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FINITE EMBEDDABILITY PROPERTY FOR RESIDUATED GROUPOIDS

A b s t r a c t. We prove Finite Embeddability Property (FEP) for the class of residuated groupoids. The problem whether the class has FEP was left open by Blok and van Alten [3]. We combine proof theoretic and algebraic methods.

1. Introduction and preliminaries

A class K of algebras has Finite Embeddability Property (FEP), if every finite partial subalgebra of a member of K can be embedded into a finite member of K. A class has Strong Finite Model Property (SFMP), if every non-valid Horn clause can be refuted in a finite member of the class. Finite Model Property (FMP) is defined analogously, but in terms of atomic formulas instead of Horn clauses. For quasi-varieties, FEP and SFMP are equivalent [2]. All the properties have an important computational meaning. Namely, if a class of algebras is finitely axiomatizable

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and has FMP/SFMP/FEP then its equational/Horn/universal theory is decidable.

FEP for integral residuated lattices, pocrims and BCK-algebras was proved by Blok and van Alten in [2]. The proof is purely algebraic and uses ideas of Okada and Terui from [9] and Dickson's lemma. The argument was refined in [3] using Higman's finite basis theorem for divisibility orders. It was applied to non-commutative and non-associative integral residuated groupoids. The assumption of integrality is essential in the proof.

Proof-theoretic constructions were used to prove FMP for some classes of residuated algebras in [4], [7] and [9]. The key tool in [7] and [9] is the cut-elimination theorem for Gentzen style axiomatization. In general, SFMP cannot be established using cut-elimination. The non-associative systems possess another important proof-theoretic property - an interpolation lemma, in a version where interpolants are of very restricted form. In [5], a lemma of this kind with a set of interpolants of polynomial size was used to prove the P-TIME decidability of the Horn theory of residuated groupoids. Using a construction of an intuitionistic phase space [9], in the spirit of [1], and the interpolation lemma for Gentzen style axiomatization, we show SFMP for the class of residuated groupoids with meet. Thus we establish FEP for this class, and consequently for the class of residuated groupoids.

The paper is organised as follows. In this section we recall some notions and definitions for residuated structures, Non-associative Lambek Calculus, FEP and SFMP. In section 2 we describe a construction of an intuitionistic phase space used to show SFMP. It is used to construct a finite model and also to prove the subformula property. The latter and the interpolation lemma are considered in section 3. In the fourth section we prove the finiteness of the constructed intuitionistic phase space model. In the last section we sketch the proof of FEP for distributive lattice-ordered residuated groupoids.

1.1 Residuated structures

A *residuated groupoid* $\mathcal{G} = (G, \cdot, \backslash, /, \leq)$ is a structure such that $(G, \cdot)$ is a groupoid, $(G, \leq)$ is a poset and $\backslash, /$ are binary operations on G satisfying the following residuation law:

$$\text{(RES)} \qquad x \cdot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x\backslash z \quad \text{for all } x, y, z \in G.$$

From (RES) the monotonicity of $\cdot$ follows. Therefore, each residuated groupoid is a partially ordered groupoid. A residuated groupoid $\mathcal{G} = (G, \cdot, \backslash, /, \leq)$ is *meet semilattice-ordered* if the poset $(G, \leq)$ is a meet-semilattice. As usual, $\wedge$ denotes the lattice meet operation.

Powerset frames are standard structures especially useful in linguistic applications. We recall the construction. Let $(G, \cdot)$ be a groupoid. If one defines operations in the powerset $P(G)$ as follows:

$$X \odot Y = \{x \cdot y \in G : x \in X, y \in Y\},$$

$$X\backslash Y = \{z \in G : X \odot \{z\} \subseteq Y\}, \qquad Y/X = \{z \in G : \{z\} \odot X \subseteq Y\},$$

for $X, Y \subseteq G$, then the structure $\mathcal{P}(G) = (P(G), \odot, \backslash, /, \subseteq)$ is a residuated groupoid, being a complete and distributive lattice. For non-distributive case one needs a more general class of structures constructed usually with the help of a *closure operator* [1]. By a closure operator over a residuated groupoid $(G, \cdot, \backslash, /, \leq)$ we mean a function $C : G \to G$ fulfilling the following conditions:

(C1) $x \leq Cx$,

(C2) $CCx \leq Cx$,

(C3) $x \leq y \Rightarrow Cx \leq Cy$,

(C4) $Cx \cdot Cy \leq C(x \cdot y)$.

The operator is also called a *nucleus* operator. An element $x \in G$ is called *C-closed* iff $x = Cx$. Let $C(G)$ be a set of all C-closed elements, $\leq_C$ be the order $\leq$ restricted to the set $C(G)$ and let $x \cdot_c y = C(x \cdot y)$. The structure $\mathcal{C}(G) = (C(G), \cdot_C, \backslash, /, \leq_C)$ is a residuated groupoid. Moreover, if $\leq$ is a (meet-semi)lattice order, then so is $\leq_C$. $sup_{\leq_C}(x, y) = C(sup_{\leq}(x, y))$, whenever $sup_{\leq}(x, y)$ exists in G.

Intuitionistic phase spaces (see [9] for a slightly different definition) are structures created with help of a combination of those two constructions. More precisely, from a given groupoid $\mathcal{G} = (G, \cdot)$ and a closure operator C over the powerset residuated groupoid $\mathcal{P}(\mathcal{G})$ one constructs a residuated groupoid $\mathcal{PS}(G, C) = (C(P(G)), \odot_C, \backslash, /, \subseteq_C)$.

1.2 Non-associative Lambek Calculus

We briefly describe the syntax of Non-associative Lambek Calculus introduced in [8]. *Formulas* of the calculus are formed out of propositional variables $p, q, r, \ldots$ by means of binary operation symbols $\otimes, \backslash, /$. Let $A, B, C, \ldots$ denote formulas. *Formula structures*, denoted by $X, Y, Z, \ldots$, are recursively defined as follows

i) all formulas are formula structures,

ii) if X, Y are formula structures, then $(X \circ Y)$ is a formula structure.

A formula structure X with a distinguished substructure Y will be denoted by $X[Y]$. A *sequent* is an expression of the form $X \to A$. It is called a *simple sequent* if X is a formula.

The axioms and rules of the system are the following:

$$\text{(Id)} \quad A \to A$$

$$\text{(L}\otimes\text{)} \quad \frac{X[A \circ B] \to C}{X[A \otimes B] \to C} \qquad \frac{X \to A; \quad Y \to B}{X \circ Y \to A \otimes B} \quad \text{(R}\otimes\text{)}$$

$$\text{(L}\backslash\text{)} \quad \frac{X[B] \to C; \quad Y \to A}{X[Y \circ (A\backslash B)] \to C} \qquad \frac{A \circ X \to B}{X \to A\backslash B} \quad \text{(R}\backslash\text{)}$$

$$\text{(L/)} \quad \frac{X[B] \to C; \quad Y \to A}{X[(B/A) \circ Y] \to C} \qquad \frac{X \circ A \to B}{X \to B/A} \quad \text{(R/)}$$

$$\text{(Cut)} \quad \frac{X[A] \to C; \quad Y \to A}{X[Y] \to C}$$

The system will be denoted by NL. We also consider a system with conjunction $\wedge$ (and the notion of formula appropriately extended). Additional rules for $\wedge$ are the following:

$$\text{(L1}\wedge\text{)} \quad \frac{X[A] \to C}{X[A \wedge B] \to C} \qquad \frac{X[B] \to C}{X[A \wedge B] \to C} \quad \text{(L2}\wedge\text{)}$$

$$\text{(R}\wedge\text{)} \quad \frac{X \to A; \quad X \to B}{X \to A \wedge B}$$

The resulting system is denoted by $\text{NL}_{\wedge}$.

We write $\vdash_{NL} X \to A$ if $X \to A$ is provable in NL, and similarly for other sequent systems. We also consider systems with an additional set Φ

of assumptions, presented as simple sequents. The resulting systems are denoted NL[Φ], $\mathrm{NL}_\wedge$[Φ].

By a model of NL ($\mathrm{NL}_\wedge$) we mean a pair $(\mathcal{G}, \mu)$, where $\mathcal{G}$ is a residuated groupoid (meet semilattice-ordered residuated groupoid) and μ is a valuation of the propositional variables in $\mathcal{G}$, recursively extended to the formulas and formulae structures by means of the following equations

$$\mu(A \otimes B) = \mu(A) \cdot \mu(B), \quad \mu(A \backslash B) = \mu(A) \backslash \mu(B), \quad \mu(A/B) = \mu(A)/\mu(B),$$

$$\mu(A \wedge B) = \mu(A) \wedge \mu(B) \quad \text{(for NL}_\wedge \text{ only)},$$

$$\mu(X \circ Y) = \mu(X) \cdot \mu(Y).$$

A sequent $X \to A$ is *true in a model* $(\mathcal{G}, \mu)$ (write $(\mathcal{G}, \mu) \models X \to A$) iff $\mu(X) \leq \mu(A)$ in $\mathcal{G}$. If Φ is a finite set of simple sequents, then the clause $\Phi \Rightarrow X \to D$ is *true* in $(\mathcal{G}, \mu)$ iff $(\mathcal{G}, \mu) \models \Phi$ implies $(\mathcal{G}, \mu) \models X \to D$. The definitions extend to the notions of truth in a structure and a class of structures in the usual manner. The following (strong) completeness theorem holds for all considered systems and respective classes of residuated structures [6].

Proposition 1. *i)* $\vdash_{NL[\Phi]} X \to A$ *iff* $(\mathcal{G}, \mu) \models \Phi \Rightarrow X \to A$ *for each model* $(\mathcal{G}, \mu)$ *of NL.*

ii) $\vdash_{NL_\wedge[\Phi]} X \to A$ *iff* $(\mathcal{G}, \mu) \models \Phi \Rightarrow X \to A$ *for each model* $(\mathcal{G}, \mu)$ *of* $NL_\wedge$.

1.3 Finite embeddability property

By a partial subalgebra of an algebra we mean a nonempty subset of its carrier with operations from signature of the algebra defined partially. The order, if present, is also restricted to the subset. More formally $\mathcal{B}$ is a partial subalgebra of an algebra $\mathcal{A}$ iff $|\mathcal{B}| \subseteq |\mathcal{A}|$ and, for each operation symbol f of arity n from the signature of $\mathcal{A}$

$$f^{\mathcal{B}}(b_1, \ldots, b_n) = \begin{cases} f^{\mathcal{A}}(b_1, \ldots, b_n) & \text{if } f^{\mathcal{A}}(b_1, \ldots, b_n) \in |\mathcal{B}| \\ undefined & \text{otherwise} \end{cases},$$

for all $b_1, \ldots, b_n \in |\mathcal{B}|$. If $\mathcal{A}$ is ordered, then $\leq^{\mathcal{B}} = \leq^{\mathcal{A}} |_{|\mathcal{B}|}$. By an embedding of a partial algebra $\mathcal{B}$ into an algebra $\mathcal{C}$ we mean an injection $e : |\mathcal{B}| \to |\mathcal{C}|$ such that, if $b_1, \ldots, b_n, f^{\mathcal{B}}(b_1, \ldots, b_n) \in |\mathcal{B}|$, then

$$e(f^{\mathcal{B}}(b_1, \ldots, b_n)) = f^{\mathcal{C}}(e(b_1), \ldots e(b_n)).$$

If $\mathcal{C}$ is ordered then e is also required to satisfy

$$x \leq^{\mathcal{B}} y \Leftrightarrow e(x) \leq^{\mathcal{C}} e(y).$$

A class $\mathcal{K}$ of algebras has Finite Embeddability Property (FEP) if for any algebra $\mathcal{A} \in \mathcal{K}$ and any finite partial subalgebra $\mathcal{B}$ of $\mathcal{A}$ there exists a finite algebra $\mathcal{C} \in \mathcal{K}$ in which $\mathcal{B}$ can be embedded.

We consider the first-order language of algebras from $\mathcal{K}$. A class $\mathcal{K}$ of algebras has Finite Model Property (FMP) if every atomic formula that fails to hold in $\mathcal{K}$ can be falsified in a finite member of $\mathcal{K}$. A class $\mathcal{K}$ of algebras has Strong Finite Model Property (SFMP) if every Horn clause that fails to hold in $\mathcal{K}$ can be falsified in a finite member of $\mathcal{K}$

In case of ordered algebras, without loss of generality, we can restrict only to inequations for FMP and to clauses built from inequations only for SFMP. The following proposition generalizes one from [2], formulated in terms of quasi-equations, not arbitary Horn clauses.

Proposition 2. *Let $\mathcal{K}$ be a class of ordered algebras of finite type, closed under finite products. If $\mathcal{K}$ has SFMP, then $\mathcal{K}$ has FEP.*

Proof. Let $\mathcal{K}$ have SFMP and let $\mathcal{B} = (B, \{f_i^{\mathcal{B}}\}_{1 \leq i \leq k}, \leq^{\mathcal{B}})$ be a finite, partial subalgebra of an algebra $\mathcal{A} = (A, \{f_i^{\mathcal{A}}\}_{1 \leq i \leq k}, \leq^{\mathcal{A}}) \in \mathcal{K}$. We can assume that $\mathcal{A}$ is infinte. Let $B = \{b_1, \ldots, b_n\}$ where $b_1, \ldots, b_n$ are different elements. We will create two finite sets of inequations Φ and Ψ. Let $x_1, \ldots, x_n$ be a set of different variables in the language of $\mathcal{K}$. For each operation symbol f_j of an arity s if

$$f_j^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_s}) \leq^{\mathcal{B}} b_{i_{s+1}},$$

then we add the inequation

$$f_j(x_{i_1}, \ldots, x_{i_s}) \leq x_{i_{s+1}}$$

to the set Φ. Similarly if $f_j^{\mathcal{B}}(b_{i_1}, \ldots, b_{i_s}) \geq^{\mathcal{B}} b_{i_{s+1}}$, then $f_j(x_{i_1}, \ldots, x_{i_s}) \geq x_{i_{s+1}}$ is added to Φ. Moreover we add to the set Φ an inequation $x_i \leq x_j$ if $b_i \leq^{\mathcal{B}} b_j$ for $i \neq j$.

The set Ψ is build as follows: for each inequation $b_i \leq^{\mathcal{B}} b_j$ not true in $\mathcal{B}$ we add $x_i \leq x_j$ to Ψ. By l we denote the cardinality of Ψ. If $n = 1$, then we can take any $a \in \mathcal{A}$ ($\mathcal{A}$ is infinite), such that $b_1 \neq a$ and add to Ψ an appropriate inequation for the pair b_1, a, where a is represented by an additional variable.

For each inequation $\psi_i \in \Psi$ we consider a clause $\Phi \Rightarrow \psi_i$. Obviously, the clause is not true in $\mathcal{K}$, as it is not true in $\mathcal{A}$. By SFMP there exists a finite algebra $\mathcal{C}_i \in \mathcal{K}$ and a valuation μ_i in which the clause fails, i.e. $(\mathcal{C}_i, \mu_i) \models \Phi$ and $(\mathcal{C}_i, \mu_i) \not\models \psi_i$. Let $\mathcal{C} = (C, \{f_i^{\mathcal{C}}\}_{1 \leq i \leq k}, \leq^{\mathcal{C}}) = \Pi_{i=1}^{l} C_i$ be the product of algebras $\mathcal{C}_i$. It is easy to see that the function $e : B \rightarrow C$ defined as

$$e(b_j) =< \mu_1(x_j), \ldots, \mu_l(x_j) >$$

for $j = 1 \ldots n$ is an embedding of partial algebra $\mathcal{B}$ into the finite algebra $\mathcal{C} \in \mathcal{K}$. □

2. A syntactical model construction

In this section we present a construction of an intuitionistic phase space model, which we will use both in the proof of the subformula property and SFMP.

We consider the system $\mathrm{NL}_\wedge$ with a fixed, finite set Φ of additional simple sequents as new axioms. Note that the new axioms from Φ are not closed under substitution. We will omit the index $\mathrm{NL}_\wedge[\Phi]$ in the provability relation $\vdash_{NL_\wedge[\Phi]}$.

Let T be a set of formulas closed under subformulas and containing all formulas appearing in sequents from Φ. By $T^\star$ we denote the set of all formula structures over T, and by $T[\diamond]^\star$ the set of all contexts over T that is, the set of all formula structures with a distinguished, exactly one occurrence of a propositional variable which does not occur in T, denoted by $\diamond$. The set $T[\diamond]^\star$ contains an empty context $[\diamond]$. Its elements will be denoted as $X[\diamond], Y[\diamond], Z[\diamond], \ldots$. By $X[Y]$ we denote a structure which arise, from the context $X[\diamond] \in T[\diamond]^\star$ and a structure $Y \in T^\star$ by substituting Y in place of $\diamond$.

Let $\prec \subseteq T^\star \times T$ be a relation satisfying the following conditions

($\prec$0) $A \prec A$,
$A \prec B$ for all $A \to B \in \Phi$,

($\prec$1) $X \prec B \;\&\; Z[B] \prec A \Rightarrow Z[X] \prec A$,

($\prec$2) $X \prec A \;\&\; Y \prec B \;\&\; A \otimes B \in T \Rightarrow X \circ Y \prec A \otimes B$,

($\prec$3) $Z[A \circ B] \prec C \;\&\; A \otimes B \in T \Rightarrow Z[A \otimes B] \prec C$,

($\prec$4) $A \circ X \prec B \;\&\; A\backslash B \in T \Rightarrow X \prec A\backslash B$,
$X \circ A \prec B \;\&\; B/A \in T \Rightarrow X \prec B/A$,

($\prec$5) $A \circ (A\backslash B) \prec B$,
$(B/A) \circ A \prec B$,

($\prec$6) $X \prec A \;\&\; X \prec B \;\&\; A \wedge B \in T \Rightarrow X \prec A \wedge B$,

($\prec$7) $A \wedge B \prec A, \quad A \wedge B \prec B$

A base set family $\mathcal{B}_{\prec}$ over $(T, \prec)$ is defined as follows:

$$\mathcal{B}_{\prec} = \{[Z[\diamond], A] : Z[\diamond] \in T[\diamond]^{\star}, A \in T\},$$

where

$$[Z[\diamond], A] = \{X \in T^{\star} : Z[X] \prec A\}.$$

Also, let $[A] = [[\diamond], A] = \{X \in T^{\star} : X \prec A\}$.

A similar kind of relations were used in the definition of a Gentzen structure [1] to give an algebraic proof of the cut elimination theorem for various sequent systems.

For a base set family $\mathcal{B}_{\prec}$ we define an operator $C^{\prec} : P(T^{\star}) \to P(T^{\star})$ over the powerset residuated groupoid $\mathcal{P}(T^{\star})$ by setting

$$C^{\prec}\alpha = \cap\{[Z[\diamond], A] \in \mathcal{B}_{\prec} : \alpha \subseteq [Z[\diamond], A]\},$$

for $\alpha \subseteq T^{\star}$.

Fact 1. *$C^{\prec}$ is a closure operator.*

Proof. It is obvious that $C^{\prec}$ satisfies the conditions (C1) - (C3). For the sake of readability, let us skip the index $\prec$ in $C^{\prec}$. We prove that C satisfies (C4).

Let $\alpha, \beta \in P(T^\star)$ and $X \circ Y \in C\alpha \odot C\beta$. Assume that $\alpha \odot \beta \subseteq [Z[\diamond], A]$ for some $[Z[\diamond], A] \in \mathcal{B}_\prec$. As $Z[X' \circ Y'] \prec A$, for each $X' \in \alpha, Y' \in \beta$, then $U[X'] \prec A$, where $U[X'] = Z[X' \circ Y']$. We have $X' \in [U[\diamond], A]$, so $\alpha \subseteq [U[\diamond], A]$. As the latter set is C-closed $C\alpha \subseteq [U[\diamond], A]$, by (C3). Therefore $X \in [U[\diamond], A]$, i.e. $Z[X \circ Y'] \prec A$. Analogically $C\beta \subseteq [V[\diamond], A]$, where $V[Y'] = Z[X \circ Y']$. This means that $Y \in [V[\diamond], A]$, i.e. $Z[X \circ Y] \prec A$. So, $X \circ Y \in C(\alpha \odot \beta)$. □

Observe that the proof of conditions (C1) - (C4) for $C^\prec$ does not rely on the special conditions for relation $\prec$. Let $\mathcal{S} = \mathcal{PS}((T^\star, \circ), C^\prec)$ be the intuitionistic phase space defined using $C^\prec$ on $(T^\star, \circ)$.

Lemma 1. *Let μ be a valuation in $\mathcal{S}$ such that $\mu(p) = [p]$, for each propositional variable $p \in T$. Then $\mu(A) = [A]$ for each $A \in T$.*

Proof. Assume μ be a valuation in $\mathcal{S}$ such that $\mu(p) = [p]$, for each propositional variable $p \in T$ and let $A \in T$. We proceed by induction on A.

Induction basis (for $A \equiv p$) holds trivially by the assumption on μ. Assume that $\mu(A_i) = [A_i]$ for $i = 1, 2$. We consider four cases.

(1) $A \equiv A_1 \otimes A_2$.

Let $X = X_1 \circ X_2 \in [A_1] \odot [A_2]$ for $X_i \in [A_i]$. Thus $X_i \prec A_i$, so $X = X_1 \circ X_2 \prec A_1 \otimes A_2$ by ($\prec$2). It means that $[A_1] \odot [A_2] \subseteq [A_1 \otimes A_2]$. As the latter set is C-closed, we get, by induction, $\mu(A) = C(\mu(A_1) \odot \mu(A_2)) = C([A_1] \odot [A_2]) \subseteq [A_1 \otimes A_2] = [A]$.

Let $X \in [A_1 \otimes A_2]$ and let $[Z[\diamond], B] \in \mathcal{B}_\prec$ be a base set, such that $\mu(A_1) \odot \mu(A_2) \subseteq [Z[\diamond], B]$. Since $A_i \in [A_i]$, by induction $A_1 \circ A_2 \in \mu(A_1) \odot \mu(A2) \subseteq [Z[\diamond], B]$. Thus $Z[A_1 \circ A_2] \prec B$, so $Z[A_1 \otimes A_2] \prec B$ by ($\prec$3). As $X \prec A_1 \otimes A_2$, we get $X \in [Z[\diamond], B]$, by ($\prec$1). We have shown $[A] \subseteq \mu(A)$.

(2) $A = A_1 \backslash A_2$

Let $X \in \mu(A_1) \backslash \mu(A_2)$. Since $A_1 \in [A_1] = \mu(A_1)$, then $A_1 \circ X \in \mu(A_2) = [A_2]$, hence $A_1 \circ X \prec A_2$. From the latter $X \prec A_1 \backslash A_2$ by ($\prec$4), which means that $X \in [A]$.

Let $X \in [A_1 \backslash A_2]$ and let $X_1 \in \mu(A_1)$. Since $\mu(A_1) = [A_1]$ then $X_1 \prec A_1$ and $X \prec A_1 \backslash A_2$ by induction. By the condition ($\prec$5) we get $A_1 \circ (A_1 \backslash A_2) \prec A_2$. Applying ($\prec$1) twice results in $X_1 \circ X \prec A_2$. Now, since $[A_2] = \mu(A_2)$ then $X_1 \circ X \in \mu(A_2)$, which means $X \in \mu(A_1) \backslash \mu(A_2) = \mu(A)$.

(3) $A = A_1 / A_2$

The argument is similar to that in (2).

(4) $A = A_1 \wedge A_2$

Let $X \in \mu(A_1) \cap \mu(A_2)$. Thus $X \in [A_1] \cap [A_2]$, so $X \prec A_i$. By the condition ($\prec$6) we get $X \prec A_1 \wedge A_2$. It means that $X \in [A]$.

Let $X \in [A_1 \wedge A_2]$, i.e. $X \prec A_1 \wedge A_2$. Since $A_1 \wedge A_2 \prec A_i$ by ($\prec$7), so $X \prec A_i$ by ($\prec$1). Therefore $X \in [A_i] = \mu(A_i)$ and we finally get $X \in \mu(A_1) \cap \mu(A_2) = \mu(A)$.

□

3. Interpolation

We will prove an interpolation lemma for $\mathrm{NL}_\wedge[\Phi]$. First we show the subformula property.

T denotes a nonempty set of formulas, closed under subformulas. By a T-sequent we mean a sequent built only upon formulas from the set T. Let $\rightarrow_T \subseteq T^\star \times T$ be a relation defined as $X \rightarrow_T A$ iff $X \rightarrow A$ has a proof consisting of T-sequents only. The next lemma (for systems without $\wedge$) has been proved in [5]; we give a different proof.

Lemma 2. *Let $X \rightarrow A$ be a T-sequent. Then $X \rightarrow_T A$ iff $\vdash X \rightarrow A$.*

Proof. The 'only if' part is obvious. For the proof of 'if' part assume that $\vdash X \rightarrow A$. Observe that the relation $\rightarrow_T$ satisfies all conditions ($\prec$0) - ($\prec$7). Consider the syntactical phase space $S = \mathcal{PS}((T^\star, \circ), C^{\rightarrow_T})$ and a valuation μ in S such that $\mu(p) = [p]$ for $p \in T$. By Lemma 1 and conditions ($\prec$0), ($\prec$1) all axioms $A \rightarrow B$ from Φ are true in (S, μ). Since $X \rightarrow A$ is provable, it must be true in (S, μ), so $\mu(X) \subseteq \mu(A) = [A]$. Observe that $X \in \mu(X)$ by Lemma 1. Thus $X \in [A]$, which means that $X \rightarrow_T A$. □

Our goal is to prove an interpolation lemma for $\mathrm{NL}_\wedge[\Phi]$. An analogous lemma for $\mathrm{NL}[\Phi]$ was proved in [5]. In the latter all interpolants belong to T. Here we need a larger set $\overline{T}$, which consists of all nonempty, finite conjunctions of formulas from T. Caution: this form of interpolation is characteristic for non-associative systems. The associative Lambek calculus admits a different form of interpolation with interpolants of a greater complexity [10].

Lemma 3. *Let $X[Z] \to C$ be a T-sequent. If $\vdash X[Z] \to C$, then there exists a formula $D \in \overline{T}$ such that $\vdash X[D] \to C$ and $\vdash Z \to D$.*

Proof. Let $X[Z] \to C$ be a T-sequent provable in a $NL_{\wedge}[\Phi]$. By Lemma 2 $X[Z] \to_T A$. Take a proof of $X[Z] \to C$ built from T-sequents only. We will proceed by induction on proof.

For axioms it is trivial, as all are in the form of simple sequents. For the multiplicative rules the proof steps are similar, and we consider only few representative cases.

(1) $\dfrac{X[A]\to C; \quad Y[Z]\to A}{X[Y[Z]]\to C}$ (Cut)

By induction there exists a formula $D \in \overline{T}$, such that $\vdash Y[D] \to A$ and $\vdash Z \to D$. Thus $\vdash X[Y[D]] \to C$, by (Cut).

(2) $\dfrac{(X[A])[Z]\to C; \quad Y\to A}{(X[Y])[Z]\to C}$ (Cut)

Similar as in (1).

(3) $\dfrac{X[Z[A]]\to C; \quad Y\to A}{X[Z[Y]]\to C}$ (Cut)

By induction there exists a formula $D \in \overline{T}$ such that $\vdash X[D] \to C$ and $\vdash Z[A] \to D$. Thus $\vdash Z[Y] \to D$, by (Cut).

(4) $\dfrac{X[A]\to C; \quad Y[Z]\to B}{(X[(A/B)\circ Y])[Z]\to C}$ (L/)

The conclusion is of course of the form $\vdash X[(A/B) \circ Y[Z]) \to C$. By induction there exists a formula $D \in \overline{T}$ such that $\vdash Y[D] \to B$ and $\vdash Z \to D$. Thus $\vdash X[(A/B) \circ Y[D]] \to C$, by (L/).

(5) $\dfrac{(X[A])[Z]\to C; \quad Y\to B}{(X[(A/B)\circ Y])[Z]\to C}$ (L/)

By induction there exists a formula $D \in \overline{T}$ such that $\vdash (X[A])[D] \to C$ and $\vdash Z \to D$. Thus $\vdash (X[(A/B) \circ Y])[D] \to C$, by (L/).

(6) $\frac{X[Z[A]]\to C; \quad Y\to B}{X[Z[(A/B)\circ Y]]\to C}$ (L/)

By induction there exists a formula $D \in \overline{T}$ such that $\vdash X[D] \to C$ and $\vdash Z[A] \to D$. Thus $\vdash Z[(A/B) \circ Y] \to D$, by (L/).

(7) $\frac{(X[A])[Z]\to C}{(X[A\wedge B])[Z]\to C}$ (L1$\wedge$)

By induction there exists a formula $D \in \overline{T}$ such that $\vdash (X[A])[D] \to C$ and $\vdash Z \to D$. Thus $\vdash (X[A \wedge B])[D] \to C$, by (L1$\wedge$)

(8) $\frac{X[Z[A]]\to C}{X[Z[A\wedge B]]\to C}$ (L1$\wedge$)

By induction there exists a formula $D \in \overline{T}$ such that $\vdash X[D] \to C$ and $\vdash Z[A] \to D$. Thus $\vdash Z[A \wedge B] \to D$, by (L1$\wedge$)

(9) $\frac{X[Z]\to A; \quad X[Z]\to B}{X[Z]\to A\wedge B}$ (R$\wedge$)

By induction there exists a formula $D_1, D_2 \in \overline{T}$ such that $\vdash X[D_1] \to A$, $\vdash Z \to D_1$, and $\vdash X[D_2] \to B$, $\vdash Z \to D_2$. Thus $\vdash Z \to D_1 \wedge D_2$ and $\vdash X[D_1 \wedge D_2] \to A$ and also $\vdash X[D_1 \wedge D_2] \to B$. So $\vdash X[D_1 \wedge D_2] \to A \wedge B$. Therefore $D = D_1 \wedge D_2$.

□

4. Strong finite model property

Let T be also finite. We consider a relation $\sim \subseteq T^\star \times T^\star$ defined as

$$X \sim Y \text{ iff } \forall_{A \in \overline{T}}(\vdash X \to A \Leftrightarrow \vdash Y \to A).$$

Fact 2. *The relation $\sim$ is an equivalence relation and its index is finite.*

Proof. It easy to see that $\sim$ is an equivalence relation and that $X \sim Y$ if and only if, for all $A \in T$, $\vdash X \to A$ iff $\vdash Y \to A$. Since T is finite, then there are only finitely many equivalence classes of $\sim$. □

Consider a syntactical phase space $PS((T^\star, \circ), C^{\to_T})$ over the finite set T and the closure operator for the relation $\to_T$. We will show that the space is finite. Let $X/_\sim$ denotes the equivalence class of a formula structure X.

Fact 3. *Let $X \in [Z[\diamond], A] \in \mathcal{B}_\prec$. Then $X/_\sim \subseteq [Z[\diamond], A]$.*

Proof. Let $X \in [Z[\diamond], A]$, i.e. $Z[X] \to_T A$. Take any $Y \in T^\star$ such that $X \sim Y$. By Lemma 3 there exist $D \in \overline{T}$ such that $\vdash Z[D] \to A$ and $\vdash X \to D$. Since $X \sim Y$,then $\vdash Y \to D$. Thus $\vdash Z[Y] \to A$, by (Cut). Since $Z[Y] \to A$ is a T-sequent, then $Z[Y] \to_T A$, by Lemma 2. It means that $Y \in [Z[\diamond], A]$ □

Since the relation $\sim$ is of finite index, and each base set is a sum of equivalence classes of $\sim$, the number of all base sets is finite, so the constructed syntactical phase space is finite.

Theorem 1. *The class of meet semilattice-ordered residuated groupoids has SFMP.*

Proof. Let $\phi_1, \ldots, \phi_k \Rightarrow \phi$ be a false clause in the language of meet semilattice-ordered residuated groupoids. Let Φ be the set of simple sequents corresponding to $\phi_1, \ldots, \phi_k$ and $A \to B$ be the sequent corresponding to ϕ. Let T be the set of all subformulas of formulas A, B and all formulas from Φ. Consider the finite phase space $\mathcal{S} = \mathcal{PS}((T^\star, \circ), C^{\to_T})$ and a valuation μ in it, such that $\mu(p) = [p]$ for all propositional variables $p \in T$. By Lemma 1, all sequents from Φ are true in $(\mathcal{S}, \mu)$. By Proposition 1 $\nvdash_{NL_\wedge[\Phi]} A \to B$, so $A \notin [B]$, but $A \in [A]$ by (Id). By Lemma 1, it means that $\mu(A) \nsubseteq \mu(B)$ in $\mathcal{S}$. □

Corollary 1. *The class of meet semilattice-ordered residuated groupoids has FEP.*

A similar reasoning, simplified by using only the set T as a source of interpolants instead of $\overline{T}$, gives us the following result.

Theorem 2. *The class of residuated groupoids has SFMP.*

Theorem 2 also follows from the fact that every residuated groupoid can be embedded into a lattice-ordered residuate groupoid (Mac Neille completion for residuated groupoids).

Corollary 2. *The class of residuated groupoids has FEP.*

In a similar way we can proove SFMP and FEP for commutative (meet-semilattice) ordered residuated groupoids and distributive lattice-ordered (commutative) residuated groupoids. It is not known whether FEP holds for arbitrary lattice-ordered residuated groupoids. In the next section we show some details for the distributive (non-commutative) case.

5. Distributive lattice-ordered residuated groupoids

Theorem 3. *The class of distributive lattice-ordered residuated groupoids has SFMP.*

Corollary 3. *The class of distributive lattice-ordered residuated groupoids has FEP.*

We sketch the proof of Theorem 3. We modify the construction used for meet semilattice-ordered residuated groupoids.

The set $\overline{T}$ is the smallest set containing T and closed under $\wedge$ and $\vee$. Let $\prec \subseteq \overline{T}^{\star} \times \overline{T}$ be a relation satisfying the following conditions:

($\prec$0) $A \prec A$,
$A \prec B$ for all $A \to B \in \Phi$,

($\prec$1) $X \prec B \;\&\; Z[B] \prec A \Rightarrow Z[X] \prec A$,

($\prec$2) $X \prec A \;\&\; Y \prec B \;\&\; A \otimes B \in T \Rightarrow X \circ Y \prec A \otimes B$,

($\prec$3) $Z[A \circ B] \prec C \;\&\; A \otimes B \in T \Rightarrow Z[A \otimes B] \prec C$,

($\prec$4) $A \circ X \prec B \;\&\; A \backslash B \in T \Rightarrow X \prec A \backslash B$,
$X \circ A \prec B \;\&\; B/A \in T \Rightarrow X \prec B/A$,

($\prec$5) $A \circ (A \backslash B) \prec B$,
$(B/A) \circ A \prec B$,

($\prec$6) $X \prec A \;\&\; X \prec B \Rightarrow X \prec A \wedge B$,

($\prec$7) $A \wedge B \prec A, \quad A \wedge B \prec B$,

($\prec$8) $X \prec A \Rightarrow X \prec A \vee B$,
$X \prec B \Rightarrow X \prec A \vee B$,

($\prec$9) $X[A] \prec C \;\&\; X[B] \prec C \Rightarrow X[A \vee B] \prec C$,

($\prec$10) $A \wedge (B \vee C) \prec (A \wedge B) \vee (A \wedge C)$.

The base set family and closure operator are defined in a similar way.

Distributive Non-associative Full Lambek Calculus, denoted DNL, arise from the system $NL_\wedge$ by adding the following axiom and rules (and extending the formula notion):

$$(D1) \quad A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$$

$$(\mathrm{L}\vee) \quad \frac{X[A] \to C \quad X[B] \to C}{X[A \vee B] \to C}$$

$$(\mathrm{R1}\vee) \quad \frac{X \to A}{X \to A \vee B} \qquad \frac{X \to B}{X \to A \vee B} \quad (\mathrm{R2}\vee)$$

The system DNL is strongly complete with respect to the class of distributive lattice-ordered residuated groupoids.

For the given set T, by T-DNL we denote the system DNL where axiom (D1) is restricted to the $\overline{T}$-sequents only. Let Φ be a finite set of simple sequents. We consider a system T-DNL (denoted T-DNL[Φ]) extended with additional axioms from Φ, assuming that all formulas from sequents from Φ are in the set T, and that T is subformula closed.

Lemma 4. *Let μ be a valuation in $\mathcal{S}$ such that $\mu(p) = [p]$, for each propositional variable $p \in T$. Then $\mu(A) = [A]$ for each $A \in \overline{T}$.*

Proof. We proceed by induction on A. The cases for $\otimes, /, \backslash, \wedge$ are analogical as in the proof of Lemma 1.

Let $A = A_1 \vee A_2$. If $X \in \mu(A_1) \cup \mu(A_2)$, then $X \in \mu(A_1)$ or $X \in \mu(A_2)$. Thus $X \in [A_1]$ or $X \in [A_2]$, by induction, so $X \prec A_1$ or $X \prec A_2$. Then $X \prec A_1 \vee A_2$, by ($\prec$8), i.e. $X \in [A_1 \vee A_2]$. We have shown that $\mu(A_1) \cup \mu(A_2) \subseteq [A_1 \vee A_2]$. As the latter set is closed, we get $\mu(A) = \mu(A_1) \cup_C \mu(A_2) \subseteq [A_1 \vee A_2]$.

Let $X \in [A_1 \vee A_2]$ and let $[Z[\diamond], D]$ be a closed set, such that $\mu(A_1) \cup \mu(A_2) \subseteq [Z[\diamond], D]$. Since $\mu(A_i) = [A_i]$, by induction, then $A_i \in \mu(A_i) \subseteq [Z[\diamond], D]$ for $i = 1, 2$. Thus $Z[A_1 \vee A_2] \prec D$, by ($\prec$9). As $X \prec A_1 \vee A_2$, then $Z[X] \prec D$, by ($\prec$1). It means that $X \in C(\mu(A_1) \cup \mu(A_2)) = \mu(A)$. $\square$

In the constructions of phase spaces from section 4 we replace the relation $\to_T$ by $\to_{\overline{T}}$ defined analogously for $\overline{T}$-sequents. It satisfies all conditions ($\prec$0) - ($\prec$10).

Lemma 5. *Let $X \to A$ be a $\overline{T}$-sequent. Then $X \to_{\overline{T}} A$ iff $\vdash X \to A$ in T-DNL[Φ].*

Proof. The 'only if' part is obvious. For the 'if' part we construct, as in the proof of Lemma 2, a phase space $\mathcal{S}$ and an assignemnt μ on it. All axioms from Φ are true in the model $(\mathcal{S}, \mu)$. Also (D1) restricted to $\overline{T}$-sequents is true in $(\mathcal{S}, \mu)$, by Lemma 4 and ($\prec$10). Thus, if $\vdash X \to A$ in T-DNL[Φ], then $X \to A$ must be true in $(\mathcal{S}, \mu)$. It means that $X \to_{\overline{T}} A$ □

We prove interpolation lemma for the restricted system T-DNL[Φ].

Lemma 6. *Let $X[Z] \to C$ be a $\overline{T}$-sequent. If $\vdash X[Z] \to C$ is provable in T-DNL[Φ], then there exists formula $D \in \overline{T}$ such that $\vdash Z \to D$ and $\vdash X[D] \to C$ are provable T-DNL[Φ].*

Proof. Induction on a proof built from $\overline{T}$-sequent only. We consider only few cases for $\vee$. All other are analogical as in the proof of Lemma 3.

(a) $\dfrac{X[Z]\to A}{X[Z]\to A\vee B}$ (R1$\vee$)

There exists, by induction, a formula $D \in \overline{T}$, such that $\vdash Z \to D$ and $X[D] \to A$. Thus, by (R1$\vee$), we get $X[D] \to A \vee B$.

(b) $\dfrac{(X[Z])[A]\to C; \quad (X[Z])[B]\to C}{(X[Z])[A\vee B]\to C}$ (L$\vee$)

There exist, by induction, formulas $D_1, D_2 \in \overline{T}$, such that $\vdash Z \to D_1$, $\vdash (X[D_1])[A] \to C$ and $\vdash Z \to D_2$, $\vdash (X[D_2])[B] \to C$. Thus $\vdash Z \to D_1 \wedge D_2$, by (R$\wedge$) and $\vdash (X[D_1 \wedge D_2])[A] \to C$, $\vdash (X[D_1 \wedge D_2])[B] \to C$, by (L1$\wedge$), (L2$\wedge$). Whence $\vdash (X[D_1 \wedge D_2])[A \vee B] \to C$, by (L$\vee$). The formula $D = D_1 \wedge D_2$ is the required interpolant.

(c) $\dfrac{X[Z[A]]\to C; \quad X[Z[B]]\to C}{X[Z[A\vee B]]\to C}$ (L$\vee$)

There exist, by induction, formulas $D_1, D_2 \in \overline{T}$, such that $\vdash Z[A] \to D_1$, $\vdash X[D_1] \to C$ and $\vdash Z[B] \to D_2$, $\vdash X[D_2] \to C$. Whence $\vdash Z[A] \to D_1 \vee D_2$, $\vdash Z[B] \to D_1 \vee D_2$, by (L1$\vee$), (L2$\vee$). Thus $\vdash Z[A \vee B] \to D_1 \vee D_2$, by (L$\vee$). Moreover $\vdash X[D_1 \vee D_2] \to C$, by (L$\vee$), so $D = D_1 \vee D_2$.

□

We define an equivalence relation $\sim$ on $\overline{T}^*$:

$$X \sim Y \text{ iff } \forall_{A\in\overline{T}}(\vdash_{\text{T-DNL}[\Phi]} X \to A \Leftrightarrow \vdash_{\text{T-DNL}[\Phi]} Y \to A).$$

Formulas A, B are said to be deductively equivalent iff both $A \to B$ and $B \to A$ are provable in T-DNL. Clearly, every formula from $\overline{T}$ is deductively equivalent to a finite conjunction of finite disjunctions of formulas from T. Then, there exists a finite set $R \subseteq \overline{T}$ such that every formula from $\overline{T}$ is deductively equivalent to some formula from R. Consequently, $X \sim Y$ iff, for all $A \in R$, $X \to A$ is provable in T-DNL[Φ] iff $Y \to A$ is so. Therefore $\sim$ has a finite index.

We prove analogue of Fact 3 and Theorem 3 (now $\vdash$ denotes provability in T-DNL[Φ]). Proofs are similar with one exception. We must show that the phase space $\mathcal{S}$ constructed like in the proof of Theorem 1 is distributive.

Evidently, the distributive law $\alpha \cap (\beta \cup_C \gamma) \subseteq (\alpha \cap \beta) \cup_C (\alpha \cap \gamma)$ is true in $\mathcal{S}$ if at least one of the sets α, β, γ is the total set or the empty set. Assume α, β, γ are non empty and non total.

Let $X/_\sim$ be an equivalence class such that $X/_\sim \subseteq \alpha$ and let $\alpha \subseteq [Z[\diamond], A]$. Then $Z[X] \to_{\overline{T}} A$. By Lemma 6 there exists $B \in \overline{T}$, such that $\vdash X \to B$ and $\vdash Z[B] \to A$. Thus, there exist $B' \in R$ such that $\vdash X \to B'$. Let $B_1, \ldots, B_n \in R$ be all formulas $B \in R$, such that $\vdash X \to B$. Denote $\overline{B} = B_1 \wedge \ldots \wedge B_n$. We claim $X \sim \overline{B}$. It suffices to show, that $\vdash X \to D$ iff $\vdash \overline{B} \to D$ for any $D \in R$. If $\vdash X \to D$, then D must be one of the formulas $B_1, \ldots, B_n$, thus $\vdash \overline{B} \to D$. If $\vdash \overline{B} \to D$, then $\vdash X \to D$, by (Cut), as $\vdash X \to \overline{B}$.

For any equivalence class $X/_\sim \subseteq \alpha$ we select a formula $\overline{A} \in X/_\sim$, in the way as $\overline{B}$ above. Let $A_1, ..., A_m$ be all formulas selected in this way. As $X/_\sim \subseteq [\overline{A}]$ and $[\overline{A}] \subseteq \alpha$ ($\overline{A} \in \alpha \Rightarrow [\overline{A}] \subseteq \alpha$, by ($\prec$1)), the set α equals the join of sets $[A_i]$, $i = 1, \ldots, m$. Whence

$$\alpha = C\alpha = C([A_1] \cup \ldots \cup [A_m]) = [A_1 \vee \ldots \vee A_m].$$

Accordingly there exists a formula $A \in \overline{T}$ such that $\alpha = [A]$. Similarly, there exist formulas $B, C \in \overline{T}$ such that $\beta = [B], \gamma = [C]$. Since $[A \wedge (B \vee C)] \subseteq [(A \wedge B) \vee (A \wedge C)]$ in $\mathcal{S}$, by ($\prec$10) and ($\prec$1), then

$$\begin{aligned} \alpha \cap (\beta \cup_C \gamma) &= [A] \cap ([B] \cup_C [C]) \\ &= [A \wedge (B \vee C)] \\ &\subseteq [(A \wedge B) \vee (A \wedge C)] \\ &= ([A] \cap [B]) \cup_C ([A] \cap [C]) \\ &= (\alpha \cap \beta) \cup_C (\alpha \cap \gamma) \end{aligned}$$

Notice that in the above proof we do not use the subformula property for DNL, which along with an interpolation lemma for DNL and $\overline{T}$-sequents, follows from the main result.

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