REPORTS ON MATHEMATICAL LOGIC
42 (2007), 87–100

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PURELY EQUIVALENTIAL PROPOSITIONAL FORMULAS IN THE INTERMEDIATE GÖDEL-DUMMETT LOGIC

A b s t r a c t. We describe effectively all purely equivalential propositional formulas in the intermediate Gödel-Dummett logic.

1. Preliminaries

Consider the variety $\mathcal{E}_\omega$ generated by the algebra $\omega := (\mathbb{N}, \cdot)$, where $i \cdot j := \max(i, j)$ for $i \neq j$, and $i \cdot j := 1$ for $i = j$, $i, j \in \mathbb{N}$[1]. These variety is a subvariety of the variety of equivalential algebras $\mathcal{E}$. By an *equivalential algebra* we mean a grupoid $\mathbf{A} = (A, \leftrightarrow)$ that is a subreduct of a Brouwerian semilattice (or, equivalently, a Heyting algebra) with the operation $\leftrightarrow$ given by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. This notion was introduced

Received 11 January 2006
Keywords: Gödel-Dummett logic, intuitionistic equivalence, free algebras, equivalential algebras
AMS Classification: Primary: 03B55, Secondary: 03G25, 08B20 December 30, 2005

[1]In this paper, we set $\mathbb{N} = 1, 2, \ldots$

by Kabziński and Wroński in [6] as an algebraic counterpart of the equivalential fragment of intuitionistic propositional logic. The class $\mathcal{E}$ of all equivalential algebras is equationally definable by the following identities: $xxy = y$, $xyzz = xz\,(yz)$, $xy\,(xzz)\,(xzz) = xy$. (We adopt the convention of associating to the left and ignoring the symbol of equivalence operation.) The variety $\mathcal{E}$ is locally finite, since so is the variety of Brouwerian semilattices. Supplementing the axioms of equivalential algebras by the identity $xyy = x$ (or by the associativity law) we obtain $\mathcal{E}_2$, the smallest non-trivial subvariety of $\mathcal{E}$, which coincides with the class of Boolean groups. The variety $\mathcal{E}_\omega$ can be equationally defined by adding to the identities of $\mathcal{E}$ the identity $(x\,(yzz)\,(yzz))\,(x\,(zyy)\,(zyy))\,(x\,(yz)\,(yz)) = x$.

The variety $\mathcal{E}_\omega$ gives the algebraic semantics for the equivalential fragment of one of the most important intermediate logics: *Gödel-Dummett intermediate logic* **LC**. The propositional logic **LC**, which naturally turns up in different areas of logic and computer science, was introduced by Kurt Gödel in 1932 [4] and axiomatized later by Michael Dummett [2] by adding the linearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha)$ to intuitionistic propositional logic.

For the basic facts of equivalential algebras we refer the reader to [6], [8], and [5]. However, to make this paper self-contained we provide in this section all the necessary information. It is easy to show that the term $1 := xx$ is the constant unit term in $\mathcal{E}$. By a *filter* of $\mathbf{A}$ we mean a non-empty subset F of A such that for all $a, x \in A$:

(i) if $a \in F$, then $axx \in F$ and

(ii) if $a \in F$ and $ax \in F$, then $x \in F$.

The maps $\Phi\,(\mathbf{A}) \ni F \to \equiv_F := \{(a, b) \in A \times A : ab \in F\} \in \mathrm{Con}\,(\mathbf{A})$ and $\mathrm{Con}\,(\mathbf{A}) \ni \alpha \to 1/\alpha \in \Phi\,(\mathbf{A})$ establish an isomorphism between $\Phi\,(\mathbf{A})$, the lattice of all filters of $\mathbf{A}$, and the lattice $\mathrm{Con}\,(\mathbf{A})$ of all congruences of $\mathbf{A}$. Hence equivalential algebras are *1-regular*. Moreover, equivalential algebras are *congruence orderable*, that is,

$$[a) = [b) \text{ implies } a = b \text{ for all } a, b \in A\ ,$$

where $[c)$ is the *principal filter* generated by $c \in A$, i.e., the smallest filter containing c. Hence, $\mathcal{E}$ is a *Fregean variety*, see [5, 3]. Clearly, this property is inherited by all its subvarieties, including $\mathcal{E}_\omega$. The natural order $\leq_E$ in

A is defined by

$$a \leq_E b \text{ if and only if } b \in [a)\,, \text{ where } a, b \in A\ .$$

We call the algebras from $\mathcal{E}_\omega$ *linear*, because a subdirectly irreducible equivalential algebra belongs to $\mathcal{E}_\omega$ if and only if it is linearly ordered by the relation '$\leq_E$'. Moreover, the only finite subdirectly irreducible algebras in $\mathcal{E}_\omega$ are $\{\mathbf{m} : 2 \leq m\}$, where $\mathbf{m}$ $(m \in \mathbb{N})$ denotes the subalgebra of ω with the subuniversum $\{1, \dots, m\}$, and the order $\leq_E$ on these algebras is dual to the natural one. For more information on linear equivalential algebras we refer the reader to [7].

We say that $x, y \in A$ are *orthogonal* if $xyy = x$ and $yxx = y$. If a subset M of A consists of pairwise orthogonal elements, then it generates an associative subalgebra of $\mathbf{A}$. In particular, if $M = \{a_1, \dots, a_k\}$ we write $\otimes M$ for $a_1 \cdot \dots \cdot a_k$. (The order of elements in the sequence is irrelevant.) We put also $\otimes \emptyset = 1$.

We define a family of retractions $\{\& X : X \text{ is a finite subset of } A\}$ putting $x \& \emptyset = x$, $x \& (X \cup \{a\}) = (x \& X)\, aa$ for $x, a \in A$. For finite nonempty sets $B \subset Y \subset A$ we set $\otimes B \& Y := \otimes \{a \& Y : a \in B\}$. This notation is proper, because for every $b, c \in B$ the elements $(b \& Y)$ and $(c \& Y)$ are orthogonal, and so $\{a \& Y : a \in B\}$ generates an associative subalgebra in $\mathbf{A}$.

We say that $a \in A \setminus \{1\}$ is *irreducible* if and only if $axx \in \{a, 1\}$ for all $x \in A$ and denote the set of all irreducible elements in $\mathbf{A}$ by $I(\mathbf{A})$. For $a, b \in I(\mathbf{A})$ we have $a <_E b$ if and only if $baa = 1$, and incomparable elements in $I(\mathbf{A})$ are orthogonal. We introduce in $I(\mathbf{A})$ an equivalence relation, by putting $a \sim b$ if and only if ($x <_E a \Leftrightarrow x <_E b$ for every $x \in I(\mathbf{A})$). The importance of this notion is justified by the following *representation theorem*:

Theorem 1 *Let $\mathbf{A} = (A, \cdot) \in \mathcal{E}$ be finite. Then for every element $x \in A$ there exists exactly one subset $R(x) \subset I(\mathbf{A})$ such that:*

- *$R(x)$ is $\leq_E$-antichain;*
- *if $a, b \in R(x)$ and $a \neq b$, then $a \nsim b$;*
- *$x = \otimes R(x)$.*

Moreover, $R(x) = \{a \in I(\mathbf{A}) : a \geq_E x \text{ and } axx = a\}$.

Thus every element of $\mathbf{A}$ can be uniquely decomposed into an $\leq_E$-antichain of irreducible elements from different $\sim$ −equivalence classes.

Let $\mathrm{Fm}(\mathbf{A})$ denote the set of completely *meet-irreducible* elements in the lattice $\Phi(\mathbf{A})$. For each element $\eta \in \mathrm{Fm}(\mathbf{A})$ there exists a unique element $\eta^+ \in \Phi(\mathbf{A})$ such that $\eta < \alpha$ implies $\eta^+ \leq \alpha$ for every $\alpha \in \Phi(\mathbf{A})$. Observe that, if α is a filter, then α is a subuniverse of $\mathbf{A}$. Moreover, if $\eta \in \mathrm{Fm}(\mathbf{A})$, then $\mathbf{A}/\eta$ is subdirectly irreducible with the monolith η^+/η and $(\eta^+/\eta, \cdot)$ is a two-element Boolean group.

We shall need the following results on irreducible elements:

Lemma 2 *Let $\eta \in \mathrm{Fm}(\mathbf{A})$ and let $a, x \in A$. Then:*
(i) $a \in \eta^+$, $x \notin \eta^+$ implies $axx \in \eta$;
(ii) $a \in I(\mathbf{A}) \cap (\eta^+ \backslash \eta)$ implies $\eta^+ = \{x \in A : axx = a\}$.

Proof. See [9] Lemma 1.(i),(ii). □

Theorem 3 *Let $\mathbf{A} = (A, \cdot) \in \mathcal{E}$, $a \in A \backslash \{1\}$. Then the following conditions are equivalent:*
(i) $a \in I(\mathbf{A})$;
(ii) $(a \in (\mu^+ \backslash \mu) \cap (\nu^+ \backslash \nu) \Rightarrow \mu^+ = \nu^+)$ for every $\mu, \nu \in \mathrm{Fm}(\mathbf{A})$;
(iii) $[a)$ is a join-irreducible element of $\Phi(\mathbf{A})$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Lemma 2. To prove $(ii) \Rightarrow (i)$ it is enough to show that $axx \in \{a, 1\}$ for every $x \in A$. On the contrary, assume that $axx \neq a$ and $axx \neq 1$ for some $x \in A$. Then there exist $\mu, \nu \in \mathrm{Fm}(\mathbf{A})$ such that $axx \in \mu^+ \backslash \mu$ and $axxa \in \nu^+ \backslash \nu$. Since $(axx)\,aa = axx \notin \mu$ and $(axxa)\,aa = axxa \notin \nu$, it follows from Lemma 2.(i) that $a \in \mu^+ \cap \nu^+$. On the other hand $a \notin \mu$ and $a \notin \nu$. From the assumption we get $\mu^+ = \nu^+$. Applying again Lemma 2.(i) we deduce that $x \in \mu^+ = \nu^+$, since otherwise $(axx)\,xx = axx \in \mu$, a contradiction. Observe that x/ν belongs to a two-element Boolean group ν^+/ν, which implies $(axxa)/\nu = (axxa/\nu)(x/\nu)(x/\nu)$. Moreover, $(axxa)\,xx = 1$, and consequently $(axxa/\nu)(x/\nu)(x/\nu) = 1/\nu$. Hence $(axxa)/\nu = 1/\nu$, and so $axxa \in \nu$, a contradiction. The equivalence of conditions (i) and (iii) follows from [8, Proposition 2.2]. □

Corollary 4 *Let $\mathbf{A} = (A, \cdot)$ be a finite equivalential algebra. Then the sets $\mathrm{Fm}(\mathbf{A})$ and $I(\mathbf{A})$ have the same cardinality.*

Proof. The variety $\mathcal{E}$ is congruence modular, and hence $\Phi(\mathbf{A}) \simeq \mathrm{Con}(\mathbf{A})$ is a finite modular lattice. It follows from the theorem of Dilworth [1] that the numbers of join- and meet-irreducible elements in $\Phi(\mathbf{A})$ are equal. Now, the assertion follows from the equivalence of conditions (i) and (iii) in Theorem 3. □

We shall consider the sequence of terms $(P_m)_{m\in\mathbb{N}}$ defined by induction on $m \in \mathbb{N}$:

$$P_1(x_1) = x_1 \ ,$$
$$P_{m+1}(x_1,\ldots,x_{m+1}) = x_{m+1}P_m(x_1,\ldots,x_m)\,P_m(x_1,\ldots,x_m)\,x_{m+1} \ .$$

In the sequel we shall need the following simple lemma characterizing the equivalence operation in the algebras $\mathbf{m}$ ($m \in \mathbb{N}$).

Lemma 5 *Let $m,n,s \in \mathbb{N}$, $2 \le s \le m$, $a,a_1,\ldots,a_n \in \{1,\ldots,m\}$, $Y \subset \{1,\ldots,m\}$. Then*

(i) $a\&Y = \begin{cases} a & \text{if } y \le a \text{ for every } y \in Y \\ 1 & \text{otherwise} \end{cases}$;

(ii) $P_n(a_1,\ldots,a_n) = \begin{cases} a_n & \text{if } a_n < a_{n-1} < \ldots < a_1 \\ 1 & \text{otherwise} \end{cases}$;

(iii) if $a_1 \cdot \cdots \cdot a_n = s$ and $a_i \le s$ for $i = 1,\ldots,n$, then $|\{i = 1,\ldots,n : a_i = s\}|$ is odd.

Proof. One can easily prove (i) by induction on $|Y|$, and then (ii) by induction on n. To prove (iii) it is enough to observe that $s = a_1ss\cdot\cdots\cdot a_nss$, and $a_iss = 1$ for $a_i < s$ ($i = 1,\ldots,n$). □

2. Construction

In [9] we obtain a recurrence formula for the free spectrum of the variety of linear equivalential algebras, and so we compute the number of purely equivalential propositional formulas in the Gödel-Dummett logic in n variables for an arbitrary $n \in \mathbb{N}$. In this paper we show how to determine the set of these formulas in a constructive way.

Let $\mathbf{F}_\omega(n)$ be the free algebra on n generators over $\mathcal{E}_\omega$, i.e., the algebra of purely equivalential terms in **LC** built from n variables up to provable equality, and let $X_n = \{v_1, \ldots, v_n\}$ be the set of its free generators. For simplicity we shall omit superscripts, writing $\mathbf{t}(v_1, \ldots, v_n)$ instead of $\mathbf{t}^{\mathbf{F}_\omega(n)}(v_1, \ldots, v_n)$ for an arbitrary n-ary term $\mathbf{t}$. The free algebra can be built constructively by the following steps. First, we construct all the irreducible elements in $\mathbf{F}_\omega(n)$. Then, we describe the natural order and the natural equivalence relation in this set. Finally, exploiting the representation theorem, we conclude that every formula in $\mathbf{F}_\omega(n)$ can be written as an equivalence of a finite antichain of irreducible elements coming from different equivalence classes. In this way we can construct effectively all propositional formulas in $(\mathbf{LC})_\leftrightarrow$.

2.1 Irreducibles

We start from describing the set $I(\mathbf{F}_\omega(n))$. Denote by $\mathcal{D}_n$ the set of all pairs (Z, D), where $D = (D_1, \ldots, D_s)$ is an ordered partition of X_n and $\emptyset \neq Z \subset D_1$. Let $(Z, D) \in \mathcal{D}_n$, $D = (D_1, \ldots, D_s)$, $(x_2, \ldots, x_s) \in D_2 \times \cdots \times D_s$. For each $i \in \{2, \ldots, s\}$ we built a finite set $Y_i := \{y \cdot x_i : y \in D_i\}$. Put $Y := D_1 \cup \bigcup_{i=2}^{s} Y_i$. Let us consider the element of $\mathbf{F}_\omega(n)$ given by the formula

$$a_{(Z,D)}(x_2, \ldots, x_s) := P_s(x_s, \ldots, x_2, \otimes Z \& Y) \ , \qquad (\star)$$

where $\otimes Z \& Y = \otimes \{v \& Y : v \in Z\}$.

In this situation we prove the following lemma:

Lemma 6 *Let $m \in \mathbb{N}$, $m \geq 2$, let f be an epimorphism from $\mathbf{F}_\omega(n)$ onto $\mathbf{m}$, and let $k \in \mathbb{N}$, $k \geq 1$. Then $f\left(a_{(Z,D)}(x_2, \ldots, x_s)\right) = k + 1$ if and only if*

(i) $s = m - k$;

(ii) $D_1 = \{v \in X : f(v) \leq k + 1\}$;

(iii) $D_i = \{v \in X : f(v) = k + i\}$ for $i = 2, \ldots, s$;

(iv) $|Z \cap \{v \in X : f(v) = k + 1\}|$ is odd.

Proof. An easy computation based on Lemma 5 shows that conditions (i)-(iv) imply $f\left(a_{(Z,D)}(x_2, \ldots, x_s)\right) = k + 1$. To prove the converse implication note that from $P_s(f(x_s), \ldots, f(x_2), f(\otimes Z \& Y)) =$

$f\left(a_{(Z,D)}\left(x_s,\ldots,x_2\right)\right) = k+1$ and from Lemma 5.(ii), it follows that $k+1 = f\left(\otimes Z\&Y\right) < f\left(x_2\right) < \ldots < f\left(x_s\right)$. Let $Z = \{z_1,\ldots,z_r\}$. Then $k+1 = f\left(\otimes Z\&Y\right) = f\left(z_1\right)\&f\left(Y\right)\cdot\ldots\cdot f\left(z_r\right)\&f\left(Y\right) = \left(f\left(z_1\right)\cdot\ldots\cdot f\left(z_r\right)\right)\&f\left(Y\right)$. By Lemma 5.(i) we get $f\left(z\right)\cdot\ldots\cdot f\left(z_r\right) = k+1$ and $f\left(y\right) \leq k+1$ for each $y \in Y$. Now, assertion (iv) follows from Lemma 5.(iii) and from the fact that $Z \subset Y$. Let $i = 2,\ldots,s$, $z \in D_i$. Then $f\left(z\right) = f\left(x_i\right)$, since otherwise $f\left(z\cdot x_i\right) = f\left(z\right)\cdot f\left(x_i\right) = \max\left(f\left(z\right), f\left(x_i\right)\right) \geq f\left(x_i\right) > k+1$ and $z\cdot x_i \in Y$, a contradiction. From $D_1 \subset Y$ we get $D_1 \subset \{v \in X : f\left(v\right) \leq k+1\}$. From surjectivity we deduce (i), (ii), and (iii), which completes the proof. □

Applying Lemma and the fact that only finite subdirectly irreducible algebras in $\mathcal{E}_\omega$ are $\{\mathbf{m} : 2 \leq m\}$ we get immediately

Corollary 7 *Let $(Z,D) \in \mathcal{D}_n$, $D = (D_1,\ldots,D_s)$, and suppose that $(x_2,\ldots,x_s)$, $(x'_2,\ldots,x'_s) \in D_2 \times \cdots \times D_s$. Then*

$$a_{(Z,D)}\left(x_2,\ldots,x_s\right) = a_{(Z,D)}\left(x'_2,\ldots,x'_s\right).$$

From now on we will write $a_{(Z,D)} := a_{(Z,D)}\left(x_2,\ldots,x_s\right)$ for an arbitrary $(x_2,\ldots,x_s) \in D_2 \times \cdots \times D_s$.

Corollary 8 *Let $D = (D_1,\ldots,D_s)$, $(Z,D) \in \mathcal{D}_n$, and let $B \subset D_1$ be such that $|Z\backslash B|$ is odd. Let $f^B_{(Z,D)} : \mathbf{F}_\omega\left(n\right) \to \mathbf{s}+\mathbf{1}$ be the only homomorphism satisfying: $f^B_{(Z,D)}\left(v\right) = 1$ for $v \in B$, $f^B_{(Z,D)}\left(v\right) = 2$ for $v \in D_1\backslash B$, and $f^B_{(Z,D)}\left(v\right) = i+1$ for $v \in D_i$, $i = 2,\ldots,s$. Then $f^B_{(Z,D)}$ is an epimorphism and $f^B_{(Z,D)}\left(a_{(Z,D)}\right) = 2$.*

Proof. It is clear that $B\backslash D_1 \neq \emptyset$, and hence $f^B_{(Z,D)}$ is an epimorphism. It is easy to show that $f^B_{(Z,D)}$ fulfills conditions (i)-(iv) from Lemma 6 for $k = 1$, which establishes the assertion. □

We shall show that the set of all elements of the form $a_{(Z,D)}$ coincides with the set of join-irreducible elements of the free algebra $\mathbf{F}_\omega\left(n\right)$.

Proposition 9 *Let $(Z,D) \in \mathcal{D}_n$. Then $a_{(Z,D)} \in I\left(\mathbf{F}_\omega\left(n\right)\right)$.*

Proof. Let $D = (D_1,\ldots,D_s)$. It follows from Corollary 8 that $a_{(Z,D)} \neq 1$. Now, it suffices to show that $a_{(Z,D)}$ satisfies condition (ii) from Theorem 3. Let $\mu, \varphi \in \mathrm{Fm}\left(\mathbf{F}_\omega\left(n\right)\right)$ be such that $a_{(Z,D)} \in (\mu^+\backslash\mu)\cap(\varphi^+\backslash\varphi)$.

Then there exist $m, p \in \mathbb{N}$ such that $\mathbf{F}_\omega(n)/\mu \simeq \mathbf{m}$ and $\mathbf{F}_\omega(n)/\varphi \simeq \mathbf{p}$. Let us consider the canonical epimorphisms $\pi_\mu : \mathbf{F}_\omega(n) \to \mathbf{m}$ and $\pi_\varphi : \mathbf{F}_\omega(n) \to \mathbf{p}$. Then $\pi_\mu\left(a_{(Z,D)}\right) = 2$, $\pi_\varphi\left(a_{(Z,D)}\right) = 2$, $\pi_\mu^{-1}(\{1,2\}) = \mu^+$, and $\pi_\varphi^{-1}(\{1,2\}) = \varphi^+$. From Lemma 6 we conclude immediately that $m = p = s+1$, $\pi_\mu(v) = \pi_\varphi(v) = i+1$ for $i = 2, \ldots, s$, $v \in D_i$, and $\pi_\mu(v), \pi_\varphi(v) \in \{1,2\}$ for every $v \in D_1$. Hence $\pi_\mu(v) \equiv_{\{1,2\}} \pi_\varphi(v)$ for every $v \in X_n$. Then for each $p(v_1, ..., v_n) \in \mathbf{F}_\omega(n)$ we have

$$\begin{aligned}\pi_\mu(p(v_1, ..., v_n)) &= p(\pi_\mu(v_1), ..., \pi_\mu(v_n)) \\ &\equiv_{\{1,2\}} p(\pi_\varphi(v_1), ..., \pi_\varphi(v_n)) \\ &= \pi_\varphi(p(v_1, ..., v_n)) \ .\end{aligned}$$

Thus $p(v_1, ..., v_n) \in \pi_\mu^{-1}(\{1,2\}) = \mu^+$ if and only if $p(v_1, ..., v_n) \in \pi_\varphi^{-1}(\{1,2\}) = \varphi^+$, and so $\mu^+ = \varphi^+$, as desired. □

Theorem 10 *The map $\mathcal{D}_n \ni (Z,D) \to a_{(Z,D)} \in I(\mathbf{F}_\omega(n))$ is a bijection.*

Proof. It follows from Proposition 9 that the map is well-defined. Let $b(n)$ denote the n-th ordered Bell number, i.e., the number of ordered partitions of an n-element set. First note that

$$\begin{aligned}|\mathcal{D}_n| &= |\{(Z,D) \in \mathcal{D}_n : Z \neq D_1\}| + |\{(Z,D) \in \mathcal{D}_n : Z = D_1\}| \\ &= |\{(Z, D_1 \backslash Z, \ldots, D_s) : (Z,D) \in \mathcal{D}_n,\ Z \neq D_1\}| + b(n) \\ &= |\{C : C \text{ is an ordered partition of } X_n,\ |C| \geq 2\}| + b(n) \\ &= (b(n) - 1) + b(n) = 2\,b(n) - 1 \ .\end{aligned}$$

By [9, Theorem 15.1], we know that the number $|\mathrm{Fm}(\mathbf{F}_\omega(n))|$ of meet-irreducible elements in $\Phi(\mathbf{F}_\omega(n))$ is equal to $2\,b(n) - 1$. Thus, it follows from Corollary 4 that the number $|I(\mathbf{F}_\omega(n))|$ of join-irreducible elements in $\mathbf{F}_\omega(n)$ is also equal to $2\,b(n) - 1$. Hence, it suffices to show that the map is one-to-one. Let $(Z,D), (Z',D') \in \mathcal{D}_n$, $a_{(Z,D)} = a_{(Z',D')}$. Let $\mu \in \mathrm{Fm}(\mathbf{F}_\omega(n))$ be such that $a_{(Z,D)} \in \mu^+ \backslash \mu$. Then there exists $m \in \mathbb{N}$ such that $\mathbf{F}_\omega(n)/\mu \simeq \mathbf{m}$. Consider canonical epimorphism $\pi_\mu : \mathbf{F}_\omega(n) \to \mathbf{m}$. Then $\pi_\mu\left(a_{(Z,D)}\right) = \pi_\mu\left(a_{(Z',D')}\right) = 2$. If $D = (D_1, \ldots, D_s)$, $D' = \left(D'_1, \ldots, D'_{s'}\right)$, we conclude from Lemma 6 immediately that $s = m - 1 = s'$ and $D_i = \pi_\mu^{-1}(i+1) \cap X_n = D'_i$ for $i = 2, \ldots, s$ and $D_1 = \pi_\mu^{-1}(\{1,2\}) \cap X_n = D'_1$. Hence $D = D'$. Suppose that $Z \neq Z'$. Then there exists $B \subset D_1$ such that $|Z \backslash B|$ is odd and $|Z' \backslash B|$ is even. From Corollary 8 we deduce that $f^B_{(Z,D)}\left(a_{(Z,D)}\right) = 2$. On the other hand, $\left|Z' \cap (f^B_{(Z,D)})^{-1}(2)\right| = |Z' \cap (D_1 \backslash B)| = |Z' \backslash B|$ is even. Thus, applying Lemma 6, we get $f^B_{(Z,D)}\left(a_{(Z',D')}\right) \neq 2$, a contradiction. □

2.2 Order and equivalence

The following theorem characterizes the natural order $\leq_E$ in $I(\mathbf{F}_\omega(n))$:

Theorem 11 *Let $D = (D_1, \ldots, D_s)$, (Z, D), $(Z', D') \in \mathcal{D}_n$. Then $a_{(Z',D')} <_E a_{(Z,D)}$ if and only if there exists $k \in \mathbb{N}$, $2 \leq k \leq s$ such that the following conditions are fulfilled:*
(i) $D' = (D_1 \cup \cdots \cup D_k, D_{k+1}, \ldots, D_s)$;
(ii) $|Z' \cap D_k|$ is odd.

We start from the following lemma:

Lemma 12 *Let $\alpha \in I(\mathbf{F}_\omega(n))$ and let $B \subset D_1$ be such that $|Z \backslash B|$ is odd. Then $\alpha <_E a_{(Z,D)}$ if and only if $f^B_{(Z,D)}(\alpha) > 2$.*

Proof of Lemma. We have either $a_{(Z,D)}\alpha\alpha = a_{(Z,D)}$ or $a_{(Z,D)}\alpha\alpha = 1$. From Corollary 8 we get, in the first case, $2 = f^B_{(Z,D)}\left(a_{(Z,D)}\right) = f^B_{(Z,D)}\left(a_{(Z,D)}\alpha\alpha\right) = 2\, f^B_{(Z,D)}(\alpha)\, f^B_{(Z,D)}(\alpha)$, and hence $f^B_{(Z,D)}(\alpha) \in \{1, 2\}$, and in the second case, $1 = 2\, f^B_{(Z,D)}(\alpha)\, f^B_{(Z,D)}(\alpha)$, and so $f^B_{(Z,D)}(\alpha) > 2$. This proves the lemma. □

Proof of Theorem. Let $B \subset D_1$ be such that $|Z \backslash B|$ is odd. From Lemma we deduce that $a_{(Z',D')} <_E a_{(Z,D)}$ if and only if $f^B_{(Z,D)}\left(a_{(Z',D')}\right) = k + 1$ for some $k \in \mathbb{N}$, $k \geq 2$. Now it is enough to apply Lemma 6. □

The natural equivalence relation in $I(\mathbf{F}_\omega(n))$ can be also easily characterized.

Theorem 13 *Let $(Z, D), (Z', D') \in \mathcal{D}_n$. Then $a_{(Z',D')} \sim a_{(Z,D)}$ if and only if $D = D'$.*

Proof. Suppose that $D = (D_1, \ldots, D_s) \neq \left(D'_1, \ldots, D'_{s'}\right) = D'$. To obtain a contradiction, it is enough to prove that $a_{(Z,D)}$ and $a_{(Z',D')}$ do not have the same predecessors. It follows from Theorem 11 that the set of predecessors of $a_{(Z,D)}$ (resp. $a_{(Z',D')}$) is non-empty if and only if D (resp. D') has at least two elements. Thus, we can assume that $s, s' \geq 2$, since otherwise the one of the elements $a_{(Z,D)}$, $a_{(Z',D')}$ has predecessors in $I(\mathbf{F}_\omega(n))$ and the other has not. Let us consider two cases:

Case 1. There exists $k \geq 2$ such that $D_1 \cup \cdots \cup D_k \neq D'_1 \cup \cdots \cup D'_k$. We take minimal $k \geq 2$ with this property. Without loss of generality we can assume that $D_1 \cup \cdots \cup D_k \subsetneq D'_1 \cup \cdots \cup D'_k$. Put $D'' = (D_1 \cup \cdots \cup D_k, D_{k+1}, \ldots, D_s)$, and choose an arbitrary $\emptyset \neq Z'' \subset D_1 \cup \cdots \cup D_k$ such that $|Z'' \cap D_k|$ is odd. Clearly, in this situation $D''_1 = D_1 \cup \cdots \cup D_k \neq D'_1 \cup \cdots \cup D'_j$ for every $j = 1, \ldots, s'$.

Case 2. Assume that $D_1 \cup \cdots \cup D_k = D'_1 \cup \cdots \cup D'_k$ for every $k \geq 2$. Then $D_2 \neq D'_2$. Take $Z'' \subset D_1 \cup D_2$ such that $|Z'' \cap D_2|$ is odd and $|Z'' \cap D'_2|$ is even. Put $D'' = (D_1 \cup D_2, D_3, \ldots, D_s)$.

In both cases it follows from Theorem 11 that $a_{(Z'',D'')} <_E a_{(Z,D)}$ and $a_{(Z'',D'')} \not<_E a_{(Z',D')}$, and so $a_{(Z',D')} \nsim a_{(Z,D)}$, a contradiction.

To prove the converse implication it again suffices to use Theorem 11.

□

2.3 Generators

In this section we describe how the generators of $\mathbf{F}_\omega(n)$ can be represented as a finite $\leq_E$-antichain of irreducible elements from different equivalence classes.

Proposition 14 *Let $\emptyset \neq Z \subset X_n$. Then*

$$R(\otimes Z \& Z) = \left\{ a_{(Z,D)} : (Z, D) \in \mathcal{D}_n \right\} ,$$

and, in particular, $\otimes Z \& Z = \otimes \left\{ a_{(Z,D)} : (Z, D) \in \mathcal{D}_n \right\}$.

Proof. Put $w := \otimes Z \& Z$. According to Theorems 1 and 10

$$\begin{aligned} R(w) &= \{a \in I(\mathbf{F}_\omega(n)) : a \geq_E w \text{ and } aww = a\} \\ &= \left\{ a_{(C,D)} : (C, D) \in \mathcal{D}_n,\ a_{(C,D)} \geq_E w \text{ and } a_{(C,D)}ww = a_{(C,D)} \right\} . \end{aligned}$$

Let $(C, D) \in \mathcal{D}_n$, $D = (D_1, \ldots, D_s)$. Then it is enough to show that $C = Z$ if and only if $a_{(C,D)} \in R(w)$. Take $\mu \in \mathrm{Fm}(\mathbf{F}_\omega(n))$ such that

$a_{(C,D)} \in \mu^{+} \backslash \mu$. There exists $m \in \mathbb{N}$ such that $\mathbf{F}_{\omega}(n)/\mu \simeq \mathbf{m}$. Let us consider the canonical epimorphism $\pi_{\mu} : \mathbf{F}_{\omega}(n) \to \mathbf{m}$. Then $\pi_{\mu}^{-1}(1) = \mu$ and $\pi_{\mu}^{-1}(2) = \mu^{+} \backslash \mu$. From Lemma 6 we deduce that $m = s + 1$ and $\mu^{+} \cap X_n = D_1$.

Assume that $C = Z$. Recall that $a_{(Z,D)}$ is given by formula ($\star$). Then, since $Z \subset Y$, we get

$$\begin{aligned} a_{(Z,D)} &= P_s(x_s, \dots, x_2, \otimes Z \& Y) \\ &= (\otimes Z \& Y)\, P_{s-1}(x_s, \dots, x_2)\, P_{s-1}(x_s, \dots, x_2)\, (\otimes Z \& Y) \\ &\geq_E \otimes Z \& Y \geq_E w \ . \end{aligned}$$

Moreover, $\pi_{\mu}\left(a_{(Z,D)}\right) = 2$, and since $Z \subset D_1$ we have $\pi_{\mu}(w) \in \{1, 2\}$. Thus $\pi_{\mu}\left(a_{(Z,D)}ww\right) = \pi_{\mu}\left(a_{(Z,D)}\right)\pi_{\mu}(w)\,\pi_{\mu}(w) = 2\ \pi_{\mu}(w)\,\pi_{\mu}(w) = 2$. This gives $a_{(Z,D)}ww \neq 1$ and, in consequence, $a_{(Z,D)}ww = a_{(Z,D)}$. Hence $a_{(Z,D)} \in R(w)$, as desired.

Suppose now that $a_{(C,D)} \in R(w)$, and so $a_{(C,D)} \geq_E w$, $a_{(C,D)}ww = a_{(C,D)}$. Then $w \notin \mu = \pi_{\mu}^{-1}(1)$ and $2 = \pi_{\mu}\left(a_{(C,D)}\right) = \pi_{\mu}\left(a_{(C,D)}ww\right) = \pi_{\mu}\left(a_{(C,D)}\right)\pi_{\mu}(w)\,\pi_{\mu}(w) = 2\ \pi_{\mu}(w)\,\pi_{\mu}(w)$. Thus $w \in \pi_{\mu}^{-1}(\{1,2\})$. Let $v \in Z$. Then $2 = \pi_{\mu}(w) = \pi_{\mu}(wvv) = \pi_{\mu}(w)\,\pi_{\mu}(v)\,\pi_{\mu}(v) = 2\ \pi_{\mu}(v)\,\pi_{\mu}(v)$. Hence $v \in \pi_{\mu}^{-1}(\{1,2\}) \cap X_n = \mu^{+} \cap X_n = D_1$. Consequently, $Z \subset D_1$, and so $(Z, D) \in \mathcal{D}_n$. We have already shown that such $a_{(Z,D)} \in R(w)$. Since also $a_{(C,D)} \in R(w)$, we deduce from Theorem 13 that $a_{(Z,D)} \sim a_{(C,D)}$. Now, Theorem 1 gives us $a_{(Z,D)} = a_{(C,D)}$, and so from Theorem 10 we get $Z = C$, as required. □

Corollary 15 *Let $v \in X_n$. Then $v = \otimes \left\{a_{(\{v\},D)} : (\{v\}, D) \in \mathcal{D}_n\right\}$.*

2.4 Examples

We illustrate the effective technique of constructing all propositional formulas in $(\mathbf{LC})_{\leftrightarrow}$ by examining two examples: $n = 2$ and $n = 3$.

Example 1 *The algebra $\mathbf{F}_{\omega}(2)$ with two free generators x, y has five irreducible elements encoded by elements of $\mathcal{D}_2$: $(\{x\}, (\{x\}, \{y\})) \to xyyx$; $(\{y\}, (\{y\}, \{x\})) \to yxxy$; $(\{x, y\}, (\{x, y\})) \to xy$; $(\{x\}, (\{x, y\})) \to xyy$; $(\{y\}, (\{x, y\})) \to yxx$, fulfilling the relation: $yxx, xy <_E xyyx$ and $xyy, xy <_E yxxy$. The four other elements of $\mathbf{F}_{\omega}(y)$ can be obtained as* *products of a finite antichain of irreducible elements coming from different equivalence classes:* $(xyyx)(yxxy)$, $(xyy)(xyyx) = x$, $(yxx)(yxxy) = y$, *and* $\otimes\emptyset = 1$.

Example 2 The algebra $\mathbf{F}_\omega(3)$ with three free generators x, y, z has 25 irreducible elements encoded by elements of $\mathcal{D}_3$ in the following way:

$$(\{x\},(\{x,y,z\})) \to x\&\{x,y,z\} = xyyzz \ ;$$
$$(\{y\},(\{x,y,z\})) \to y\&\{x,y,z\} = yxxzz \ ;$$
$$(\{z\},(\{x,y,z\})) \to z\&\{x,y,z\} = zxxyy \ ;$$
$$(\{x,y\},(\{x,y,z\})) \to xy\&\{x,y,z\} = xyzz \ ;$$
$$(\{x,z\},(\{x,y,z\})) \to xz\&\{x,y,z\} = xzyy \ ;$$
$$(\{y,z\},(\{x,y,z\})) \to yz\&\{x,y,z\} = yzxx \ ;$$
$$(\{x,y,z\},(\{x,y,z\})) \to xyz\&\{x,y,z\} = xyzxxyyzz \ ;$$

$$(\{x\},(\{x\},\{y,z\})) \to; (xyyzz)(x(yz)(yz)) \ ;$$
$$(\{x\},(\{x,y\},\{z\})) \to (xyy)zz(xyy) = (xzzx)yy \ ;$$
$$(\{y\},(\{x,y\},\{z\})) \to (yxx)zz(yxx) = (yzzy)xx \ ;$$
$$(\{x,y\},(\{x,y\},\{z\})) \to (xy)zz(xy) \ ;$$
$$(\{z\},(\{z\},\{x,y\})) \to (zxxyy)(z(xy)(xy)) \ ;$$
$$(\{x\},(\{x,z\},\{y\})) \to (xzz)yy(xzz) = (xyyx)zz \ ;$$
$$(\{z\},(\{x,z\},\{y\})) \to (zxx)yy(zxx) = (zyyz)xx \ ;$$
$$(\{x,z\},(\{x,z\},\{y\})) \to (xz)yy(xz) \ ;$$
$$(\{y\},(\{y\},\{x,z\})) \to (yxxzz)(y(xz)(xz)) \ ;$$
$$(\{y\},(\{y,z\},\{x\})) \to (yzz)xx(yzz) = (yxxy)zz \ ;$$
$$(\{z\},(\{y,z\},\{x\})) \to (zyy)xx(zyy) = (zxxz)yy \ ;$$
$$(\{y,z\},(\{y,z\},\{x\})) \to (yz)xx(yz) \ ;$$

$$(\{x\},(\{x\},\{y\},\{z\})) \to x(yzzy)(yzzy)x \ ;$$
$$(\{x\},(\{x\},\{z\},\{y\})) \to x(zyyz)(zyyz)x \ ;$$
$$(\{y\},(\{y\},\{x\},\{z\})) \to y(xzzx)(xzzx)y \ ;$$
$$(\{y\},(\{y\},\{z\},\{x\})) \to y(zxxz)(zxxz)y \ ;$$
$$(\{z\},(\{z\},\{x\},\{y\})) \to z(xyyx)(xyyx)z \ ;$$
$$(\{z\},(\{z\},\{y\},\{x\})) \to z(yxxy)(yxxy)z \ .$$

Points in the diagram below (Fig. 1) correspond to all elements of the poset $I(\mathbf{F}_\omega(3))$. Each dot denotes an element $a_{(Z,D)}$ labelled with a set Z, and a solid circle or ellipse shows an equivalence class in $I(\mathbf{F}_\omega(3))$ labelled with an ordered partition D. Straight lines directed outside the diagram represent a partial ordering in the set $I(\mathbf{F}_\omega(3))$. Dotted circles and ellipses inside solid ones are used to reduce the number of lines needed. All the elements labelled in our diagram with $\{x\}$ (resp. $\{y\}$, $\{z\}$) form an antichain. It follows from Corollary 15 that the equivalence of these elements is equal to the generator x (resp. y, z). Note that the algebra $\mathbf{F}_\omega(3)$ has 6380 elements [9].

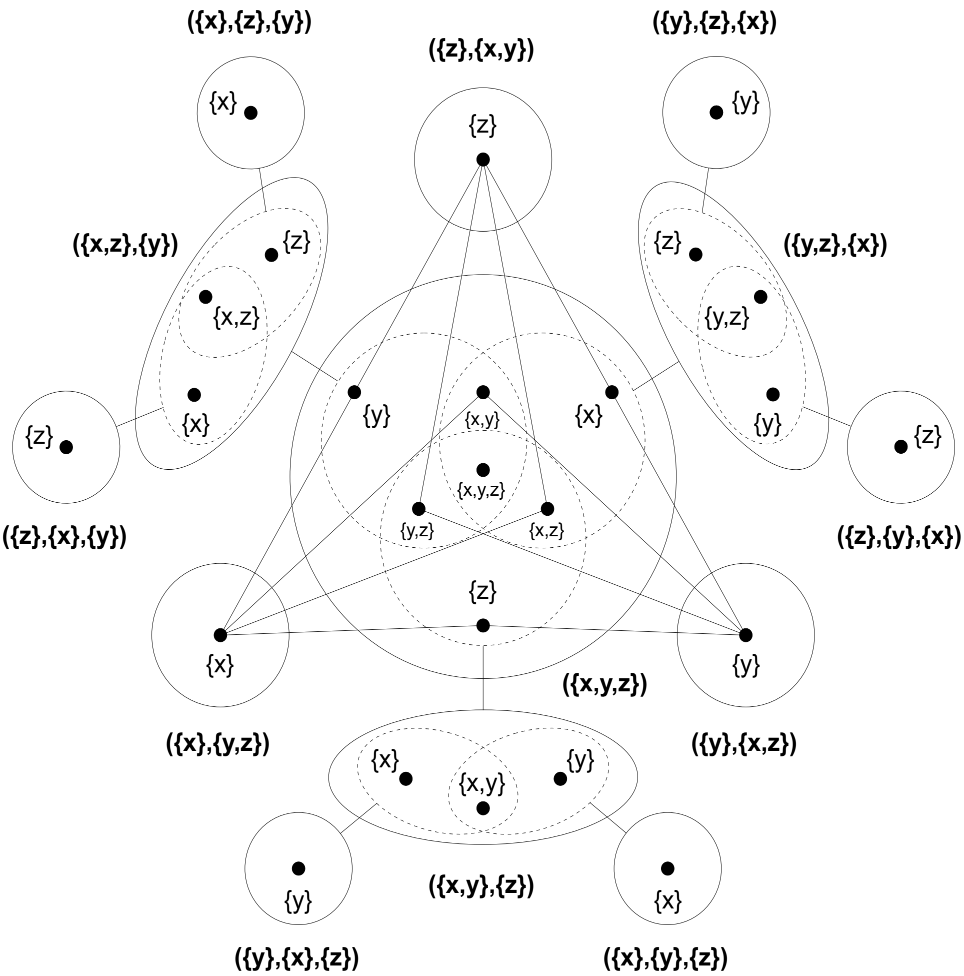

Figure 1: $(I(\mathbf{F}_\omega(\mathbf{3})), \leq_E, \sim)$

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