*, Studia Logica, **30** (1972), pp. 45–52.

[17] R. Veroff, M. Spinks, *On a homomorphism property of hoops* Bull. of the Section of Logic **33** (2004), pp. 135–142.

[18] A. Wroński, *Transparent Unification Problem*, Reports on Mathematical Logic **29** (1995), pp. 105–107.

[19] A. Wroński, *Unitary Unification for Equivalential Algebras and other structures related to Logic*, Volume of Abstracts, 11th Internat. Congress of Logic, Methodology, Phil. of Sci., Cracow, Poland, 1999, pp. 63–76.

Institute of Mathematics,
Silesian University, Katowice, Poland

dzikw@silesia.top.pl