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ON THE VARIETY GENERATED BY INVOLUTIVE POCRIMS

A b s t r a c t. An *involutive pocrim* (a.k.a. an L_0–*algebra*) is a residuated integral partially ordered commutative monoid with an involution operator, considered as an algebra. It is proved that the variety generated by all involutive pocrims satisfies no nontrivial idempotent Maltsev condition. That is, no nontrivial $\langle \wedge, \vee, \circ \rangle$-equation holds in the congruence lattices of all involutive pocrims. This strengthens a theorem of A. Wroński. The result survives if we restrict the generating class to *totally ordered* involutive pocrims.

1. Involutive Pocrims and their Subreducts

The purely intensional fragments of affine linear logic (i.e., linear logic with the weakening axiom $p \to (q \to p)$) are known to be algebraizable in the sense of [1]. The equivalent algebraic semantics for these fragments are

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the quasivarieties of pocrims, of involutive pocrims, and of BCK–algebras. These algebras can be defined as follows.

Consider a commutative monoid $\langle A; \cdot, t\rangle$ whose universe A is partially ordered by a relation $\leq$. Suppose, moreover, that $\leq$ is compatible with $\cdot$, i.e., for all $a, b, c \in A$, if $a \leq b$ then $a \cdot c \leq b \cdot c$. The structure $\mathcal{A} = \langle A; \cdot, t; \leq\rangle$ is said to be *residuated* if for any $a, b \in A$, there is a *largest* $c \in A$ such that $a \cdot c \leq b$. The largest c with this property is then denoted by $a \to b$, so $\langle A; \cdot, \to, t; \leq\rangle$ satisfies

$$x \cdot z \leq y \iff z \leq x \to y$$

and in particular, $x \leq y \iff t \leq x \to y$. If in addition, t is the *greatest* element of $\langle A; \leq\rangle$, we say that $\mathcal{A}$ is *integral.* In this case, the partial order $\leq$ is equationally definable by

$$x \leq y \iff t \approx x \to y,$$

so $\mathcal{A}$ is first order definitionally equivalent to the algebra $\mathbf{A} = \langle A; \cdot, \to, t\rangle$.

An algebra $\mathbf{A}$ which arises in this way is called a *pocrim.* (This is an acronym for 'partially ordered commutative residuated integral monoid'.) It is well known that the class POCR of all pocrims is axiomatized by the identities

(M1) $\quad (x \cdot y) \to z \approx y \to (x \to z)$

(M2) $\quad t \to x \approx x$

(M3) $\quad x \to t \approx t$

(M4) $\quad (x \to y) \to ((y \to z) \to (x \to z)) \approx t$

together with the single quasi-identity

(M5) $\quad (x \to y \approx t \ \textsf{and}\ \ y \to x \approx t) \Rightarrow x \approx y$

[15, 16], so POCR is a quasivariety. For a general study of pocrims, see [4]. An *involutive pocrim* is an algebra $\mathbf{A} = \langle A; \cdot, \to, \neg, t\rangle$ such that $\langle A; \cdot, \to, t\rangle$ is a pocrim, $\neg$ is a unary operation on A, and $\mathbf{A}$ satisfies

(M6) $\quad \neg\neg x \approx x$

(M7) $\quad x \to \neg y \approx y \to \neg x.$

The class IPOC of all involutive pocrims is therefore also a quasivariety. Every involutive pocrim has a least element $f = \neg\, t$ and satisfies

$$x \to f \;\approx\; \neg\, x, \qquad x \cdot y \;\approx\; \neg\,(x \to \neg\, y), \qquad x \to y \;\approx\; \neg\,(x \cdot \neg\, y).$$

Up to term equivalence, IPOC is just the class of all pocrims with a distinguished constant f satisfying $(x \to f) \to f \approx x$.

Involutive pocrims were introduced under the name *pre-Boolean algebras* by Wroński and Krzystek in [39]. They were studied in [21], and independently by Grishin [10], who called them L_0-*algebras*. Every MV-algebra in the sense of Chang [5] is termwise equivalent to an involutive pocrim. It is known that

> every pocrim $\mathbf{A}$ is a $\{\cdot, \to, t\}$–*subreduct* of an involutive pocrim,

i.e., $\mathbf{A}$ is a subalgebra of the pocrim reduct $\langle B; \cdot, \to, t\rangle$ of some involutive pocrim $\langle B; \cdot, \to, \neg, t\rangle$, see [38].

The quasivariety of pure $\{\to, t\}$–subreducts of pocrims (or equivalently, of involutive pocrims) turns out to be axiomatized by the laws (M2)–(M5). Thus, it coincides with the older class of *BCK–algebras* of Iséki [14], which we denote by BCKA. Proofs of this relationship appear in [30, 27, 7]; there are some related results in [25, 13]. So, as observed in [30],

> every BCK–algebra is a $\{\to, t\}$–subreduct of an *involutive* pocrim (and conversely).

In general, the variety generated by a class K of similar algebras will be denoted by $\mathrm{V}(\mathsf{K})$. Wroński [37] proved that BCKA is not a variety. Krzystek's unpublished 1983 dissertation [21] established the stronger fact that IPOC is not a variety, i.e., its homomorphic closure $\mathrm{V}(\mathsf{IPOC})$ does not satisfy (M5). Independently, Grishin [10] gave a different proof that IPOC is not a variety. It follows that POCR is not a variety; this was shown directly by Higgs in [11].

The aim of the present note is to point out a stronger conclusion, viz.:

> No nontrivial equation in the signature $\wedge, \vee, \circ$ holds in the congruence lattices of all involutive pocrims.

Here $\circ$ corresponds to the relational product operation—see Section 2 for precise definitions. The analogous results for pocrims and for BCK–algebras turn out to be corollaries (in view of Theorem 2 below).

The following result was already proved in [38]:

Theorem 1. (Wroński) *An equation in the language* $\wedge, \vee$ *holds in the congruence lattice of every BCK-algebra only if it holds in every lattice.*

In general, a variety satisfying no nontrivial congruence equation in $\wedge, \vee$ may still satisfy a nontrivial congruence equation in $\wedge, \vee, \circ$. The standard example is the variety of semilattices (see the remarks after Corollary 8).

The absence of congruence equations in IPOC and its subreduct classes contrasts with the desirable *relative* congruence properties of these quasivarieties. (For a stipulated quasivariety K, the *relative* congruences of an algebra $\mathbf{A} \in \mathsf{K}$ are the congruences θ such that $\mathbf{A}/\theta \in \mathsf{K}$. They form an algebraic lattice.) For instance, IPOC etc. are *relatively* congruence distributive and relatively point regular at t, whence every *variety* that *consists* of [involutive] pocrims is congruence distributive [29] and congruence n–permutable for some finite $n \geq 2$: see the references in [4]. In [38], Wroński exhibited BCK–algebras whose congruences are neither modular nor n–permutable for any finite $n \geq 2$. (These examples are not *reducts* of pocrims. In general, when we extend a BCK–algebra to an [involutive] pocrim, the size of the congruence lattice may be reduced, as for instance in [2].)

2. Congruence Conditions

Definition 1. A *congruence equation* is a formal equation in the binary symbols $\wedge, \vee, \circ$. It is *satisfied* by an algebra $\mathbf{A}$ if it becomes true whenever we interpret the variables of the equation as congruence relations of $\mathbf{A}$, and for arbitrary binary relations θ and ϕ on A, we interpret $\theta \wedge \phi$, $\theta \vee \phi$ and $\theta \circ \phi$ as the intersection, the congruence of $\mathbf{A}$ generated by the union, and the relational product of θ and ϕ, respectively.

A congruence equation is *satisfied* by a class of similar algebras if it is satisfied by every member of the class.

A congruence equation is called *nontrivial* if some algebra fails to satisfy it.

If a congruence equation is satisfied by a quasivariety K then it is satisfied by the variety $\mathrm{V}(\mathsf{K})$ also. (This follows from the Correspondence Theorem of universal algebra, extended in the obvious way to handle relational products of congruences.) Note that congruence n-permutability defines a nontrivial congruence equation, and so does congruence modularity. These are examples of nontrivial idempotent Mal'cev conditions. The general definition is as follows.

Definition 2. We say that a variety V *satisfies a nontrivial idempotent Maltsev condition* if some idempotent finitely based variety of finite signature can be interpreted into V and cannot be interpreted into every variety (or equivalently, into the variety of sets).

Recall here that an *idempotent* variety is one that satisfies $\alpha(x, x, \ldots, x) \approx x$ for each of its fundamental operation symbols α. An idempotent variety U can be *interpreted into* a variety V (of possibly different signature) if there is a homomorphism from the clone of term operations of the free $\aleph_0$–generated algebra in U to the corresponding clone of V. Such a map is required to preserve composition of terms and to fix all projections.

A variety that satisfies a nontrivial congruence equation must satisfy a nontrivial idempotent Maltsev condition. This follows by a standard argument that can be found in [32, 36]. The converse was proved more recently in [19]. Thus, conditions (i) and (ii) in the next theorem are equivalent. The equivalence of (ii) and (iii) is a direct consequence of [34, Cor. 5.3].

Theorem 2. (Kearnes and Szendrei ; Taylor) *For any variety V, the following conditions are equivalent.*

(i) V *satisfies a nontrivial congruence equation.*

(ii) V *satisfies a nontrivial idempotent Maltsev condition.*

(iii) *There exist an integer $n > 1$, an n-ary term α that is idempotent over V and a choice of (not necessarily distinct) variables $x_{ij}, y_{ij} (1 \leq i, j \leq n)$ such that $x_{ii} \neq y_{ii}$ for each i and V satisfies*

$$\begin{aligned} \alpha(x_{11}, \ldots, x_{1n}) &\approx \alpha(y_{11}, \ldots, y_{1n}) \,; \\ &\cdots\cdots \\ \alpha(x_{n1}, \ldots, x_{nn}) &\approx \alpha(y_{n1}, \ldots, y_{nn}) \,. \end{aligned}$$

Recall that a variety V is said to be *locally finite* if its finitely generated members are finite algebras. In particular, every variety generated by a finite set of finite algebras is locally finite. An *n–finite* variety is one in which all n–generated algebras are finite.

The lattice $\mathbf{D}_1$ of all convex subsets of a totally ordered three-element set is depicted below. Observe that $\mathbf{D}_1$ has a pentagon sublattice and so cannot be embedded into any modular (in particular, any distributive) lattice.

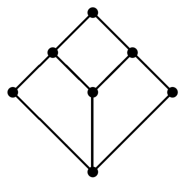

Figure 1. The Lattice $\mathbf{D}_1$

Theorem 3. (Kearnes and Kiss) *A variety* V *satisfies no nontrivial idempotent Maltsev condition iff* $\mathbf{D}_1$ *is a sublattice of the congruence lattice of some algebra* $\mathbf{B} \in \mathsf{V}$.

For the case of locally finite varieties V, Theorem 3 was proved earlier by Hobby and McKenzie [12, Thm. 9.6]. Moreover, when a locally finite variety satisfies the equivalent conditions of Theorem 3 then $\mathbf{B}$ can be chosen finite. This follows from the proof of [12, Thm. 7.9].

The arguments in [12] made definite use of local finiteness, but Kearnes and Kiss [18] have shown that this assumption can be eliminated[1] and that various other small lattices can play the role of $\mathbf{D}_1$. Using a direct universal algebraic argument, they establish that if V satisfies a nontrivial idempotent Maltsev condition then the congruence lattices of all algebras in V satisfy

$$(x \wedge y \approx w \quad \textsf{and} \quad x \wedge z \approx w \quad \textsf{and} \quad x_{[2]} \approx w) \implies x \wedge (y \vee z) \approx w,$$

where $x_{[2]} := x \wedge (y \vee z) \wedge [(y \wedge (x \vee z)) \vee (z \wedge (x \vee y))]$. To see that $\mathbf{D}_1$ violates this quasi-identity, take $\{x, y, z\}$ to be the unique three-element

[1] announced in [17]

generating set for $\mathbf{D}_1$, with x as the meet-reducible generator. Theorem 3 is powerful because containing $\mathbf{D}_1$ is a purely lattice-theoretic demand, whereas the relational product operation $\circ$ is not redundant in the claim that no nontrivial congruence equation is satisfied (witness the variety of semilattices).

Recall that every finite lattice can be embedded in the partition lattice $\mathbf{\Pi}_X$ of a *finite* set X [33]. Of course when $|X| \leq |Y|$ then $\mathbf{\Pi}_X$ can be embedded into $\mathbf{\Pi}_Y$. In particular, $\mathbf{D}_1$ is isomorphic to a sublattice of $\mathbf{\Pi}_4$ (see [12, p. 23] for the Hasse diagram of $\mathbf{\Pi}_4$). So Theorem 3 has the following consequence:

Corollary 4. *For any variety* V*, if* $\mathbf{\Pi}_\kappa$ *embeds in* $\mathbf{Con\,A}$ *for some* $\mathbf{A} \in \mathsf{V}$ *and some cardinal* $\kappa \geq 4$*, then* V *satisfies no nontrivial idempotent Maltsev condition.*

The converse is an open problem posed in [18]; it is open even for locally finite varieties.

3. Congruence Conditions and IPOC

Since the subvarieties of IPOC have desirable congruence properties, the claim that V(IPOC) satisfies no nontrivial congruence equation strengthens the claim that IPOC is not a variety. An algebraic proof of either claim must involve infinite algebras, in view of the following result, essentially established in [6].

Theorem 5. *If a locally finite variety is generated by involutive pocrims then it* consists *of involutive pocrims, and is therefore congruence distributive.*

The proof of Theorem 5 uses the fact that in a variety generated by a class K of similar algebras, the free algebras belong to the *quasivariety* generated by K, hence they are involutive pocrims if $\mathsf{K} \subseteq$ IPOC. Further, since IPOC satisfies $x^{n+1} \leq x^n$ for all $n \in \omega$ (where $x^0 := t$ and $x^{n+1} := x^n \cdot x$), it follows that every *finite* involutive pocrim satisfies $x^n \approx x^{n+1}$ for some $n \in \omega$. So, in a variety V generated by involutive pocrims, if the 1–generated free algebra is finite, then V satisfies $x^n \approx x^{n+1}$ for some finite

n. Now any involutive pocrim satisfying $x^n \approx x^{n+1}$ also satisfies

$$(x \to y) \to^n (y \to x) \to^n x \approx (y \to x) \to^n (x \to y) \to^n y$$

(where $x \to^0 y := y$ and $x \to^{n+1} y := x \to (x \to^n y))$, see [6]. As the 2–generated free algebra in V belongs to IPOC, the above equation holds throughout V, and it clearly entails (M5), whence V consists of involutive pocrims. Finally, as we observed in Section 1, every variety of involutive pocrims is congruence distributive.

In [10], Grishin gave an algebraically non-constructive and quite complex proof that IPOC is not a variety. The argument shows that the free 1–generated involutive pocrim $\mathbf{F}$ must map onto a certain four-element algebra $\mathbf{G}$ that violates (M5), but almost nothing is known about the structure of the infinite algebra $\mathbf{F}$. Krzystek's proof of the same result is algebraically constructive and elegant, but the reference [21] does not seem to be well known. The possibility of a really simple proof is blocked by the following still-open problem:

Problem 1. *Are the varieties* $\mathrm{V}(\mathsf{IPOC})$, $\mathrm{V}(\mathsf{POCR})$ *and* $\mathrm{V}(\mathsf{BCKA})$ *finitely based*?

These questions were raised for $\mathrm{V}(\mathsf{POCR})$ and $\mathrm{V}(\mathsf{BCKA})$ in [11, 20, 31].

A generalization of the argument from [21] is presented below. The extra generality will allow us to explain the main result about congruence equations in a direct way. Our adaptation of [21] is in the spirit of the 'distension' construction from [38].

Let Z, Z^- and Z^+ denote the respective sets of all integers, of all non-positive integers, and of all non-negative integers. Given a nonzero ordinal $\tau \leq \omega$ and distinct entities $\bot, \top \notin Z \cup \tau$, we shall construct an involutive pocrim $\mathbf{A}_\tau$ with universe

$$A_\tau = (\{\bot\} \times Z^+) \cup (\tau \times Z) \cup (\{\top\} \times Z^-).$$

We shall use i, j, k to denote integers, α, β to denote elements of τ, and x, y to denote elements of A_τ. Let $\leq$ be the partial order of A_τ depicted in Figure 2 and note that

$$\langle \alpha, i \rangle \leq \langle \beta, j \rangle \quad \text{iff} \quad i + |\,\alpha - \beta\,| \leq j.$$

We define

$$\neg\langle\top,i\rangle=\langle\bot,-i\rangle,\quad \neg\langle\alpha,j\rangle=\langle\alpha,-j\rangle,\quad \neg\langle\bot,k\rangle=\langle\top,-k\rangle\quad (i\le 0\le k).$$

The operation $\cdot$ on A_τ is defined thus:

$$\begin{array}{lll}
x\cdot y=y\cdot x & & \\
\langle\top,i\rangle\cdot\langle\top,j\rangle = \langle\top,i+j\rangle & & (i,j\le 0)\\
\langle\top,i\rangle\cdot\langle\alpha,j\rangle = \langle\alpha,i+j\rangle & & (i\le 0)\\
\langle\top,i\rangle\cdot\langle\bot,j\rangle = \langle\bot,\max\{0,\ i+j\}\rangle & & (i\le 0\le j)\\
\langle\alpha,i\rangle\cdot\langle\beta,j\rangle = \langle\bot,\max\{0,\ i+j+|\,\alpha-\beta\,|\}\rangle & & \\
\langle\alpha,i\rangle\cdot\langle\bot,j\rangle = \langle\bot,k\rangle\cdot\langle\bot,j\rangle = \langle\bot,0\rangle & & (0\le j,k)
\end{array}$$

This makes $\langle A_\tau;\cdot,0\rangle$ a commutative monoid which is residuated with respect to the partial order $\le$.

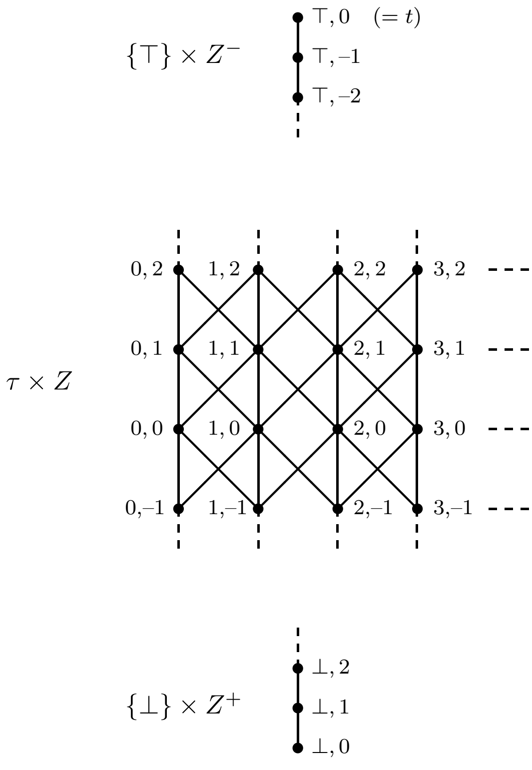

Figure 2. Hasse Diagram of $\langle A_\tau;\le\rangle$

The resulting pocrim $\langle A_\tau; \cdot, \to, \langle\top, 0\rangle\rangle$ has the following residuation properties:

$$
\begin{array}{lcll}
x \leq y \ \text{ iff } \ x \to y & = & \langle \top, 0\rangle & \\
\langle \top, i\rangle \to \langle \top, j\rangle & = & \langle \top, \min\{0,\, j-i\}\rangle & (i, j \leq 0) \\
\langle \top, i\rangle \to \langle \alpha, j\rangle & = & \langle \alpha, j-i\rangle & (i \leq 0) \\
\langle \top, i\rangle \to \langle \bot, j\rangle & = & \langle \bot, j-i\rangle & (i \leq 0 \leq j) \\
\langle \alpha, i\rangle \to \langle \beta, j\rangle & = & \langle \top, \min\{0,\, j-i-|\,\alpha-\beta\,|\}\rangle & \\
\langle \alpha, i\rangle \to \langle \bot, j\rangle & = & \langle \alpha, j-i\rangle & (0 \leq j) \\
\langle \bot, i\rangle \to \langle \bot, j\rangle & = & \langle \top, \min\{0,\, j-i\}\rangle & (0 \leq i, j)
\end{array}
$$

Clearly $\mathbf{A}_\tau = \langle A_\tau; \cdot, \to, \neg, \langle\top, 0\rangle\rangle \in \mathsf{IPOC}$. The algebra $\mathbf{A}_2$ appears in [21].

Let 0_τ and 1_τ be the finest and coarsest partitions of τ, respectively. That is, $0_\tau = \{\{\alpha\} : \alpha \in \tau\}$ and $1_\tau = \{\tau\}$. For any partition π of τ,

$$\{\{\bot\} \times Z^+,\ \{\top\} \times Z^-\} \cup \{Y \times Z : Y \in \pi\}$$

is the set of equivalence classes of a congruence $\theta(\pi)$ of $\mathbf{A}_\tau$. It follows easily that the function $\pi \mapsto \theta(\pi)$ is an injective lattice homomorphism from the partition lattice $\mathbf{\Pi}_\tau$ of τ into the interval sublattice $[\theta(0_\tau),\, \theta(1_\tau)]$ of $\mathbf{Con}\,\mathbf{A}_\tau$. (The main point is that joins are preserved, by definition of $\theta(\pi)$, because the congruence lattice of an algebra is always a sublattice of the equivalence lattice of its universe.) So the lattice $\mathbf{\Pi}_\tau$ also embeds into this interval of $\mathbf{Con}\,\mathbf{A}_\tau$. Choosing $\tau \geq 4$, we may deduce from Corollary 4 that

Theorem 6. *The variety* $\mathrm{V}(\mathsf{IPOC})$ *satisfies no nontrivial idempotent Maltsev condition, i.e., it satisfies no nontrivial congruence equation* (*in* $\wedge, \vee, \circ$).

Corollary 7. $\mathrm{V}(\mathsf{IPOC})$ *is not congruence meet semi-distributive, i.e., it is not congruence neutral.*

Recall that a variety V is said to be *congruence meet semi-distributive* if the quasi-identity $x \wedge y \approx x \wedge z \implies x \wedge y \approx x \wedge (y \vee z)$ holds in the congruence lattices of all algebras from V. This is equivalent to *congruence neutrality*, i.e., the demand that $[x, y] = x \cap y$ should hold in the congruence lattices of all algebras in V, where $[\,,\,]$ is the TC commutator: see [19, 23]. When these equivalent conditions hold then V satisfies a nontrivial idempotent Maltsev condition; see [35] for a direct proof of this.

Corollary 8. *No semilattice operation is termwise definable over all involutive pocrims.*

Here a semilattice operation is not assumed to be compatible with the pocrim order or operations. The corollary follows because semilattices form a congruence meet semi-distributive variety (although they do not satisfy a nontrivial congruence equation in $\wedge, \vee$ only) [9]. Among the 2–finite varieties, the congruence meet semi-distributive ones are just those that satisfy the congruence equation $\theta \wedge (\phi \circ \psi) \subseteq \phi \vee (\theta \wedge (\psi \vee (\theta \wedge \phi)))$, see [19].

We have noted that congruence n–permutable varieties satisfy nontrivial idempotent Maltsev conditions, so Theorem 6 entails the next corollary.

Corollary 9. *There is no integer $n \geq 2$ such that all involutive pocrims are congruence n-permutable.*

In fact, Lipparini [22] has shown that for each integer $n \geq 2$, every congruence n-permutable variety satisfies a nontrivial congruence equation in $\wedge, \vee$ (without $\circ$). See [24] and its bibliography for alternative proofs, and [18] for generalizations of this result.

Imitating an argument from [38], we can easily see directly that for every $n \geq 2$, the involutive pocrim $\mathbf{A}_\omega$ fails to be congruence n-permutable. And when $4 \leq \tau \leq \omega$ then $\mathbf{A}_\tau$ is not congruence modular, because its own congruence lattice contains a copy of $\mathbf{\Pi}_4$ (and therefore $\mathbf{D}_1$).

Of course these results witness the fact that IPOC is not a variety. More directly, for $\tau \geq 2$, the algebra $\mathbf{B}_\tau = \mathbf{A}_\tau/\theta(0_\tau)$ belongs to $\mathrm{V}(\mathsf{IPOC})$ but not to IPOC. Indeed, $\{\top\} \times Z^-$ interprets t in $\mathbf{B}_\tau$, and if $\tau \geq 2$ then

$$a := \{0\} \times Z \quad \text{and} \quad b := \{1\} \times Z$$

are distinct elements of B_τ with $a \to b = t$ and $b \to a = t$, so $\mathbf{B}_\tau$ violates (M5). This, for $\tau = 2$, is the proof given in [21] that IPOC is not a variety.

The four-element algebra $\mathbf{G} \in \mathrm{V}(\mathsf{IPOC}) \setminus \mathsf{IPOC}$ exhibited by Grishin in [10] is different from $\mathbf{B}_2$. In $\mathbf{B}_2$, we have $\neg\, a = a$ and $\neg\, b = b$, whereas $\neg$

has no fixed point in $\mathbf{G}$. Apparently this makes it harder to construct a transparent involutive pocrim that maps onto $\mathbf{G}$.

The $\langle\rightarrow\rangle$-reduct of $\mathbf{A}_\tau$ has just the same congruences as $\mathbf{A}_\tau$. Thus, Theorem 6 through Corollary 9 hold for the varieties $\mathrm{V}(\mathsf{POCR})$ and $\mathrm{V}(\mathsf{BCKA})$ as well, and this is predictable from Theorem 2.

4. Alternative Proofs and Ramifications

In the particular case of $\mathrm{V}(\mathsf{BCKA})$, one can give a syntactic proof of the absence of congruence equations in $\wedge, \vee, \circ$.

We adopt the convention that $x \rightarrow y \rightarrow z$ abbreviates $x \rightarrow (y \rightarrow z)$. The identity (M1) and the associativity and commutativity of $\cdot$ invite us to define

$$(\Pi_{i<n}\, \alpha_i) \rightarrow y \;:=\; \alpha_0 \rightarrow \alpha_1 \rightarrow \ldots \rightarrow \alpha_{n-1} \rightarrow y \qquad (1)$$

for any variable y, any $n \in \omega$ and any $\langle\rightarrow\rangle$-terms α_i, $i < n$. If $n = 0$ we interpret the left hand side of (1) as y. We call y the *right-most variable* of the term in (1). Clearly every $\langle\rightarrow\rangle$-term is of this form. The following result, conjectured by Wroński, was proved in [26].

Lemma 10. (Nagayama) *Let $\alpha \approx \beta$ be a $\langle\rightarrow\rangle$-equation that is satisfied by every BCK-algebra. Then either the terms α and β have the same right-most variable or every BCK-algebra satisfies $\alpha \approx t$ (and therefore $\beta \approx t$).*

This, together with Theorem 2, gives a different proof of

Theorem 11. *The variety* $\mathrm{V}(\mathsf{BCKA})$ *satisfies no nontrivial idempotent Maltsev condition.*

Proof. Since t is definable over $\mathrm{V}(\mathsf{BCKA})$ as $x \rightarrow x$, it is enough to prove the result for the variety V of t-free reducts of members of $\mathrm{V}(\mathsf{BCKA})$. Suppose V satisfies a nontrivial idempotent Maltsev condition. Then V satisfies a scheme of equations as displayed in Theorem 2 (iii) for some $n > 1$, some term $\alpha(z_1, \ldots, z_n)$ (in *distinct* variables $z_1, \ldots, z_n$) that is idempotent over V and some choice of variables x_{ij}, y_{ij}, $1 \leq i, j \leq n$, where $x_{ii} \neq y_{ii}$ for each i. Now we must have

$$\alpha(z_1, \ldots, z_n) = (\Pi_{k<m}\, \beta_k(z_1, \ldots, z_n)) \rightarrow z_j$$

for some fixed $j \in \{1, \ldots, n\}$ and some finite family $\{\beta_k : k < m\}$ of $\langle\rightarrow\rangle$-terms. The j-th equation in the scheme is therefore

$$(\Pi_{k<m}\,\beta_k(x_{1j}, \ldots, x_{nj})) \rightarrow x_{jj} \;\approx\; (\Pi_{k<m}\,\beta_k(y_{1j}, \ldots, y_{nj})) \rightarrow y_{jj}.$$

The right-most variables x_{jj} and y_{jj} in this equation are different, by assumption. Therefore, by Lemma 10, the term on the left hand side of the equation is equal to t in all BCK-algebras. Of course this remains true when all of its variables are set equal, i.e., all BCK-algebras satisfy $\alpha(x, x, \ldots, x) \approx t$. This contradicts the idempotence of α over V(BCKA). □

For pocrim terms we have no normal forms and no notion of 'right-most variable', so no useful variant of Lemma 10 can be expected. Thus, the above proof is not adaptable to V(POCR) or V(IPOC).

A polynomial time algorithm implemented in the Algebra Calculator Program [8] computes the 'type set' of any finite algebra **A**. This is a non-empty subset of $\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$. (The program also computes the type set of any free algebra in V(**A**) on finitely many free generators.) The following result is [12, Thm. 9.6].

Theorem 12. (Hobby and McKenzie) *A locally finite variety* V *satisfies no nontrivial idempotent Maltsev condition iff* **1** *belongs to the type set of some finite algebra in* V*.*

Applying this result to V(IPOC) and to the finite algebra $\mathbf{B}_2$ from the previous section, we get a very swift (but less transparent) proof of Theorem 6, since the type set of $\mathbf{B}_2$ is $\{\mathbf{1}, \mathbf{3}\}$. The same is true of Grishin's algebra **G**, and likewise the reducts of these algebras that retain $\rightarrow$.

Definition 3. An [involutive] pocrim is said to be *representable* if it is a subdirect product of totally ordered [involutive] pocrims.

The class of all representable involutive pocrims is the quasivariety axiomatized relative to IPOC by the identity

$$((x \rightarrow y) \rightarrow z) \rightarrow ((y \rightarrow x) \rightarrow z) \rightarrow z \approx t \tag{2}$$

(see [28]). Another analogy with [38] is that the subalgebra of $\mathbf{A}_2$ generated by $\{\langle 0, 0\rangle, \langle 1, 1\rangle\}$ is a totally ordered involutive pocrim with the same significant properties as $\mathbf{A}_2$. Factoring out the restriction of $\theta(0_2)$, we get an algebra isomorphic to $\mathbf{B}_2$ again. So, from the type set of $\mathbf{B}_2$ and Theorem 12, we can deduce the following strengthening of Theorem 6.

Theorem 13. *The variety generated by all representable involutive pocrims satisfies no nontrivial idempotent Maltsev condition.*

The use of Algebra Calculator Program in Theorem 13 could be eliminated. Along the lines of [3, pp. 73–74], we can construct a finite subdirect power of $\mathbf{B}_2$ whose congruence lattice contains the partition lattice of a four-element set.

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