REPORTS ON MATHEMATICAL LOGIC
42 (2007), 117–144

Gemma ROBLES and José M. MÉNDEZ

MINIMAL NON-RELEVANT LOGICS WITHOUT THE K AXIOM

A b s t r a c t. The logic B_+ is Routley and Meyer's basic positive logic. The logic B_{K+} is B_+plus the K rule. We add to B_{K+} four intuitionistic-type negations. We show how to extend the resulting logics within the modal and relevance spectra. We prove that all the logics defined lack the K axiom.

1. Introduction

The aim of this paper is to study the effect of adding the K rule and the paradoxes of consistency to relevance logics in the context of intuitionistic-type negations with respect to the presence (or absence) of the K axiom in the resulting systems.

In the literature on relevance logics, paradoxes of implication are classified into paradoxes of relevance and paradoxes of consistency (see, e.g., [2], p. 349). A distinctive paradigm of the former is the K axiom

$$\text{(a).} \quad \vdash A \to (B \to A)$$

Received 23 August 2006

or the K rule

$$\text{(b).}\ \vdash A \Rightarrow\ \vdash B \to A$$

and a typical example of the latter is the ECQ axiom ("E contradictione quodlibet" axiom)

$$\text{(c).}\ \vdash (A \wedge \neg A) \to B$$

and related theses such as

$$\text{(d).}\ \vdash A \to (\neg A \to B)$$

and

$$\text{(e).}\ \vdash \neg A \to (A \to B)$$

Well, the purpose of this paper is to prove that a wide range of relevance logics can be extended with the K rule and the paradoxes of consistency without the resulting logics having the formula (a) cited above (the so-called K axiom) as a theorem.

Now, relevance logicians have always been interested in exploring the frontiers between relevance and non-relevance logics. The most prominent example is, no doubt, the considerable attention conceded to the non-relevance, paradoxical R-Mingle in Entailment I (see [1]). But not less interesting is the work of Routley, Meyer and others on the logics KR, CR and CE (see [7], [8], [4] and [11]). The logic KR is the result of adding the axiom ECQ (c) to the Logic of Relevance R and, on the other hand, the logic CR and the logic CE are obtained by adding a boolean negation to R and to the Logic of Entailment E, respectively.

Now, it is to be stressed that the investigations just mentioned have been carried in the context of the standard negation in relevance logics, which is a De Morgan negation , as it is known. Well, the objective of this paper is to study this type of questions in the context of intuitionistic-type negations.

In particular, we are interested (as it was remarked above) in adding to relevance logics the K rule and the paradoxes of consistency. But let us now note some previous results.

In [10], the K rule is added to relevance logics in the presence of a weak constructive negation. It is proved that neither the K axiom nor the ECQ axioms are provable. In [5], the same results are obtained in the context of a non-constructive negation. But let us be more explicit about or results.

By B_+, we refer to Routley and Meyer's basic positive logic (see [11]). Next, B_{K+} is the result of adding the K rule to B_+. Two are the principal logics presented here. The logic B_{Kcdnr} is the result of adding the constructive contraposition, double negation and reductio axioms. The logic B_{Kj} is obtained by adding the ECQ axioms to B_{Kcdnr}.

Our logics are clearly subsystems of intuitionistic logic or of minimal intuitionistic logic, as it is the case. But let us remark that they are not included in Lewis's modal logic S5 (so, they are not included in Lewis's S4 or in the Logic of Entailment either): A8 and A11 are not valid in S5 (the arrow is read as S5 strict implication, of course). On the other hand, our logics are not included in the logics KR and CR (so neither are they in CE) mentioned above consequently providing a different perspective (from that considered until now) on the borderlines between relevance and non-relevance logics.

The structure of the paper is as follows. In sections 2-4, the logic B_{K+} is introduced. In section 5, we define the logic B_{Km}. This logic is obtained by adding a minimal negation to B_{K+} by way of extending the positive language with the propositional falsity constant F. In sections 6-8, the logic B_{Kcdn} is introduced. This logic is the result of adding the (constructive) contraposition and double negation axioms to B_{Km}. In sections 9-11. The logic B_{Kcdnr} is defined. It is the result of adding the (constructive) reductio axioms to B_{Kcdn}. In these sections, we thoroughly investigate the relations between the different contraposition (reductio) axioms, on the one hand, and between the contraposition (reductio) postulates, on the other. We also look into the relations maintained by axioms (contraposition or reductio) and postulates (contraposition or reductio) between each other. In sections 12-14, we add the ECQ axioms to B_{Kcdnr} defining the logic B_{Kj}. Finally, we include two appendices and some concluding remarks. The first appendix provides some notorious theorems of the logics studied. The second one places these logics in the spectra of well-known modal and relevance logics, respectively. In the concluding remarks, we discuss some possibilities for extending them.

2. The positive logic B_{K+}

B_{K+} is axiomatized with

Axioms

A1. $A \to A$
A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of derivation are

Modus ponens (MP): $(\vdash A \ \& \ \vdash A \to B) \Rightarrow \ \vdash B$
Adjunction (Adj.): $(\vdash A \ \& \ \vdash B) \Rightarrow \ \vdash A \wedge B$
Suffixing (Suf.): $\vdash A \to B \Rightarrow \ \vdash (B \to C) \to (A \to C)$
Prefixing (Pref.): $\vdash A \to B \Rightarrow \ \vdash (C \to A) \to (C \to B)$
K: $\vdash A \Rightarrow \ \vdash B \to A$

Therefore, B_{K+} is B_{+} with the addition of the K rule.

We note the following two lemmas and conjecture:

Lemma 2.1 *The implicational fragment of B_{K+} does not have the variable sharing property.*

Proof. For any propositional variable p, $p \to p$ is a theorem of B_{K+}, and by rule K, so is $q \to (p \to p)$. □

Obviously, the same is true for B_{K+}.

Lemma 2.2 *A paradox of implication, namely, $A \to (B \to B)$ is a theorem of B_{K+}. (Indeed, it is a theorem already of its implicational fragment).*

Proof. Immediate from the previous proof. □

Conjecture 2.1 *There are denumerably many distinct logics between B_{+} and B extended by the axiom K, that do not posses the variable sharing property.*

3. Semantics for $\mathbf{B}_{K+}$

A B_{K+} *model* is a triple ¡K, R, $\models$¿ where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all a, b, c, $d \in K$ with quantifiers ranging over K:

$$\text{d1. } a \leq b =_{df} \exists x Rxab$$

$$\text{d2. } R^2abcd =_{df} \exists x(Rabx \ \& \ Rxcd)$$

$$\text{P1. } a \leq a$$

$$\text{P2. } (a \leq b \ \& \ Rbcd) \Rightarrow Racd$$

$$\text{P3. } (b \leq d \ \& \ Radc) \Rightarrow Rabc$$

Finally, $\models$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i). $(a \leq b \ \& \ a \models p) \Rightarrow b \models p$

(ii). $a \models A \wedge B$ iff $a \models A$ and $a \models B$

(iii). $a \models A \vee B$ iff $a \models A$ or $a \models B$

(iv). $a \models A \to B$ iff for all b, $c \in K$, $(Rabc \ \& \ b \models A) \Rightarrow c \models B$

A formula A is B_{K+} *valid* ($\models_{B_{k+}} A$) iff $a \models A$ for all $a \in K$ in all models.

Note that the postulates

$$\text{P4. } Rabc \Rightarrow b \leq c$$

$$\text{P5. } (a \leq b \ \& \ b \leq c) \Rightarrow a \leq c$$

and

$$\text{P6. } R^2abcd \Rightarrow Rbcd$$

are immediate in all B_{K+} models.

Regarding semantic consistency (soundness), the proof that all theorems of B_{K+} are valid is left to the reader (see, for example, [2] or [6] for a general strategy).

A final note. As it is known, there is a set of "designated points" in the standard semantics for relevance logics (see the two items just quoted above). It is in respect of this set that d1 is introduced and wff are evaluated. The absence of this set in B_{K+} semantics (and the corresponding changes in d1 and the definition of validity) are the only (but crucial) differences between B_{+} models and B_{K+} models.

4. Completeness of $\mathbf{B}_{K+}$

We begin by recalling some definitions:

A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all the theorems of B_{K+} belong to a. Finally, a is *null* iff no wff belongs to a.

Now, we define the B_{K+} canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and a, b, $c \in K^T$, $R^T abc$ iff if $A \rightarrow B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime non-null theories and R^C be the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the B_{K+} canonical model is the triple ¡K^C, R^C, $\vDash^C$¿.

Next, we sketch a proof of the completeness theorem.

Lemma 4.1 *If a is a non-null theory, then a is regular.*

Proof. Let $A \in a$ and B be a theorem. By the K rule, $A \rightarrow B$ is a theorem. So, $B \in a$. □

Lemmas 4.2-4.6 below are an easy adaptation of the corresponding B_+ lemmas (see, e.g., [6]) to the case of non-null theories (as it is known, theories are not necessarily non-null in the B_+ canonical model and, in fact, in the canonical model of any standard relevance logic).

Lemma 4.2 *Let A be any wff, a, a non-null element in K^T and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4.3 *Let a be a non-null element in K^T, $b \in K^T$ and c a prime member in K^T such that $R^T abc$. Then, $R^T xbc$ for some $x \in K^C$ such that $a \subseteq x$.*

Lemma 4.4 *Let $a \in K^T$, b a non-null element in K^T and c a prime member in K^T such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Now, we set

Definition 4.1 *Let a, $b \in K^T$. Then, $a \leq^T b$ iff $R^T xab$ and $x \in K^C$.*

We have

Lemma 4.5 *$a \leq^T b$ iff $a \subseteq b$.*

And consequently,

Lemma 4.6 *$a \leq^C b$ iff $a \subseteq b$.*

Note that b and c in lemma 4.3 and a and c in lemma 4.4 need not be non-null. On the other hand, lemma 4.7 below follows immediately from lemma 4.2.

Lemma 4.7 *If $\nvdash_{B_{K+}} A$, then there is some $x \in K^C$ such that $A \notin x$.*

Lemma 4.8 *Let a, b be non-null theories. The set $x = \{B \mid \exists A[A \to B \in a$ and $A \in b]\}$ is a non-null theory such that $R^T abx$.*

Proof. It is easy to prove that x is a theory such that $R^T abx$. We prove that x is non-null. Let $A \in b$. By lemma 4.1, $A \to A \in a$. So, $A \in x$ by $R^T abx$. □

The following three lemmas are proved similarly as in the standard semantics (use lemma 4.8 in the proof of the canonical adequacy of clause (iv)).

Lemma 4.9 *The postulates P1, P2 and P3 hold in the B_{K+} canonical model.*

Lemma 4.10 *$\vDash^C$ is a valuation relation satisfying conditions (i)-(iv) above.*

Lemma 4.11 *The canonical model B_{K+} is in fact a model.*

By lemmas 4.7 and 4.11, we have

Theorem 4.1 (Completeness of B$_{K+}$) *If $\vDash_{B_{K+}} A$, then $\vdash_{B_{K+}} A$.*

5. $\mathbf{B}_{K+}$ with minimal negation: the logic $\mathbf{B}_{Km}$

The logic B_{Km} is an extension of the language of B_{K+} with the propositional falsity constant F. We add the constant F to the positive language and define

$$\neg A =_{df} A \to F$$

No new axioms, however, are added.

A B_{Km}-*model* is a quadruple ¡K, S, R, $\vDash$¿ where K, R and $\vDash$ are defined similarly as in a B_{K+} model and S is a non-empty subset of K. The clauses

$$\text{(v).}\ \ (a \leq b \ \&\ a \vDash F\ ) \Rightarrow b \vDash F$$
$$\text{(vi).}\ \ a \vDash F \text{ iff } a \notin S$$

are added to (i)-(iv). A is B_{Km} valid ($\vDash_{B_{Km}} A$) iff $a \vDash A$ for all $a \in K$ in all models. Semantic consistency of B_{Km} follows immediately from that of B_{K+}. Moreover, we note that F is not valid (in fact, it is insatisfiable). Let $\mathcal{M}$ be any model and $a \in S$. Then, $a \nvDash F$.

Turning to completeness, we define the canonical model as the structure ¡K^C, S^C, R^C, $\vDash^C$¿ where K^C, R^C, $\vDash^C$ are defined similarly as in the B_{K+} canonical model, and S^C is interpreted as the set of all consistent prime non-null theories, a theory being consistent if $F \notin a$. In order to prove completeness, we have to prove that clauses (v) and (vi) are canonically valid and that S^C is not empty. Now, clauses (v) and (vi) are

$$\text{(v').}\ \ (a \subseteq b \ \&\ F \in a\ ) \Rightarrow F \in b$$
$$\text{(vi').}\ \ F \in a \text{ iff } F \in a$$

when read canonically (cf. definition of B_{K+} canonical model and lemma 4.6). So, there is nothing to prove. On the other hand, let B_{Km} be the set of its theorems. As $\nvDash_{B_{Km}} F$, $\nvdash_{B_{Km}} F$ by the soundness theorem, i.e., $F \notin \mathrm{B}_{Km}$. Then, by lemma 4.2, there is a consistent prime theory x such that $F \notin x$. So, we have

Theorem 5.1 (Completeness of $\mathbf{B}_{Km}$) *If $\vDash_{B_{Km}} A$, then $\vdash_{B_{Km}} A$.*

On the meaning of the constant F in B_{Km}, we prove

Proposition 5.1 *A theory a is inconsistent iff for some theorem $\neg B$, $B \in a$.*

Proof. (1) Suppose a inconsistent. Then, $F \in a$. But $\vdash \neg F$, by A1. (2) Suppose $B \in a$ for some theorem $\neg B$. By definition, $\vdash B \to F$. So, $F \in a$. □

In other words, a is inconsistent if it contains the argument of a negative formula that is a theorem.

6. $\mathbf{B}_{K+}$ with weak contraposition and weak double negation. The logic $\mathbf{B}_{Kcdn}$

The logic B_{Kcdn} is an extension of B_{Km}. In order to axiomatize B_{Kcdn}, we consider the following axioms

$$\begin{aligned}
&\text{A7. } (A \to B) \to [(B \to F) \to (A \to F)]\\
&\text{A8. } (B \to F) \to [(A \to B) \to (A \to F)]\\
&\text{A9. } A \to [(A \to F) \to F]\\
&\text{A10. } [A \to (B \to F)] \to [B \to (A \to F)]\\
&\text{A11. } B \to [[A \to (B \to F)] \to (A \to F)]
\end{aligned}$$

and the rule (Permutation F)

$$\text{Perm } F\text{: If } \vdash A \to (B \to F)\text{, then } \vdash B \to (A \to F)$$

Then, the result of adding any of the six following groups (a-f) to B_{Km} axiomatize B_{Kcdn} in an equivalent way:

a: A8, A10
b: A10, A11
c: A7, A8, A9
d: A7, A8, Perm F
e: A7, A9, A11
f: A7, A11, Perm F

We prove

Proposition 6.1 *The six possibilities (a-f) just mentioned provide equivalent axiomatizations of B_{Kcdn}.*

Proof. We leave to the reader the proof of the following facts:

1. Given B_{Km} and A7, A9 and Perm F are equivalent.
2. Given B_{Km} and A9, A8 and A11 are equivalent.
3. Given B_{Km} and A10, A9 is derivable.
4. Given B_{Km}, A10 and A9, A7 is derivable.
5. Given B_{Km}, A7 and A9, A10 is derivable.

Then, c and d, and e and f are equivalent by 1; moreover, c, d, e and f are equivalent by 2. Now, it follows immediately from 2 y 3 that A8 and A11 are equivalent, given B_{Km} and A10. So, a and b are equivalent and, on the other hand, b $\Rightarrow$ e by 3 and 4. Finally, e $\Rightarrow$ b, by 5. Therefore, The six possibilities are equivalent. □

7. Semantics for $\mathbf{B}_{Kcdn}$

Consider the following list of semantical postulates

P7. $R^2abcd \mathbin{\&} d \in S \Rightarrow (\exists x \in K)(\exists y \in S)(Racx \mathbin{\&} Rbxy)$
P8. $R^2abcd \mathbin{\&} d \in S \Rightarrow (\exists x \in K)(\exists y \in S)(Rbcx \mathbin{\&} Raxy)$
P9. $Rabc \mathbin{\&} c \in S \Rightarrow (\exists x \in K)Rbax$
P10. $R^2abcd \mathbin{\&} d \in S \Rightarrow (\exists x \in S)R^2acbx$
P11. $R^2abcd \mathbin{\&} d \in S \Rightarrow (\exists x \in S)R^2bcax$

Consider now the following four groups formed by combining the postulates in the list:

a: P8, P10
b: P10, P11
c: P7, P8, P9
d: P7, P9, P11

Next, we define B_{Kcdn} models. A B_{Kcdn} *model* is exactly like a B_{Km} model save for the addition of any of the four groups of postulates above. We prove that the four possibilities are equivalent.

Lemma 7.1 *Let a, b be non-null members in K^T, c, a consistent member in K^T and $R^T abc$. Then, a and b are consistent too.*

Proof.

1. By reductio, suppose that a is not consistent. Then, $F \in a$. Let $A \in b$. By T10, $A \to F \in a$, and so, by $R^T abc$, $F \in c$ contradicting the consistency of c.

2. Suppose that b is not consistent. Then, $F \in b$. By A1 and lemma 4.1, $F \to F \in a$. So, by $R^T abc$, $F \in c$ contradicting the consistency of c.

□

We now prove the equivalence of the four possibilities

Proposition 7.1 *Given B_{Km} semantics, the groups a-d referred to above are equivalent.*

Proof. We first prove that b and c are equivalent.

1. b ⇒ c

   (a) *P9 is derivable*: suppose $Rabc$, $c \in S$. By P1, d1, $Rxaa$ for some $x \in K$. By d2, R^2xabc; by P11, R^2xbay for some $y \in S$ and, by P11, R^2baxz for some $z \in S$. Then, $Rbau$, $Ruxz$ by d2. Finally, by lemma 7.1, $u \in S$.

   (b) *P7 is derivable*: suppose R^2abcd, $d \in S$. By P10, $Racx$, $Rxby$ for some $y \in S$. Then, by P9, $Rbxz$ for some $z \in S$.

   (c) *P8 is derivable*: proof similar to the previous one.

2. c ⇒ b

   (a) *P10 is derivable*: suppose R^2abcd, $d \in S$. By P7, $Racx$, $Rbxy$ for some $y \in S$. By P9, $Rxbz$ for some $z \in S$. So, R^2acbz by d2.

   (b) *P11 is derivable*: proof similar to the previous one. Use P8.

   Next, we prove that a and b are equivalent

3. a ⇒ b

(a) *P9 is derivable*: suppose $Rabc$, $c \in S$. By P1, d1, $Rxaa$ for some $x \in K$. By d2, R^2xabc; by P10, R^2xbay for some $y \in S$. Then, $Rbaz$, $Rxzu$ for some $u \in S$ by P8. Finally, $z \in S$ by lemma 7.1.

(b) *P11 is derivable*: suppose R^2abcd, $d \in S$. By P8, $Rbcx$, $Raxy$ for some $y \in S$. By P9, $Rxaz$ for some $z \in S$. So, R^2bcaz by d2.

4. b $\Rightarrow$ a

   (a) *P8 is derivable*: suppose R^2abcd, $d \in S$. By P11 and d2, $Rbcy$, $Ryax$ for some $x \in S$. Then, by P9, $Rayz$ for some $z \in S$ (note that P9 is derivable, cf. 1(a)).

Finally, we prove c $\Leftrightarrow$ d

5. c $\Rightarrow$ d. We have to prove that P11 is derivable (cf. 2(b))

6. d $\Rightarrow$ c. We have to prove that P8 is derivable (cf. 4(a)).

$\square$

Now, A is B_{Kcdn} valid ($\vDash_{B_{Kcdn}} A$) iff $a \vDash A$ for all $a \in K$ in all models. The proof that all theorems of B_{Kcdn} are valid is left to the reader. Anyway, we note that the validity of A7, A8, A9, A10, A11 and Perm F is proved with P7, P8, P9, P10, P11 and P9, respectively.

8. Completeness of $\mathbf{B}_{Kcdn}$

The definition of the B_{Kcdn} canonical model is similar to that of the B_{Km} canonical model. Next, we prove a useful lemma:

Lemma 8.1 *Let $a \in K^T$. Then, a is inconsistent iff a contains the negation of a theorem.*

Proof.

1. Suppose a is inconsistent. Then, $F \in a$, and by A9, $(F \to F) \to F \in a$.

2. Let A be a theorem and $A \to F \in a$. By A9, $(A \to F) \to F$ is a theorem, so $F \in a$.

□

Now, as in B_{Km}, it can be proved

Lemma 8.2 *S^C is not empty.*

Next, we have

Lemma 8.3 *Let a, b, c be non-null members in K^T and d a consistent member in K^T. Further, let R^2abcd. Then, for some x in K^C and $y \in S^C$, we have*

1. *R^Tacx and R^Tbxy*
2. *R^Tbcx and R^Taxy*
3. *R^Tacx and R^Txby*
4. *R^Tbcx and R^Txay*

Now, let a, b be non-null members in K^T, c a consistent element in K^T and R^Tabc. Then, for some x in S^C, we have

5. *$c \subseteq x$ and R^Tbax.*

Proof.

1. Assume the hypothesis of the lemma. By d2, we have R^Tabz and R^Tzcd for some $z \in K^T$. Define (cf. lemma 4.8) the non-null theories

$$u = \{B| \ \exists A[A \to B \in a \text{ and } A \in c]\}$$

and

$$w = \{B| \ \exists A[A \to B \in b \text{ and } A \in u]\}$$

such that R^Tacu and R^Tbuw. Next, we prove that w is consistent. Suppose $F \in w$. By definitions of u and w, $B \to A \in a$, $A \to F \in b$ and $B \in c$. By A7, $(A \to F) \to (B \to F) \in a$. Given R^Tabz and $A \to F \in b$, $B \to F \in z$. Given R^Tzcd and $B \in c$, $F \in d$ contradicting the hypothesis. Now, apply lemmas 4.2 and 4.4 to define x in K^C and y in S^C such that R^Tacx and R^Tbxy.

Proof of cases 2-5 are similar using A8, A10, A11 and A9, respectively. □

Lemma 8.4 *The canonical postulates hold in the B_{Kcdn} canonical model.*

Proof. Immediate by lemma 8.3. □

Lemma 8.5 *Clauses (v) and (vi) hold in the canonical model.*

Proof. By definitions, as in B_{Km}. □

Lemma 8.6 *The B_{Kcdn} canonical model is indeed a model.*

Proof. Lemmas 4.9, 4.10, 8.2, 8.4 and 8.5 □

Now, we can prove

Theorem 8.1 (Completeness of $\mathbf{B}_{Kcdn}$) *If $\vDash_{B_{Kcdn}} A$, then $\vdash_{B_{Kcdn}} A$.*

Proof. Given that an analogue of lemma 4.7 for B_{Kcdn} is immediate, completeness follows, by lemma 8.6. □

9. $\mathbf{B}_{Kcdn}$ with the reductio axiom: the logic $\mathbf{B}_{Kcdnr}$

In order to axiomatize B_{Kcdnr}, we add to B_{Kcdn} any of the following axioms or rules of inference:

A12. $[A \wedge (A \to F)] \to F$
A13. If $\vdash A \to B$ and $\vdash A \to (B \to F)$, then $\vdash A \to F$
A14. $[A \to (A \to F)] \to (A \to F)$
A15. $A \to [[A \to (A \to F)] \to F]$
A16. If $\vdash A \to B$, then $\vdash [A \to (B \to F)] \to (A \to F)$
A17. If $\vdash A \to (B \to F)$, then $\vdash (A \to B) \to (A \to F)$
A18. $[A \to (B \to F)] \to [(A \wedge B) \to F]$
A19. $(A \to B) \to [[A \wedge (B \to F)] \to F]$
A20. $(A \wedge B) \to [[A \to (B \to F)] \to F]$
A21. $[(A \wedge B) \to F)] \to [(A \to B) \to F]$
A22. $[A \to (B \to F)] \to [(A \to B) \to (A \to F)]$
A23. $(A \to B) \to [[A \to (B \to F)] \to (A \to F)]$

We prove

Proposition 9.1 *A12-A23 are equivalent, given B_{Kcdn}.*

Proof. We prove

1. *A23 is derivable from A12*: We can proceed as follows. Using A3 and the K rule prove

$$(a).\ \vdash (A \to B) \to [A \to (A \wedge B)]$$

By (a) and A7

$$(b).\ \vdash [A \to (A \to F)] \to [[[A \wedge (A \to F)] \to F] \to (A \to F)]$$

On the other hand, by A9 and A10,

$$(c).\ \text{If } \vdash A, \text{ then } \vdash [A \to (B \to F)] \to (B \to F)$$

By (b), (c) and A12

$$\text{(d).} \ \vdash [A \to (A \to F)] \to (A \to F)$$

which easily give us

$$\text{(e). If } \vdash A \to B\text{, then } \vdash [A \to (B \to F)] \to (A \to F)$$

and

$$\text{(f).} \ \vdash (A \wedge B) \to [[A \to (B \to F)] \to F]$$

Finally, A23 is easily derivable from (a) and (f).

2. *A12 is derivable from any of A13-A22*: All cases are easy, and the proof is left to the reader.

3. *A12-A22 are derivable from A23*: All cases are easy except maybe A22, which is proved as follows:

   1. $[A \to (B \to F)] \to [[A \to [(B \to F) \to F]] \to (A \to F)]$ A23
   2. $[(B \to F) \to (A \to F)] \to [A \to [(B \to F) \to F]]$ A10
   3. $\{[A \to [(B \to F) \to F]] \to (A \to F)\} \to$ $\{[(B \to F) \to (A \to F)] \to (A \to F)\}$ Suffixing, 2
   4. $[A \to (B \to F)] \to [[(B \to F) \to (A \to F)] \to (A \to F)]$ Transitivity, 1, 2

Now, by A7 and 4

$$5. \ [A \to (B \to F)] \to [(A \to B) \to (A \to F)]$$

□

10. Semantics for $\mathbf{B}_{Kcdnr}$

Models for B_{Kcdnr} are defined similarly as those for B_{Kcdn} but with the addition of any of the following postulates

P12. $a \in S \Rightarrow (\exists x \in S)Raax$

P13. $Rabc$ & $c \in S \Rightarrow (\exists x \in S)Rcbx$

P14. $Rabc$ & $c \in S \Rightarrow (\exists x \in S)R^2baax$

P15. $Rabc$ & $c \in S \Rightarrow (\exists x \in S)Rbcx$

P16. $Rabc$ & $c \in S \Rightarrow (\exists x \in S)R^2abbx$

P17. $Rabc$ & $c \in S \Rightarrow (\exists x \in K)(\exists y \in S)(Rabx$ & $Rbxy)$

P18. $Rabc$ & $c \in S \Rightarrow (\exists x \in K)(\exists y \in S)(Rbax$ & $Raxy)$

P19. R^2abcd & $d \in S \Rightarrow (\exists x, y \in K)(\exists z \in S)(Racx$ & $Rbcy$ & $Rxyz)$

P20. R^2abcd & $d \in S \Rightarrow (\exists x, y \in K)(\exists z \in S)(Racx$ & $Rbcy$ & $Ryxz)$

These postulates are equivalent in the presence of B_{Kcdn} models, as we prove below.

Proposition 10.1 *The postulates P12-P20 are equivalent given B_{Kcdn} semantics.*

Proof. We prove

1. *P12 $\Rightarrow$ P19, P20*

   We start by proving:

   (a) P12 $\Rightarrow$ P15: suppose $Rabc$ and $c \in S$. By P12, $Rccx$ for some $x \in S$. By d2, R^2abcx. Applying P8, $Rbcy$, $Rayz$ for some $y \in K$, $z \in S$. But $y \in S$ by lemma 7.1.

   (b) P15 $\Rightarrow$ P18: suppose $Rabc$ and $c \in S$. By P9, $Rbax$ for some $x \in S$. Then, $Raxy$ for some $y \in S$, by P15.

   Now,

   (c) P12 $\Rightarrow$ P19: suppose R^2abcd and $d \in S$. By P10, there is some $x \in K$ and $y \in S$ such that $Racx$ and $Rxby$. By $Rxby$ and P18, we have $Rbxz$, $Rxzu$ for some $u \in S$; by $Racx$ and d1, $c \leq x$, that is, $Rbcz$ by P3 and $Rbxz$. Therefore, we have $Racx$, $Rbcz$ and $Rxzu$ with $u \in S$ as it was required.

(d) P12 $\Rightarrow$ P20: The proof is similar.

2. *P12 $\Rightarrow$ P13-P18*

(a) P12 $\Rightarrow$ P13: suppose $Rabc$ and $c \in S$. By P12, $Rccx$ for some $x \in S$. Applying d2, R^2abcx. By P8, $Rbcy$, $Rayz$ for some $z \in S$. Finally, $Rcbz$ for some $z \in S$ by $Rbcy$ and P9.

(b) P12 $\Rightarrow$ P14: Suppose $Rabc$ and $c \in S$. By P9, $Rbay$ for some $y \in S$. By P13, $Ryax$ for some $x \in S$.

(c) P12 $\Rightarrow$ P15: cf. 1(a).

(d) P12 $\Rightarrow$ P16: proof similar to that of 2(b).

(e) P12 $\Rightarrow$ P17: proof similar to that of 2(b).

(f) P12 $\Rightarrow$ P18: cf. 1(b).

3. *P13 (P14, P15, P16, P17, P18) $\Rightarrow$ P12:*

(a) P13 $\Rightarrow$ P12: suppose $a \in S$. By P1 and d1, $Rxaa$ for some $x \in K$. Then, by P13, $Raaz$ for some $z \in S$.

(b) P14 $\Rightarrow$ P12: suppose $a \in S$. By P1 and d1, $Rxaa$ fro some $x \in K$. By P9, $Raxz$ for some $z \in S$. Applying P14, $Rxay$, $Ryau$ for some $u \in S$. By $Rxay$ and d1, $a \leq y$. So, $Raau$ by P3 and $Ryau$.

(c) P15 $\Rightarrow$ P12: proof similar to that of 3(a).

(d) P16 $\Rightarrow$ P12: proof similar to that of 3(b)

(e) P17 $\Rightarrow$ P12: proof similar to that of 3(b)

(f) P18 $\Rightarrow$ P12: proof similar to that of 3(b)

4. *P19 $\Rightarrow$ P12 (P20 $\Rightarrow$ P12):* we start by proving:

(a) P19 $\Rightarrow$ P18: suppose $Rabc$ and $c \in S$. By P1 and d1, $Rxcc$ for some $x \in K$. By P9, $Rcxy$ for some $y \in S$. So, by d2, R^2abxy whence R^2bxaz for some $z \in S$, by P11. Applying P19, we have $Rbau$, $Rxaw$, $Ruww'$ for some $w' \in S$. As $a \leq w$ ($Rxaw$, d1), $Ruaw'$ ($Ruww'$); then by P9, $Rauw''$ for some $w'' \in S$. Therefore, we have $Rbau$, $Rauw''$ with $w'' \in S$ as it was required.

Now,

(b) P19 $\Rightarrow$ P12: it follows from 4(a) and 3(f)

(c) P20 $\Rightarrow$ P12: proof similar to that of 4(b).

It remains to prove that P19 and P20 are equivalent, which is easy by P9. □

Now, A is B_{Kcdnr} valid ($\vDash_{B_{Kcdnr}} A$) iff $a \vDash A$ for all $a \in K$ in all models. The proof that all the theorems of B_{Kcdnr} are valid is left to the reader. Anyway, we remark that the validity of A12-A23 may be proved as follows (but there are more possibilities, cf. proposition 10.1):

1. A12, A13: P12
2. A14, A16: P13
3. A15, A20: P14
4. A17: P15

Next, A18, A19, A21, A22 and A23 are proved with P16, P17, P18, P19 and P20, respectively.

11. Completeness of $\mathbf{B}_{Kcdnr}$

The B_{Kcdnr} canonical model is defined similarly as the corresponding one for B_{Kcdn}. Then, we prove

Lemma 11.1 *1. Let $a \in S^C$, then for some x in S^C, $R^C aax$.*

2. Let a, b be non-null members in K^T and c a consistent element in K^T. Further, let $R^T abc$. Then, for some y in K^T and x in S^C, we have

(a) $R^T cbx$

(b) $R^{T2} baax$

(c) $R^T bcx$

(d) $R^{T2} abbx$

(e) $R^T aby$ and $R^T byx$

(f) $R^T bay$ and $R^T ayx$

3. *Let a, b, c be non-null members in K^T and d a consistent element in K^T. Further, let $R^{T2}abcd$. Then, for some x, y in K^T and z in S^C, we have*

   (a) *$R^T acx$ and $R^T bcy$ and $R^T xyz$*

   (b) *$R^T acx$ and $R^T bcy$ and $R^T yxz$*

Proof. Similar to that of lemma 8.3. Case 1 can be proved with A16, A20, A17, A18, A19 and A21, respectively. But there are other possibilities (cf. proposition 9.1). Finally, let us show how to prove cases 3(a) and 3(b).

We prove case 3(a): Let a, b, c be non-null members in K^T and d a consistent member in K^T. Further, let $R^{T2}abcd$, i.e, $R^T abx$ and $R^T xbd$ for some $x \in K^T$. Define the theories y, z and u such that $R^T acy$, $R^T bcz$ and $R^T yzu$. Note that y, z and u are non-null (cf. lemma 4.8). We prove that u is consistent. Suppose it is not. Then, $B \to (A \to F) \in a$, $C \to A \in b$ for some wffs A and $B \in c$, $C \in c$. By A1, $(C \to A) \to (C \to A) \in a$. So, $C \to A \in x$ ($Rabx$, $C \to A \in b$) and next $A \in d$ ($Rxcd$, $B \in c$). On the other hand, $B \to B \in x$ (A1). Then, $B \in d$ ($Rxcd$, $B \in c$) and consequently, $A \wedge B \in d$.

We now prove $\neg(A \wedge B) \in d$ contradicting the consistency of d. Given that $B \to (A \to F) \in a$, we have $A \to (B \to F) \in a$ by A10 and, by A18, $(A \wedge B) \to F \in a$. Now, let $D \in b$. By T10, $(A \wedge B) \to \neg D \in a$. By A7, $\neg\neg D \to \neg(A \wedge B) \in a$. As $\neg\neg D \in b$ (A9), $\neg(A \wedge B) \in x$ ($Rabx$). By T10, $(A \wedge B) \to \neg C \in x$. Next, by A7, $\neg\neg C \to \neg(A \wedge B) \in x$. As $\neg\neg C \in c$ (A9), by $Rxcd$, $\neg(A \wedge B) \in d$.

Case 3(b) of lemma 11.1 is proved similarly. □

Now, given that an analogue of lemma 8.4 is immediate, we have

Theorem 11.1 (Completeness of $\mathbf{B}_{Kcdnr}$) *If $\vDash_{B_{Kcdnr}} A$, then $\vdash_{B_{Kcdnr}} A$.*

We finish this section with a proposition simplifying the standard postulates for relevance logics P19 and P20 (see, e.g., [9]).

Proposition 11.1 *Given B_{Kcdnr}, the postulates*

P19'. $Rabcd \,\&\, d \in S \Rightarrow (\exists x \in K)(\exists y \in K)(\exists z \in S)(Racx \,\&\, Rbxy \,\&\, Rxyz)$

and

P20'. $Rabcd \,\&\, d \in S \Rightarrow (\exists x \in K)(\exists y \in K)(\exists z \in S)(Racx \,\&\, Rbxy \,\&\, Ryxz)$

are equivalent to P19 and P20, respectively.

Proof.

1. *P19' ⇒ P19*: suppose R^2abcd, $d \in S$. By P19', $Racx$, $Rbxy$, $Rxyz$ with $z \in S$. By P4, $c \leq x$, and by P3, $Rbcy$. So, $Racx$, $Rbxy$, $Rxyz$ and $z \in S$ as it was required.

2. *P19 ⇒ P19'*: As it is shown in section 10, P19 implies postulates P12-P20. We now prove P19'. Suppose R^2abcd and $d \in S$. By P7, $Racx$, $Rbxy$ and $y \in S$. By P15, $Rxyz$ for some $z \in S$. Therefore, $Racx$, $Rbxy$, $Rxyz$ for some $x \in K$, $y \in K$, $z \in S$, as it was required.

□

12. $\mathbf{B}_{K+}$ with intuitionistic negation: the logic $\mathbf{B}_{Kj}$

To define B_{Kj}, we add to B_{Kcdnr} the axiom

$$\text{A24. } F \to A$$

We note that, in addition to T1-T20, T21-T28 are now provable (see Appendix A).

13. Semantics for $\mathbf{B}_{Kj}$

A B_{Kj} *model* is a triple ¡K, R, $\vDash$¿ where K is a non-empty set, R is a ternary relation defined on K and $\vDash$ is valuation relation from K to the sentences of the positive language extended with F. The following definitions and postulates hold for all a, b, c, $d \in K$: d1, d2, P1-P3 (as in B_{Kcdnr} models) and also

$$\text{P21. } R^2abcd \Rightarrow (\exists x \in K)R^2acbx$$
$$\text{P22. } R^2abcd \Rightarrow (\exists x \in K)R^2bcax$$
$$\text{P23. } (\exists x \in K)Raax$$

On the other hand, clauses (i)-(iv) hold, but clause (vi) is replaced with

(vii). $a \nvDash F$ for all $a \in K$ in all models

A is valid ($\vDash_{B_{Kj}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

Semantic consistency is proved similarly as before, but now, it is obvious that A24 is valid. It is easy to see that a B_{Kj} model is like a B_{Kcdnr} model save for the omission of all references to the set S: now, $S = K$.

14. Completeness of $\mathbf{B}_{Kj}$

The B_{Kj} canonical model is the structure ¡K^C, R^C, $\vDash^C$¿ where R^C and $\vDash^C$ are defined as in the B_{Kcdnr} canonical model, but K^C is now the set of all non-null consistent prime theories, a theory being consistent as before (a is inconsistent iff $F \in a$). We begin by proving some previous lemmas.

Lemma 14.1 *a is consistent iff $A \wedge \neg A \notin a$ for some wff A.*

Proof. Suppose a is inconsistent. Then, $F \in a$. By A24, $F \to A$, $F \to \neg A$ are theorems. By A3, $F \to (A \wedge \neg A)$ is also a theorem. So, $A \wedge \neg A \in a$. Suppose now $A \wedge \neg A \in a$. Then, $F \in a$ by T23. Therefore, a is inconsistent. □

Next, we define

Definition 14.1 *a is degenerate if all wffs belong to a.*

We prove

Lemma 14.2 *a is consistent iff a is non-degenerate.*

Proof. Suppose a is consistent. Then, $F \notin a$. So, a is non-degenerate. Suppose now a is non-degenerate. If $F \in a$, then, by A24, any wff belongs to a. Therefore, a is consistent. □

We note that by lemmas 8.1, 14.1 and 14.2, we have

Lemma 14.3 *a is inconsistent ($F \in a$) iff a contains the negation of a theorem iff for some wff A, $A \wedge \neg A \in a$ iff a is degenerate.*

Now, it is clear that in order to prove the completeness of B_{Kj}, most of the previous lemmas must be modified: given the B_{Kj} model, we have to prove that all theories defined in these lemmas are consistent. In particular, lemmas 4.2-4.6, 8.3-8.6 and 11.1 must be modified, and, on the other hand, note that 4.8 does not hold generally (if x has to be consistent) and that 8.2 is dropped.

Well, with the aid of lemma 14.3, it is not difficult to prove that the required modifications of the above lemmas hold. Let us, for instance, prove lemma 4.3. Now, it would read

Lemma 4.3': Let $R^T abc$, a a maximal non-null consistent element in K^T, $b \in K^T$ and c a prime member in K^T. Then, $R^T xbc$ for some x in K^T such that $a \subseteq x$.

Proof. Define from a a maximal non-null consistent theory x such that $a \subseteq x$ and $R^T xbc$. Suppose x is not prime. Then, $A \vee B \in x$, $A \notin x$, $B \notin x$ for some wffs A, B. Define

$$[x, A] = \{C | \ \exists D[D \in x \text{ and } \vdash (A \wedge D) \to C]\}$$

Define $[x, B]$ similarly. It is easy to prove that $[x, A]$ and $[x, B]$ are non-null theories strictly including x. By the maximality of x, there are three possible situations:

1. $[x, A]$ and $[x, B]$ are inconsistent
2. not - $R^T[x, A]bc$ and not - $R^T[x, B]bc$
3. not - $R^T[x, A]bc$ and $[x, B]$ is inconsistent or not - $R^T[x, B]bc$ and $[x, A]$ is inconsistent.

We prove that each one of these situations is untenable:

1. $[x, A]$ and $[x, B]$ are inconsistent. Then, $F \in [x, A]$, $F \in [x, B]$ (lemma 14.3) whence it is easy to show $F \in x$.
2. not - $R^T[x, A]bc$ and not - $R^T[x, B]bc$. But, then, it is easy to show that c is not prime.
3. not - $R^T[x, A]bc$ and $[x, B]$ is inconsistent or not - $R^T[x, B]bc$ and $[x, A]$ is inconsistent: We prove that the first alternative is untenable

(proof of the second one is similar): suppose not - $R^T[x, A]bc$. By definitions, $\vdash (A \wedge C) \to (D \to E)$ with $C \in x$, $D \in b$, $E \notin c$. Suppose $[x, B]$ is inconsistent. Then, $D \to E \in [x, B]$ (lemma 14.3). By definitions, $\vdash (B \wedge C\prime) \to (D \to E)$ with $C\prime \in x$. Then, it is easy to show that $D \to E \in x$. But given $R^T xbc$ and $D \in b$, $E \in c$, a contradiction. Therefore, x is prime.

□

Now, given that the canonical clause (vi) is immediate, we have

Theorem 14.1 (Completeness of $\mathbf{B}_{Kj}$) *If* $\vDash_{B_{Kj}} A$, *then* $\vdash_{B_{Kj}} A$.

A. Appendix. Theorems of $\mathbf{B}_{K+}$, $\mathbf{B}_{Km}$, $\mathbf{B}_{Kcdn}$, $\mathbf{B}_{Kcdnr}$ and $\mathbf{B}_{Kj}$

Theorems of $\mathbf{B}_{K+}$

t1. $B \to (A \to A)$	A1, K
t2. $(A \to B) \to [A \to (A \wedge B)]$	A1, A3, t1

Theorems of $\mathbf{B}_{Km}$

T1. $\vdash A \to B \Rightarrow \vdash \neg B \to \neg A$	Suf.
T2. $\vdash \neg B \Rightarrow \vdash (A \to B) \to \neg A$	Pref.
T3. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$	By B_+
T4. $(\neg A \vee \neg B) \to \neg(A \wedge B)$	By B_+
T5. $\neg F$	A1
T6. $A \to \neg F$	A1, K

Theorems of $\mathbf{B}_{Kcdn}$

T7. $(A \vee B) \to \neg(\neg A \wedge \neg B)$	T3, A7
T8. $(A \wedge B) \to \neg(\neg A \vee \neg B)$	T4, A7

T9.	$A \to [(A \to \neg B) \to \neg B]$	A10, A11
T10.	$F \to \neg A$	T6, A10
T11.	$\neg A \to (A \to \neg B)$	T10
T12.	$A \to (\neg A \to \neg B)$	A9, T11
T13.	$\neg A \to (B \to \neg A)$	T11, A10
T14.	$(A \vee \neg B) \to (\neg A \to \neg B)$	T12, T13
T15.	$(\neg A \vee \neg B) \to (A \to \neg B)$	T11, T13
T16.	$\neg A \to [(A \vee \neg B) \to \neg B]$	T11, K
T17.	$A \to [(\neg A \vee \neg B) \to \neg B]$	T12, K
T18.	$\neg\neg\neg A \to \neg A$	A7, A9

Theorems of $\mathbf{B}_{Kcdnr}$

In addition to T1-T18, A7-A23, we have

T19.	$\neg\neg(A \vee \neg A)$	A4, A7, A23
T20.	$(A \wedge \neg A) \to \neg B$	A12, T10

Theorems of $\mathbf{B}_{Kj}$

In addition to T1-T20 and A7-A24, we have

T21.	$\neg A \to (A \to B)$	A24
T22.	$A \to (\neg A \to B)$	A9, A24
T23.	$(A \wedge \neg A) \to B$	A12, A24
T24.	$(A \wedge \neg A) \leftrightarrow F$	A24, T23
T25.	$(A \vee B) \to (\neg A \to \neg B)$	T13, T22
T26.	$(\neg A \vee B) \to (A \to \neg\neg B)$	T13, T21
T27.	$\neg A \to [(A \vee B) \to B]$	T21, K
T28.	$A \to [(\neg A \vee B) \to B]$	T22, K

B. Appendix. Matrices

Consider the following axioms and rules of inference:

a1. $(A \to B) \to (B \to C) \to (A \to C)$
a2. $(B \to C) \to [(A \to B) \to (A \to C)]$

a3. $[A \to (A \to B)] \to (A \to B)$
a4. $A \to [(A \to B) \to B]$
a5. $[[(A \to B) \to C] \to (A \to B)] \to (A \to B)$
a6. $\vdash A \Rightarrow \vdash (A \to B) \to B$

Now, Lewis's positive modal logic $\mathrm{S5}_+$ can be axiomatized (see [3]) with adjunction, modus ponens, the K rule, a1, a2, a3, a5, a6 and A1-A6 of §2. And Anderson and Belnap's positive logic of relevance R_+ (without the connectives $\circ$ and t), can be axiomatized with adjunction, modus ponens, a1, a3, a4 and A1-A6 of §2.

Let R_{K+} be the result of adding the K rule to R_+; and let $\mathrm{S5}_j$ and R_j be the result of adding A8, A10, A12 and A24 to $\mathrm{S5}_+$ and R_{K+}, respectively. We note the following proposition:

Proposition B.1 *1. R_j is equivalent to intuitionistic logic.*

2. $S5_j$ is not included in Lewis's modal logic S5.

3. The axiom K, i.e.,

$$A \to (B \to A)$$

is not derivable in $S5_j$.

Proof. The proofs of 1 and 2 are left to the reader. A simple proof of 3 is as follows. Consider the following set of matrices where 2 is the only designated value and F is assigned the value 0:

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

This set of matrices verifies $S5_j$ but falsifies the K axiom when $v(A) = 1$ and $v(B) = 2$. So, note that any logic included in $S5_j$ (see Concluding remarks) lacks the K axiom. We find among these logics, e.g., E_j, The Positive Logic of Entailment E_+ with intuitionistic negation. □

C. Concluding remarks. Extending the logics

A number of logics can be defined from B_{Km}, B_{Kcdnr} or B_j. By way of example, consider the following semantic postulates:

p1. $R^2abcd \Rightarrow (\exists x \in K)(Racx \ \& \ Rbxd)$

p2. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \ \& \ Raxd)$

p3. $Rabc \Rightarrow R^2abbc$

p4. $Rabc \Rightarrow Rbac$

p5. $(\exists x \in K)Raxa$

The logic ML_{K+} is minimal positive logic of Anderson and Belnap (see [5]) ML_+ with the distributive laws and the K rule. It can be axiomatized by adding a1, a2 to B_{K+}. ML_{K+} models can be defined by adding p1 and p2 to B_{K+} models. Then, ML_{Km}, ML_{Kcdnr} and ML_{Kj} can be defined similarly as B_{Km}, B_{Kcdnr} and B_{Kj} were defined. The logic T_{K+} is positive Ticket Entailment T_+ with the K rule, it can be axiomatized by adding a3 to ML_{K+}. T_{K+} models are defined similarly as ML_{K+} models but with the addition of the postulate p3. The logic RW_{K+} is contractionless positive relevance logic RW_+ with the K rule. It can be axiomatized by adding a4 to ML_{K+}. RW_{K+} models are defined by adding p4 to ML_{K+} models. A last example. The logic E_{K+} is positive entailment logic E_+ with the K rule. E_{K+} models are defined by adding p5 to T_{K+} models. In fact, the logic E_{K+} is Lewis's modal logic S4 (see [3]). So, actually, E_j is $S4_+$ with intuitionistic negation. But we shall not pursue the study of the negation extensions of the logics here defined any further.

Notes

Work supported by the research project HUM2005-05707 of the Spanish Ministry of Education and Science.

We sincerely thank a referee of RML for his (her) comments on a previous version of this paper that helped to improve the presentation of the material.

References

[1] A.R. Anderson and N.D. Jr. Belnap, Entailment. The Logic of Relevance and Necessity, vol. I, Princeton University Press, Princeton 1975.

[2] A.R. Anderson, N.D. Jr. Belnap and J.M. Dunn, Entailment. The Logic of Relevance and Necessity, vol. II, Princeton University Press, Princeton 1992.

[3] I. Hacking, *What is strict implication*, Journal of Symbolic Logic **28** (1963), pp. 51–71.

[4] E.D. Mares, *CE is not a conservative extension of E*, Journal of Philosophical Logic **29** (2000), pp. 263–275.

[5] J.M. Méndez and G. Robles, *Relevance logics, paradoxes of consistency and the K rule. A quasi De Morgan negation* (submitted).

[6] J.M. Méndez, F. Salto, G. Robles, *Anderson and Belnap´s minimal positive logic with minimal negation*, Reports on Mathematical Logic **36** (2002), pp. 117–130.

[7] R.K. Meyer and R. Routley, *Classical Relevant Logics I*, Studia Logica **32** (1973), pp. 51–66.

[8] R.K. Meyer and R. Routley, *Classical Relevant Logics II*, Studia Logica **33** (1974), pp. 183–194.

[9] G. Robles, J.M. Méndez, F. Salto, *Minimal negation in the ternary relational semantics*, Reports on Mathematical Logic **39** (2005), pp. 47–65.

[10] G. Robles, J.M. Méndez, F. Salto, *Relevance logics, paradoxes of consistency and the K rule* (contributed paper presented at the Logic Colloquium 2006, Nijmegen, The Netherlands July 27-August 2) Bulletin of Symbolic Logic (abstract accepted).

[11] R. Routley et alii, Relevant Logics and their Rivals, vol. 1, Ridgeview Press, Ridgeview 1982.

Department of Philosophy and Logic
Universidad de Salamanca
Campus Unamuno, Edificio FES
E-37007 Salamanca, Spain

`gemm@usal.es` `sefus@usal.es`