REPORTS ON MATHEMATICAL LOGIC
42 (2007), 47–69

Roland HINNION

INTENSIONAL SOLUTIONS TO THE IDENTITY PROBLEM FOR PARTIAL SETS

A b s t r a c t. A general forcing method is developed that allows to construct pure term models for (roughly speaking) predicative first-order partial set theory, where specific identification/differentiation rules hold. The applications bring new results about the links with intensionality and extensionality.

1. Introduction

Partial sets can be seen in several ways: as objects that code partial information; as non-classical sets allowing to escape the paradoxes; as naive sets in a paracomplete (or partial) logic; as duals of paraconsistent sets, etc. The list is probably not exhaustive.

From the very beginning, it appeared that, in contrast to other forms of set

Received 25 November 2005
Math. Subject Classification: 03E35, 03E40, 03E70, 03C15, 03C25, 03C90.
Keywords: Abstraction, Partial Set Theory, Paracomplete Set Theory, Intensionality, Extensionality.

theory linked to positive comprehension (or abstraction), i.e. the "positive set theories" (see e.g. [2], [6]) and the "paraconsistent set theories" (see e.g. [3], [9]), the partial set theories present incompatibilities with identification/differentiation criteria (be they axioms or rules). As we don't want to repeat here those matters that have already been clearly explained elsewhere, we refer the reader to previous publications, for more details; in particular:

- [7] [8] for the initial formalisations of partial set theories, and the discovery of their incompatability with extensionality,
- [9] [17] for the links with naive set theory in paracomplete logics, and the duality aspects with respect to paraconsistent set theories,
- [10] for the relation to "partial information", and an introduction to the forcing method that is adapted here.

Section 2 presents the languages and theories, with remarks about identity criteria. Section 3 describes the general forcing method used to construct pure term models for a partial set theory that we denote PPST ("predicative partial set theory"). Section 4 finally explores some consistency and independence results (concerning PPST with respect to intensionality and/or extensionality rules and/or axioms) that are obtained via the forcing method.
Our metatheory is Zermelo-Fraenkel set theory with the axiom of choice.

2. Partial set theories

We will call $\mathcal{L}_G$ the language introduced by P.C. Gilmore in [7] (see also [8]). It is based on 4 primitive relational symbols $\in$, $\notin$, $=$, $\neq$ together with the abstraction operator $\{ \mid \}$. The positive terms, positive formulas and (general) formulas are then build up by following the rules:

(1) any variable (small letter) is a positive term,

(2) if τ, τ' are positive terms, then $\tau = \tau'$, $\tau \neq \tau'$, $\tau \in \tau'$, $\tau \notin \tau'$ are positive formulas ("atomic formulas"),

(3) if φ, ψ are positive formulas, then $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists x\varphi$, $\forall x\varphi$ are positive formulas,

(4) if φ is a positive formula, then $\{x|\varphi\}$ is a positive term,

(5) if φ is a formula, then $\neg\varphi$ is a formula.

We will first consider the theory PST^+ (one of the variants of Gilmore's partial set theories), whose axioms are (the universal closure of):

1. the scheme expressing that $=$ is a congruence for $\mathcal{L}_G$,
2. the "partiality axioms":
   - $\neg(x \in y \wedge x \notin y)$,
   - $\neg(x = y \wedge x \neq y)$.
3. the abstraction scheme:
   For any positive term $\{x|\varphi(x,\vec{y})\}$,

   $$(t \in \{x|\varphi(x,\vec{y})\} \leftrightarrow \varphi(t,\vec{y})) \wedge (t \notin \{x|\varphi(x,\vec{y})\} \leftrightarrow \overline{\varphi}(t,\vec{y})),$$

   where $\overline{\varphi}$ (the "positive negation" of φ) is defined inductively by:

   - $\overline{\tau \in \tau'}$ is $\tau \notin \tau'$,
   - $\overline{\tau = \tau'}$ is $\tau \neq \tau'$,
   - $\overline{\overline{\psi}}$ is ψ,
   - $\overline{\psi \vee \theta}$ is $\overline{\psi} \wedge \overline{\theta}$,
   - $\overline{\exists x\psi}$ is $\forall x\overline{\psi}$.

   (An immediate consequence of this is that $\overline{\tau \notin \tau'}$ is $\tau \in \tau'$, $\overline{\tau \neq \tau'}$ is $\tau = \tau'$, $\overline{\psi \wedge \theta}$ is $\overline{\psi} \vee \overline{\theta}$, and $\overline{\forall x\psi}$ is $\exists x\overline{\psi}$.)

It should be noticed that $\notin$ is not the classical "complement" of $\in$, i.e. that we do <u>not</u> expect $x \notin y \leftrightarrow \neg x \in y$; naturally the same remark holds for $\neq$ with respect to $=$. Precisely, this allows to avoid Russell's paradox, as from $\rho \in \rho \leftrightarrow \rho \notin \rho$, obtained via abstraction for the positive term $\rho := \{x|x \notin x\}$, we simply get $\neg\rho \in \rho$ and $\neg\rho \notin \rho$, which is not contradictory here.

It was shown in [8] that PST^+ disproves the extensionality axiom EXT, whose natural form in this context is: EXT : $x \underset{\cdot}{=} y \rightarrow x = y$, where $\underset{\cdot}{=}$ is

defined by $x \doteqdot y$ iff $\forall t[(t \in x \leftrightarrow t \in y) \wedge (t \notin x \leftrightarrow t \notin y)]$.
We briefly sketch the proof here, as we will have to refer to it in further remarks about intensionality. Consider the following positive terms in $PST^+ + EXT$:

- $\rho := \{x | x \notin x\}$
- $\phi := \{x | \rho \in \rho\}$
- $\tau(t) := \{x | t \in t\}$
- $\tau^\star := \{t | \{y | y \in \tau(t) \wedge y \in \phi\} = \tau(t)\}$.

One can easily check successively that:

- ϕ is the "double empty set", i.e. $\forall x \neg x \in \phi$, and $\forall x \quad \neg x \notin \phi$
- $\tau(\tau^\star) \doteqdot \tau(\tau^\star) \cap \phi$; where $a \cap b$ is the term $\{x | x \in a \wedge x \in b\}$
- so by $EXT : \tau(\tau^\star) = \tau(\tau^\star) \cap \phi$
- and by abstraction: $\tau^\star \in \tau^\star$
- then, by the definition of $\tau(t) : \phi \in \tau(\tau^\star)$, which contradicts $\tau(\tau^\star) = \tau(\tau^\star) \cap \phi$.

So it is natural to try to make identifications (i.e. $\tau = \tau'$) on another basis than EXT (i.e. via $\tau \doteqdot \tau'$); and an apparently "reasonable" basis could be the equivalence of the formulas defining the concerned positive terms.
For example it is natural to expect $\{x | x \in x\} = \{x | x \in x \wedge x \in x\}$, as the formulas "$x \in x$" and "$x \in x \wedge x \in x$" are simply logically equivalent!
But due to the "non-classical" situation here, and more particularly to the form of the abstraction axioms, is it important to notice that, even if φ and ψ are equivalent, we do not necessarily have that $\overline{\varphi}$ and $\overline{\psi}$ are equivalent!
In order to take that into account, we will consider the following stronger bi-implication:

$$\varphi \overset{st}{\leftrightarrow} \psi \text{ iff (def)} (\varphi \leftrightarrow \psi) \wedge (\overline{\varphi} \leftrightarrow \overline{\psi}),$$

where φ, ψ are positive formulas.
And a kind of basic intensionality rule could then be:

$$\frac{\vdash \varphi \overset{st}{\leftrightarrow} \psi}{\{x | \varphi\} = \{x | \psi\}}$$

(where $\vdash$ is the provability-symbol in classical logic with equality).
More generally, we can introduce rules of the type:

$$\frac{\Gamma \vdash \varphi \overset{st}{\leftrightarrow} \psi}{\{x|\varphi\} = \{x|\psi\}}$$

where Γ is some theory (in the concerned language).
We have now also to think about the differentiation relation $\neq$, which also has lost here its classical connexion with its counterpart $=$. For classical sets, $\neq$ is ruled once $=$ is, simply because $x \neq y$ is $\neg x = y$! For partial sets, however, rules of identification and rules of differentiation are rather independent. It is even so that many partial set theories are simply unable to prove that the $\neq$ relation is not empty (i.e. $\exists x, y \quad x \neq y$)! This is the case for PST^+. Indeed, just interpret $\in^\star$, $\notin^\star$, $=^\star$, $\neq^\star$ in PST^+ via the definitions:

- $x \in^\star y$ iff $x \in y$
- $x \notin^\star y$ iff $x \notin y$
- $x =^\star y$ iff $x = y \vee \rho \in \rho$
- $x \neq^\star y$ iff $x \neq y \wedge \rho \notin \rho$

where $\rho := \{x|x \notin x\}$.
This gives an interpretation of $PST^+ + \forall x, y\, \neg x \neq y$ in the theory PST^+ itself, so that $PST^+ \nvdash \exists x, y\, x \neq y$ (assuming PST^+ is consistent, of course).
To summarize (and roughly speaking): a partial set theory without $\neq$-rules presents the risk of having an empty $\neq$.
Several differentiation rules have already been proposed in the past. In [7], for example, one finds the axiom $x \mathrel{\dot{\neq}} y \leftrightarrow x \neq y$, where $x \mathrel{\dot{\neq}} y$ is the abbreviation of the positive formula $(\exists t \ni x \quad y \notin t) \vee (\exists t \not\ni x \quad y \in t))$; while in [9] we rather studied the axiom $x \mathrel{\underset{\cdot}{\neq}} y \leftrightarrow x \neq y$, where $x \mathrel{\underset{\cdot}{\neq}} y$ stands for $(\exists t \in x \;\, t \notin y) \vee (\exists t \notin x \;\, t \in y))$. Here, we will be led to also consider intensionality rules (for $\neq$) of the type:

$$\frac{\Gamma \vdash \exists x \;\neg \varphi \overset{st}{\leftrightarrow} \psi}{\{x|\varphi\} \neq \{x|\psi\}}.$$

Now, as was remarked for positive set theories in [10], intensionality (instead of extensionality) is in itself not a guarantee against paradoxes

(see also [7]). This will actually depend on the theory Γ used in such intensionality rules! For example, PST^+ is incompatible with the intensionality rule for = that would use PST^+ itself as Γ. Indeed, consider the four terms defined before $(\rho, \phi, \tau(t), \tau^\star)$ and observe that, modulo the abstraction scheme, one can prove that the formulas defining $\tau(\tau^\star)$ and $\tau(\tau^\star) \cap \phi$ are strongly equivalent ($\overset{st}{\leftrightarrow}$), so that the concerned intensionality rule produces $\tau(\tau^\star) = \tau(\tau^\star) \cap \phi$, and so the same contradiction follows as before. We notice that the term $\tau^\star$ used in that version of Russell's paradox presents the particularity that it abstracts on a variable ("t") that is free in a proper (i.e. not just a pure variable) term. If (as in the language that we will consider below) we forbid that possibility, we have also to forbid quantification on such variables, because otherwise we can pass round the previous interdiction, with "tricks" like: replacing the concerned variable "t" by another one "z" (not used elsewhere in the term), for which we indicate at the adequate place in the term that "$\exists z \quad z = t \wedge \ldots$"! For example could we "simulate" the term $\tau^\star$ (defined before) by
$\tau' := \{t | \ \exists z \ (z = t \wedge \tau(z) = \tau(z) \cap \phi)\}$; and in τ' we indeed don't abstract on variables free in a proper term!
This leads us to formulate
Rule I: never abstract nor quantify on a variable that is free in a proper term. In [10] it is shown that taking this rule in account in the construction of positive terms suffices to get compatibility with intensionality rules, in the case of "positive set theories".
The situation here however brings in new difficulties, due to the fact that partial sets have to satisfy the extra-axioms of partiality; and in order to let our forcing technique work, will we have to add the following
Rule II: abstraction and quantification are not allowed on a variable "at the right" of an abstracted variable (we say that y is at the right of x in the situations: $x \in y$, $x \notin y$).
We will call "predicative" the formulas and terms of $\mathcal{L}_G$ that follow the Rules I and II; this terminology is probably not the ideal one, but suggests rather well the idea of restrictions on abstraction and quantification in formulas used to define sets.
We denote $\mathcal{L}_p$ the fragment of the language $\mathcal{L}_G$ where only **predicative** (as defined before) positive terms are allowed.
The construction of the class U_ω of the closed terms in $\mathcal{L}_p$ (that follows

some preliminary definitions) shows that $\mathcal{L}_p$, however not **strictu sensu** a first-order language, is very close to $\mathcal{L}$ (the first-order fragment of $\mathcal{L}_G$, obtained by dropping the abstraction operator).

Definitions. A positive formula $\psi(x, \vec{y})$ in $\mathcal{L}$ will be called "predicative with respect to x" (notation: pred/x) when any variable at the right of x is distinct from x and is free in ψ (as usually does the notation $\varphi(\vec{z})$ indicate that any free variable in φ is in the list $\vec{z} = z_1, z_2, \ldots, z_k$)

$$\begin{aligned}
U_0 &:= \{\{x|\psi(x)\}|\psi(x) \text{ is a positive formula in } \mathcal{L} \text{ and } \psi(x) \text{ is pred}/x\} \\
U_{k+1} &:= \{\{x|\psi(x, \vec{b})\}|\psi(x, \vec{y}) \text{is a positive formula in } \mathcal{L},\ \psi(x, \vec{y}) \text{ is} \\
&\qquad \text{pred}/x \text{ and each } b_i \text{ is in } U_k\}. \\
U_\omega &:= \cup_{k \in \omega} U_k
\end{aligned}$$

Actually the "pure term models" that we construct in section 3 will all have U_ω as universe. For τ in U_ω, the "rank of τ" will be the least $k \in \omega$ such that $\tau \in U_k$.
Further, if τ is $\{x|\psi(x, \vec{b})\}$, we denote $\varphi_\tau(x)$ the formula $\psi(x, \vec{b})$; notice that $\psi(x, \vec{y})$ is in the language $\mathcal{L}$, while $\varphi_\tau(x)$ is in $\mathcal{L}_p$.
We will call "predicative partial set theory" (notation: $PPST$) the fragment of PST^+ restricted to predicative positive terms; more precisely: the axioms of $PPST$ are like those of PST^+, except that the scheme (in axiom 1) now expresses that $=$ is a congruence for $\mathcal{L}_p$ (instead of $\mathcal{L}_G$) and that the abstraction scheme is restricted to positive terms in $\mathcal{L}_p$ (i.e. predicative positive terms in $\mathcal{L}_G$).

3. Pure term models

We call "U_ω-structure" any structure M of the type (U_ω, $\in_M$, $\notin_M$, $=_M$, $\neq_M$), where $\in_M$, $\notin_M$, $=_M$, $\neq_M$ are arbitrary binary relations on U_ω. On the class of these structures we put the following partial order:
$M \leq N$ ("N is an extension of M")
iff (def)
$R_M \subset R_N$, for $R :\in, \notin, =, \neq$.
The theory ADM (for "admissible") has as axioms (in the language $\mathcal{L}_p$):

(1) $=$ is a congruence for $\mathcal{L}$

(2) the partiality axioms

(3) for $\psi(x,\vec{y})$ a positive formula in $\mathcal{L}$, predicative with respect to x.:

$$\forall\vec{y}\forall t\left[(t\in\{x|\psi(x,\vec{y})\}\rightarrow\psi(t,\vec{y}))\wedge(t\notin\{x|\psi(x,\vec{y})\}\rightarrow\overline{\psi}(t,\vec{y}))\right]$$

We call "admissible" any U_ω-structure that is a model for ADM. Notice that (3) could equivalently (i.e. without modifying the concept of "admissible structure") be replaced by: $(\forall x\in\tau\ \ \varphi_\tau(x))\wedge(\forall x\notin\tau\ \ \overline{\varphi}_\tau(x))$, for each τ in U_ω.
The theory $ABST$ (for "abstraction") is like ADM, except that in (3) the implications become bi-implications.

Comments.

- The reader could be astonished to see that ADM (in axiom (1)) only asks for a congruence for $\mathcal{L}$, and not necessarily for $\mathcal{L}_p$; actually will we see that this suffices to get, in fine, structures G where $=_G$ is indeed a congruence for $\mathcal{L}_p$.
- In terms of "partial information" (see also [10]) does the axiom (3), that we will call "pre-abstraction", express the fact that the information, however probably incomplete, is at least reliable: only those objects "x" satisfying the "criterion" $\varphi_\tau(x)$ will be listed positively in τ (i.e. $x\in\tau$); and the analogue for $\overline{\varphi}_\tau$ and $\notin$. Abstraction then reflects the ideal situation where the available information has been encoded completely (all the x's such that $\varphi_\tau(x)$ are then $\in$-members of τ, etc).

Some more definitions and notations:
We will let Adm denote the class of all admissible structures. The notation $M\overset{\leq}{\underset{X}{}}N$ will indicate that $M\leq N$, that both M,N are in X and that X is a non-empty part of Adm.
A "chain" in Adm will be a sequence of the type $(M_\alpha)_{\alpha<\beta}$, increasing for $\leq$ (i.e. $\alpha\leq\alpha'\rightarrow M_\alpha\leq M_{\alpha'}$; the indices are ordinals).
The "limit" (or "union") of such a chain is simply the (still admissible; cf. Fact 1 below) structure:

$$(U_\omega,\cup_{\alpha<\beta}\in_{M_\alpha},\cup_{\alpha<\beta}\notin_{M_\alpha},\cup_{\alpha<\beta}=_{M_\alpha},\cup_{\alpha<\beta}\neq_{M_\alpha}).$$

We will consider a "forcing relation", $\Vdash$ that will appear in expressions of the type $M \Vdash_X \tau = \tau'$, and $M \Vdash_X \tau \neq \tau'$ (we only force on $=$, and on $\neq$), where M is admissible, $X \subset Adm$ (X non-empty) and $\tau, \tau' \in U_\omega$.
And we expect $\Vdash$ to satisfy the four following conditions (Section 4 contains several examples of such forcing conditions):

Cond 1: (for σ of the type $\tau = \tau'$, or $\tau \neq \tau'$)

$$N \geq_{Adm} M \Vdash_X \sigma \to N \Vdash_X \sigma$$

Cond 2: (for M, X fixed).
The binary relation: $x \sim y$ iff (def) $M \Vdash_X x = y$, is an equivalence relation on U_ω

Cond 3:

$$M \Vdash_X \tau = \tau' \to \exists N \geq_X M \;\; N \models \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x))$$

and

$$M \Vdash_X \tau \neq \tau' \to \exists N \geq_X M \;\; N \models \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x))$$

Cond 4:

$$\vec{a} =_M \vec{b} \to M \Vdash_X \{x | \psi(x, \vec{a})\} = \{x | \psi(x, \vec{b})\}.$$

Comments. Cond 1 corresponds to the classical so-called "extension lemma" (see e.g. [1]). Cond 2 guarantees that $=$ will be a congruence for $\mathcal{L}$. Cond 3 is a more technical one, that will be used in the proofs. Cond 4 guarantees that $=$ will be a congruence for $\mathcal{L}_p$.

Definitions. ($X \subset Adm$; X non-empty).

- $X^+ := \{M \in X |$(for any σ of the type $\tau = \tau'$, or $\tau \neq \tau'$) $M \models \sigma \to M \Vdash_X \sigma\}$.
- $\begin{cases} A_0 := Adm \\ A_{\alpha+1} := A_\alpha^+ \\ A_{\gamma(limit)} := \bigcap_{\beta<\gamma} A_\beta \end{cases}$
- X is "closed under limits" iff the limit of any chain in X stays in X.

Actually we will exclusively use parts X of Adm that are indeed closed under limits; the advantage of these is that they always admit maximal elements (for $\leq$), via the usual Zorn-argument (remember that our metatheory is ZF with the axiom of choice).

Some elementary facts.

Fact 1 ("positive preservation"):
If M, N are U_ω-structures, $\theta(\vec{x})$ is a positive formula in $\mathcal{L}$, and $\vec{b}$ is in U_ω, then $N \geq M \models \theta(\vec{b})$ implies $N \models \theta(\vec{b})$.

Fact 2:
If X is closed under limits, then so is X^+.
Consequence: each A_α (as defined before) is closed under limits.

Fact 3:
Each A_α is non-empty.
Indication: just consider the minimum element in $Adm, \leq$:
$(U_\omega, \phi, \phi, id, \phi)$, where "$id$" is the identity relation on U_ω, and ϕ is the empty set.

Fact 4:
For some ordinal δ, we have $A_{\delta+1} = A_\delta$.
Indication: use the classical fixed-point argument, on the $\subset$-decreasing chain $(A_\alpha)_{\alpha ordinal}$.

So from here on will "δ" be supposed to be the least ordinal such that $A_{\delta+1} = A_\delta$ (i.e. $A_\delta^+ = A_\delta$). And we fix some G, maximal in $A_\delta, \leq$.
Further do we adopt the following simplified notations:

$$\begin{cases} \Vdash_\alpha & \text{for } \Vdash_{A_\alpha} \\ \leq_\alpha & \text{for } \leq_{A_\alpha} \end{cases}$$

Lemma 1 *G is a model for ABST.*

Proof. As G belongs to Adm, it suffices obviously to show that G satisfies the abstraction scheme; more precisely: fix some term τ (element of U_ω) and some element x (also in U_ω), and suppose $G \models \varphi_\tau(x)$.
We should prove that $x \in_G \tau$.

In order to achieve this, we construct a new structure G', that extends G: the universe is still U_ω, and only "$\in$" is modified, via the definition:

$$a \in_{G'} b \quad \text{iff} \left[(x =_G a \wedge b =_G \tau) \vee a \in_G b\right].$$

We prove in 3 steps (via an induction on α) that: $\forall\alpha \quad G' \in A_\alpha$; so in particular do we get: $G' \in A_\delta$.

First step: we prove that $G' \in A_0 := Adm$. The only less trivial point is to show that:

$$(x' =_G x \wedge \tau' =_G \tau) \to G' \models \varphi_{\tau'}(x').$$

As $G \models \tau = \tau'$ and $G \in A_{\delta+1}$, we have $G \Vdash_\delta \tau = \tau'$; and so, by cond 3:

$$\exists N \geq_\delta G \quad N \models \forall t \left(\varphi_\tau(t) \overset{st}{\leftrightarrow} \varphi_{\tau'}(t)\right).$$

As G is maximal in A_δ, this N is G, so that:

$$G \models \forall t \left(\varphi_\tau(t) \overset{st}{\leftrightarrow} \varphi_{\tau'}(t)\right).$$

With the hypotheses $x' =_G x$, $\tau' =_G \tau$, $G \models \varphi_\tau(x)$ we get then directly: $G \models \varphi_{\tau'}(x')$; and, by fact 1: $G' \models \varphi_{\tau'}(x')$.

Second step: using cond 1, it is easy to check that:

$$G' \in A_\alpha \to G' \in A_{\alpha+1}.$$

Third step: $\forall\beta < \gamma$ (limit) $G' \in A_\alpha$ trivially implies that $G' \in A_\gamma$.

By induction on α, the 3 preceding steps show that: $\forall\alpha \quad G' \in A_\alpha$; so also: $G' \in A_\delta$; and as $G \leq G'$ and G is maximal in A_δ, we get $G = G'$, and then obviously (as $x \in_{G'} \tau$, by the definition of $\in_{G'}$) the desired result: $x \in_G \tau$. □

A similar proof would show that $x \notin_G \tau$, once $G \models \overline{\varphi_\tau}(x)$; so that G indeed satisfies abstraction.

Lemma 2 *If τ, τ' are terms in U_ω, then $\neg(\tau \neq_G \tau' \wedge G \models \tau \doteq \tau')$.*

Proof. Suppose $\tau \neq_G \tau'$; so, as $G \in A_{\delta+1}$, we get $G \Vdash_\delta \tau \neq \tau'$, and (by cond 3): $\exists N \geq_\delta G\ \neg N \models \forall x \left(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)\right)$. As G is maximal in A_δ, this N is G, and we have: $\neg G \models \forall x \left(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)\right)$.
This, by lemma 1 (abstraction), obviously contradicts $G \models \tau \doteqdot \tau'$. □

Lemma 3 *If τ, τ' are terms in U_ω, and $\psi(x, \vec{y})$ is a positive formula in $\mathcal{L}$, predicative with respect to x, then*

$$\neg G \models \exists \vec{y}(\psi(\tau, \vec{y}) \wedge \overline{\psi}(\tau', \vec{y}) \wedge \tau \doteqdot \tau').$$

Proof. Let us call "catastrophe" a 4-tuple $(\psi(x, \vec{y}), \tau, \tau', \vec{b})$ such that $G \models \psi(\tau, \vec{b}) \wedge \overline{\psi}(\tau', \vec{b}) \wedge \tau \doteqdot \tau'$.
We have precisely to show that this cannot exist, under the hypotheses of Lemma 3.
We will call parametric rank of such a "catastrophe" the maximum rank (the rank of a term in U_ω is defined in section 2) of the parameters b_i for which the corresponding variable y_i is found somewhere at the right of x in the formula ψ; if there are no such b_i's, we put zero for this parametric rank.
Now, suppose that a catastrophe exists, and consider those of minimum parametric rank, say n_0. Choose one catastrophe of parametric rank n_0, that presents a minimally complex formula ψ (as first-order formula). One can easily check that this ψ cannot be of the type $\theta \vee \mathcal{N}$, or $\theta \wedge \mathcal{N}$, because otherwise one could easily get a catastrophe of same parametric rank, but with a strictly less complex formula.
We prove now that ψ cannot be of the type: $\forall t \quad \theta$ (a similar argument eliminates also the type $\exists t\ \theta$).
Indeed, suppose that $\psi(x, \vec{y})$ is $\forall t\, \theta(x, \vec{y}, t)$; and that we have $\tau, \tau', \vec{b}$ in U_ω, such that:

$$G \models \psi(\tau, \vec{b}) \wedge \overline{\psi}(\tau', \vec{b}) \wedge \tau \doteqdot \tau',$$

i.e.

$$G \models (\forall t\, \theta(\tau, \vec{b}, t)) \wedge (\exists t\, \overline{\theta}(\tau', \vec{b}, t)) \wedge \tau \doteqdot \tau'.$$

Choose some t_0 such that $\overline{\theta}(\tau', \vec{b}, t_0)$. As the variable "$t$" was quantified in ψ, we know (because ψ is pred/x) that "t" is not the variable "x" and that "t" is nowhere (in θ) at the right of "x". So that t_0 plays no role at all in the evaluation of the parametric rank of this (new) catastrophe, and is this

parametric rank still n_0; but this time with a strictly less complex formula (than before, for ψ): a contradiction.
At this point we can conclude that ψ has to be an atomic formula.
So we review the possible cases:

- $y \in x$: then the catastrophe is of the type: $G \models b \in \tau \wedge b \notin \tau' \wedge \tau \doteqdot \tau'$; an obvious contradiction!

- $x = y$; $x = x$: immediately excluded by Lemma 2.

- $x \in y$: here the catastrophe would look like this:
  $G \models \tau \in b \wedge \tau' \notin b \wedge \tau \doteqdot \tau'$. But b itself is of the type $\{y|\Delta(z, \vec{c})\}$, so that we get: $G \models \Delta(\tau, \vec{c}) \wedge \overline{\Delta}(\tau', \vec{c}) \wedge \tau \doteqdot \tau'$.
  And this is a catastrophe with a parametric rank $< n_0$: a contradiction.

- $y \in z$; $y \in y$; $z \in z$: are all trivially excluded.

- the other cases (concerning $\notin$, $\neq$) are eliminated in similar ways. □

Comment. The proof of Lemma 3 is the place where the role of the restrictions that led from $\mathcal{L}_G$ to $\mathcal{L}_p$ is the most evident.

Define $\Vdash_{On}$ by: $M \Vdash_{On} \sigma$ iff $\forall\alpha$ (ordinal) $M \Vdash_\alpha \sigma$.

Lemma 4 *(for σ of the type $\tau = \tau'$, or $\tau \neq \tau'$) $G \Vdash_{On} \sigma$ iff $G \models \sigma$.*

Comment. This shows that G has really the behaviour of a "generic" structure (as in more classical forcings; see e.g. [1]), in the sense that in such structures "true" and "forced" do coincide.

Proof of Lemma 4. That $G \models \sigma$ implies $G \Vdash_{On} \sigma$ is rather obvious from the definitions.
So we should only prove that:

$$G \Vdash_{On} \sigma \rightarrow G \models \sigma.$$

First case: σ is of the type $\tau = \tau'$. We fix some element τ in U_ω, and define the following extension $G^\star = (U_\omega, \in^\star, \notin^\star, =^\star, \neq^\star)$ of G:

$$\begin{cases} \bullet \quad x =^\star y & \text{iff} \quad x =_G y \quad \text{or} \quad G \Vdash_{On} x = y = \tau \\ \bullet \quad x \neq^\star y & \text{iff} \quad \exists x', y' \qquad x =^\star x' \neq_G y' =^\star y \\ \bullet \quad x \in^\star y & \text{iff} \quad \exists x', y' \qquad x =^\star x' \in_G y' =^\star y \\ \bullet \quad x \notin^\star y & \text{iff} \quad \exists x', y' \qquad x =^\star x' \notin_G y' =^\star y \end{cases}$$

To summarize: $G^\star$ is just the quotient $G/=^\star$.
Let us first check that $G^\star$ is still admissible.

(1) $=^\star$ is a congruence on $G^\star$, for the language $\mathcal{L}$, thanks to Cond 2.

(2) The partiality axioms are very easy to check, as situations like: $a =^\star b \wedge a \in_G c \wedge b \notin_G c$, as well as: $a =^\star b \wedge d \in_G a \wedge d \notin_G b$ are excluded by Lemmas 1, 2, 3.

(3) The argument that shows that $G^\star$ satisfies pre-abstraction is the analogue of the one developed in [10] for (classical) positive sets; fundamentally, the only problem is to prove that if $\tau =^\star \tau'$ and $\tau \in_G \tau''$, then $G^\star \models \varphi_{\tau''}(\tau')$; so consider the (first-order, i.e. for the language $\mathcal{L}$) automorphism of $G^\star$ that exchanges τ and τ'; one easily checks that this is really an automorphism; then, as $G \models \varphi_{\tau''}(\tau)$, one has by fact 1: $G^\star \models \varphi_{\tau''}(\tau)$, and by the automorphism: $G^\star \models \varphi_{\tau''}(\tau')$.
The $\notin$-case is solved in a similar way.
Conclusion. $G^\star$ is in $A_0 := Adm$.
Further can one very easily check that: $G^\star \in A_\alpha \rightarrow G^\star \in A_{\alpha+1}$; and that: $(\forall \beta < \gamma \text{ limit } G^\star \in A_\beta) \rightarrow G^\star \in A_\gamma$.
By induction on α, all this shows that $\forall\alpha$ (ordinal) $G^\star \in A_\alpha$; in particular: $G^\star \in A_\delta$, and (again) the maximality of G is A_δ then implies: $G = G^\star$; and finally: $G \models \tau = \tau'$, for any τ' such that $G \Vdash_{On} \tau = \tau'$.

Second case: σ is of the type $\tau \neq \tau'$.
This time, we just fix τ and τ' such that $G \Vdash_{On} \tau \neq \tau'$; and then construct (another) extension $G^\star$ of G, where only $\neq$ is modified, via the definition:

$$x \neq^\star y \quad \text{iff} \quad x \neq_G y \vee (x =_G \tau \wedge y =_G \tau').$$

We check now that this $G^\star$ (again) is in Adm. The only problem that could really arise is a situation that can be brought back to:

$$G \Vdash_{On} \tau \neq \tau' \quad \text{and} \quad \tau =_G \tau'.$$

But then we would have $G \Vdash_\delta \tau \neq \tau'$ and $\tau =_G \tau'$, and so by cond 3 and the maximality of G in A_δ:

$$G \models \neg\forall x \left(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)\right) \quad \text{and} \quad \tau =_G \tau';$$

and this obviously contradicts Lemma 1 (in particular abstraction in G).
Further can one (again) easily check that:

- $G^\star \in A_\alpha \to G^\star \in A_{\alpha+1}$,
- $(\forall\beta < \gamma \text{ limit } \; G^\star \in A_\beta) \to G^\star \in A_\gamma$.

By induction on α, we conclude: $\forall\alpha \quad G^\star \in A_\alpha$; so $G^\star \in A_\delta$; so $G = G^\star$ and finally: $G \models \tau \neq \tau'$. □

Lemma 5 *$=_G$ is a congruence for $\mathcal{L}_p$.*

Proof. Suppose $\vec{a} =_G \vec{b}$, and consider a positive formula in $\mathcal{L}$: $\psi(x, \vec{y})$, supposed pred/x. Obviously, by cond 4: $\forall\alpha$ ordinal $G \Vdash_\alpha \{x|\psi(x, \vec{a})\} = \{x|\psi(x, \vec{b})\}$. Then, by Lemma 4:

$$G \models \{x|\psi(x, \vec{a})\} = \{x|\psi(x, \vec{b})\}.$$

□

We summarize now the obtained results in our

Generic Theorem. If $\Vdash$ is a forcing (as defined before), then the generic structure G (as constructed before) is a pure term model for $PPST$, realizing the extra condition:

$$G \Vdash_{On} \sigma \leftrightarrow G \models \sigma$$

(for σ of the type $\tau = \tau'$, or $\tau \neq \tau'$; and τ, τ' in U_ω).

Comments about intensionality and extensionality.

We mentioned already that the “natural” form of extensionality generally adopted for partial sets, namely $EXT \equiv x \underset{\cdot}{=} y \rightarrow x = y$, is incompatible with PST^+; we conjecture (and this is a weakening of a conjecture in [9]) that, a contrario, $PPST$ is compatible with EXT. As abstract system, $PPST + EXT$ can be interesting; but as a description of partial information processes is this very unnatural: it is indeed not reasonable to identify, at a given moment (say in an admissible model M, which reflects the idea of a “concrete” list of data, i.e. incomplete but reliable data), two terms τ, τ' on the basis of their extensions, as these can indeed grow in the future (i.e. in admissible extensions N of M: $N \geq M$). Technically the somewhat irrelevant aspect of EXT with respect to partial sets is clear via the fact that $x \underset{\cdot}{=} y$ is not preserved under extensions.
Other criteria, that one can also see as particular versions of the classical extensionality axiom, are in a far better situation. The relation $\underset{\cdot}{\neq}$ for example, where $x \underset{\cdot}{\neq} y$ is defined as $(\exists t \in x \quad t \underset{\cdot}{\notin} y) \vee (\exists z \underset{\cdot}{\notin} x \quad z \in y)$, can be seen as the “positive translation” of $(\exists t \in x \quad \neg t \in y) \vee (\exists z \in y \quad \neg z \in x)$, clearly linked to the classical extensionality axiom, written (in the language $\{\in, =\}$) as:

$$[(\exists t \in x \quad \neg t \in y) \vee (\exists z \in y \quad \neg z \in x)] \leftrightarrow \neg x = y.$$

And $\underset{\cdot}{\neq}$ is preserved under extensions.
Notice that the axiom: $x \underset{\cdot}{\neq} y \leftrightarrow x \neq y$ can also be seen as an intensionality axiom, as $\tau \underset{\cdot}{\neq} \tau'$ will indicate that there is an object x such that $\varphi_\tau(x) \wedge \overline{\varphi_{\tau'}}(x)$, bringing in a “positive” distinction between τ and τ', linked to the “meaning” of these terms.

Another example is given by $\dot{\neq}$, that is also preserved under extensions and can also be seen as an “intensional distinction”.
All this somehow explains that (in section 4) we get nice results for $\underset{\cdot}{\neq}$ and $\dot{\neq}$, but rather nothing for $\underset{\cdot}{=}$ (there is an exception for EXT restricted to “classical” sets).

4. Applications

We give here some examples of applications. Probably one can imagine several others, or even generalize the forcing method (e.g. force also on

$\in$ and $\notin$). In spite of some efforts we could find no forcing producing a generic model G satisfying EXT; nor we could prove that such a forcing cannot exist. After all, in the light of the final comments of section 3, this is probably not so astonishing...
We discuss first some cases of identification and differentiation that can appear:

- any forcing will lead to a G where $\{x|x = x\} = \{x|\phi = \phi\}$; more generally will we have $\tau =_G \tau'$ whenever $ADM \vdash \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x))$. A slightly less trivial example: $\{x|r \in r\} =_G \{x|r \notin r\}$, where $r := \{t|\exists z(z = t \wedge z \notin t)\}$.
(“r is a “predicative” version of Russell's set $\{x|x \notin x\}$, which is not itself a predicative term);
- some forcings (like the one in application 1) produce a G where $\{x|x = x\} \neq_G \{x|r \in r\}$.
This is an interesting case, as $\{x|x = x|\} \not\doteq \{x|r \in r\}$ does not hold in G.

Application 1. We define a first forcing $\Vdash^1$ by:

$$\begin{cases} M \Vdash^1_X \tau = \tau & \text{iff} \quad \forall N \geq_X M \quad N \vdash \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)) \\ M \Vdash^1_X \tau \neq \tau & \text{iff} \quad \forall N \geq_X M \quad N \vdash \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)) \end{cases}$$

With this forcing, the decreasing sequence of the A_α quickly stops, actually already at $\delta = 1$! One can easily check that $\Vdash^1$ indeed satisfies conditions 1 to 5. So that G will have all the qualities described in our Generic Theorem. Further will G also obey the intensionality rules of the type

$$\frac{\Gamma \vdash \forall \vec{y} \forall x(\varphi(x,\vec{y}) \overset{st}{\leftrightarrow} \psi(x,\vec{y}))}{\forall \vec{y}\{x|\varphi(x,\vec{y})\} = \{x|\psi(x,\vec{y})\}}$$

and

$$\frac{\Gamma \vdash \forall \vec{y} \neg \forall x(\varphi(x,\vec{y}) \overset{st}{\leftrightarrow} \psi(x,\vec{y}))}{\forall \vec{y}\{x|\varphi(x,\vec{y})\} \neq \{x|\psi(x,\vec{y})\}}$$

where we can take for Γ any “theory” which axioms are in $\mathcal{B} := \{\theta|\theta$ is a sentence in $\mathcal{L}_p$ and $\forall N \geq_{Adm} G \quad N \models \theta\}$.
We don't however accept $\mathcal{B}$ itself as a potential theory Γ, as it depends on G! One can for example take simply $\Gamma = ADM$. But stronger Γ's are

possible, as we show now.
Consider the extensionality axiom restricted to "classical" sets, i.e.:

$$EXT_{Cl} \equiv \forall x, y[(Cl(x) \wedge x \underset{\bullet}{=} y) \rightarrow x = y],$$

where $Cl(x) \equiv \forall t(t \in x \vee t \notin x)$.
Notice that $Cl(x) \wedge x \underset{\bullet}{=} y$ implies $Cl(y)$. It is also easy to see that, if $G \models Cl(\tau) \wedge \tau \underset{\bullet}{=} \tau'$, then, in any $N \geq_{Adm} G$, both τ and τ' stay classical and satisfy abstraction; so, by the definition of $\Vdash^1$ and the Generic Theorem, we get $\tau =_G \tau'$. So that EXT_{Cl} is in $\mathcal{B}$, and (a fortiori) $G \models EXT_{Cl}$. An acceptable Γ' is thus $ADM + EXT_{Cl}$. Further, by fact 1, do we know also that any positive sentence θ in $\mathcal{L}_p$, such that $G \models \theta$, will stay true in each $N \geq_{Adm} G$; so that we can also propose $\Gamma'' \equiv ADM + EXT_{Cl} + \Sigma$, where $\Sigma = \{\theta | \theta$ is a positive sentence in $\mathcal{L}_p$, such that $PPST + EXT_{Cl} \vdash \theta\}$. Γ'' is indeed somewhat complicated and/or artificial, but its formulation does not depend on $G \ldots$

Application 2. An apparently more refined forcing (in the style of the one used in [10]) can be defined by:

$$M \Vdash^2_X \tau = \tau' \ \text{ iff } \ \forall N \geq_X M \, \exists N' \geq_X N \; N' \models \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x))$$

and

$$M \Vdash^2_X \tau \neq \tau' \ \text{ iff } \ \forall N \geq_X M \, \exists N' \geq_X N \; N' \models \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)).$$

It leads to the same conclusions as application 1; further (and however we have no proof for it) do we suspect that here δ is infinite and that really more identifications and differentiations take place (than in application 1). All we can say so far is that for any generic G for $\Vdash^1$, one can construct a generic G' for $\Vdash^2$, such that $G \leq_{Adm} G'$ (same remark as in [10]); and the question is: do we have $G \neq G'$?

Application 3. One can imagine that (for some reason) one wants to have exactly $G \models x \underset{\bullet}{\neq} y \leftrightarrow x \neq y$. Notice that in the generic G for $\Vdash^1$ (in application 1) one has $G \models x \underset{\bullet}{\neq} y \rightarrow x \neq y$, but not $G \models x \underset{\bullet}{\neq} y \leftarrow x \neq y$, as is shown by the counter-example (already mentioned at the beginning of this section): $\{x | x = x\} \neq_G \{x | r \in r\}$, but $\neg G \models \{x | x = x\} \underset{\bullet}{\neq} \{x | r \in r\}$.

So, start with any forcing $\Vdash$, and construct $\Vdash^\star$ by keeping the "= aspect", while modifying the "$\neq$ aspect":

$$\begin{cases} \text{(def)} \\ M \Vdash^\star_X \tau = \tau' \leftrightarrow M \Vdash_X \tau = \tau' \\ M \Vdash^\star_X \tau \neq \tau' \leftrightarrow M \models \tau \underset{.}{\neq} \tau' \end{cases}$$

Then $\Vdash^\star$ satisfies completely obviously conditions 1, 2, 4 and the first part of condition 3: for the second part: if $M \models \tau \underset{.}{\neq} \tau'$, take as $N \geq_X M$ simply $N = M$; then there exists some t such that for instance $t \in_M \tau$ and $t \notin_M \tau'$; so by admissibility: $M \models \varphi_\tau(t) \wedge \overline{\varphi}_\tau(t)$, and a fortiori: $M \models \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \overline{\varphi}_\tau(x))$.

Conclusion. $\Vdash^\star$ is indeed a forcing. The generic $G^\star$ for $\Vdash^\star$ will satisfy the same identification rules as the G for $\Vdash$, but will satisfy as differentiation rule: $x \underset{.}{\neq} y \leftrightarrow x \neq y$.

One can easily check that a similar construction can be made with $\dot{\neq}$ in the place of $\underset{.}{\neq}$, with corresponding results.

Application 4. Suppose that Δ is some theory (in the concerned language) and that we wish to gain some more control on the identifications/ differentiations that will take place in the generic structure G; for example, something like:

$$\begin{cases} G \models \tau = \tau' \leftrightarrow \Delta \vdash \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)), \\ G \models \tau \neq \tau' \leftrightarrow \Delta \vdash \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x)). \end{cases}$$

The idea that comes immediately in mind is then to define $\Vdash^\Delta$ as follows:

$$\begin{cases} G \Vdash^\Delta_X \tau = \tau' \leftrightarrow \Delta \vdash \forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x), \\ G \Vdash^\Delta_X \tau \neq \tau' \leftrightarrow \Delta \vdash \neg\forall x(\varphi_\tau(x) \overset{st}{\leftrightarrow} \varphi_{\tau'}(x). \end{cases}$$

Conditions 1 and 2 are then rather obviously satisfied; but the theory Δ should be chosen so that condition 3 holds: we give examples below, where it is the case. The main problem however lies in condition 4, where one expects:

$$\vec{a} =_M \vec{b} \rightarrow \Delta \vdash \forall \vec{y}(\psi(x, \vec{y}, \vec{a}) \overset{st}{\leftrightarrow} \psi(x, \vec{y}, \vec{b}))$$

(for the adequate types of formulas ψ).
That "links" the theory Δ to arbitrary admissible structures M; so we

won't accept that, as we expect Δ to be "really" a theory, in the sense: well-defined list of axioms... The way out of this situation is (for example) to accept to weaken our requirement "$=_G$ is a congruence for $\mathcal{L}_p$", i.e. to replace it by the equality-rule (that G should satisfy):

$$\frac{\Delta \vdash \vec{a} = \vec{b}}{\forall \vec{y}\{x|\psi(x, \vec{y}, \vec{a})\} = \{x|\psi(x, \vec{y}, \vec{b})\}}.$$

To obtain this, it suffices to ask Cond'4 (in place of Cond 4):

$$\Delta \vdash \vec{a} = \vec{b} \to \Delta \vdash \forall \vec{y}(\psi(x, \vec{y}, \vec{a}) \overset{st}{\leftrightarrow} \psi(x, \vec{y}, \vec{b})).$$

Then everything goes as before, except that Lemma 5 now states that G satisfies the equality-rule just discussed (instead of :$=_G$ is a congruence for $\mathcal{L}_p$): this is easy to check, via Cond'4 and Lemma 4.

Synthesis. if Δ satisfies Cond 3 and Cond'4, then $\Vdash^{\Delta}$ produces a generic G "as before", except that Lemma 5 replaces the expectation that $=_G$ is a congruence for $\mathcal{L}_p$, by the fact that G satisfies the equality-rule discussed before. And the control over $=_G$ and $\neq_G$ is the one that we wished initially.

Some examples (easy to check):

- $\Delta_1 = ADM$,
- $\Delta_2 =$ the axioms (1) and (2) of ADM.

Remark. the theories modelized by the generics G_1 and G_2 (corresponding respectively to these Δ_1 and Δ_2) are really different; indeed: consider the term $\Lambda := \{x|x \neq x\}$, and the formulas $\varphi :\equiv \Lambda \in \{\Lambda\}$ and $\psi :\equiv \exists t\, t \in \{\Lambda\}$.
Naturally is $\{\Lambda\}$ just the abbreviation for $\{y|y = \Lambda\}$. Then is it easy to see that: $\Delta_1 \models \varphi \overset{st}{\leftrightarrow} \psi$, while $\Delta_2 \nvdash \psi \to \varphi$, so that, a fortiori: $\Delta_2 \nvdash \varphi \overset{st}{\leftrightarrow} \psi$.

Conclusion. $\{x|\varphi\} = \{x|\psi\}$ is true in G_1, but false in G_2.

Application 5. One could want, on the basis of a given $\Vdash$, to bring in some "restrictions" (as size-restrictions, complexity-restrictions, etc...). If however we expect $=_G$ to be an $\mathcal{L}_p$-congruence, we have to satisfy Cond 4, which obliges necessarily some kinds of identifications, namely those of the type $\{x|\psi(x,\vec{a})\} = \{x|\psi(x,\vec{b})\}$ when $\vec{a} =_M \vec{b}$; we will say that these terms are "isosyntactic over M". With this in mind can we modify $\Vdash$, via the use of a subset $\mathcal{R}$ of U_ω; we define $\Vdash^{\mathcal{R}}$ by:

$$\begin{cases} M \Vdash_X^{\mathcal{R}} \tau = \tau' \text{ iff } [(\tau, \tau' \in \mathcal{R} \;\&\; M \Vdash_X \tau = \tau') \text{ or } \tau, \tau' \text{ are isosyntactic} \\ \qquad\qquad\qquad\qquad \text{over } M] \\ M \Vdash_X^{\mathcal{R}} \tau \neq \tau' \text{ iff } (\tau, \tau' \in \mathcal{R} \;\&\; M \Vdash_X \tau \neq \tau'). \end{cases}$$

Remark. We don't have for $\neq$ the limitation that Cond 4 puts on $=$.

Now can one easily check that $\Vdash^{\mathcal{R}}$ is again a forcing, i.e. satisfies conditions 1-4. And the corresponding generic $G_{\mathcal{R}}$ will take in account the desired "restrictions", i.e.:

$$\begin{cases} G_{\mathcal{R}} \models \tau = \tau' \leftrightarrow [(\tau, \tau' \in \mathcal{R} \;\&\; G_{\mathcal{R}} \Vdash_{On} \tau = \tau') \text{ or } \tau, \tau' \text{ are isosyntactic} \\ \qquad\qquad\qquad\qquad \text{over } G] \\ G_{\mathcal{R}} \models \tau \neq \tau' \leftrightarrow (\tau, \tau' \in \mathcal{R} \;\&\; G_{\mathcal{R}} \Vdash_{On} \tau \neq \tau'). \end{cases}$$

Example. take for $\mathcal{R}$ those terms $\{x|\psi(x,\vec{a})\}$ (of U_ω), where ψ is an $\mathcal{L}$-formula, such that ψ is of length $\leq k$ (a fixed natural number).

Application 6. We can walk another path than the preceding one, if we accept to replace the expectation " $=_G$ is a congruence for $\mathcal{L}_p$" by axioms (that G should satisfy) of type: $\vec{a} = \vec{b} \to \{x|\psi(x,\vec{a})\} = \{x|\psi(x,\vec{b})\}$; for $\{x|\psi(x,\vec{a})\}$ and $\{x|\psi(x,\vec{b})\}$ in $\mathcal{R}$. In that case, we replace Cond 4 by:
Cond*4: if $\vec{a} =_M \vec{b}$ & $\{x|\psi(x,\vec{a})\}$ and $\{x|\psi(x,\vec{b})\}$ are in $\mathcal{R}$, then $M \Vdash_X \{x|\psi(x,\vec{a})\} = \{x|\psi(x,\vec{b})\}$.
N.B.: ψ here is in $\mathcal{L}$!
Starting with an arbitrary forcing $\Vdash$ (i.e. satisfying conditions 1-4), we define $\Vdash^\star$ by:

$$\begin{cases} M \Vdash_X^\star \tau = \tau' \text{ iff } (\tau, \tau' \in \mathcal{R} \;\&\; M \Vdash_X \tau = \tau'), \\ M \Vdash_X^\star \tau \neq \tau' \text{ iff } (\tau, \tau' \in \mathcal{R} \;\&\; M \Vdash_X \tau \neq \tau'). \end{cases}$$

This time $\Vdash^\star$ will satisfy conditions 1-3, and Cond* 4; the corresponding generic $G^\star$ will be "as before", except that Lemma 5 will now state that

$\{x|\psi(x,\vec{a})\} =_{G^\star} \{x|\psi(x,\vec{b})\}$ whenever both terms are isosyntactic over $G^\star$, and both belong to $\mathcal{R}$; finally will $G^\star$ take in account our initial wish, i.e.:

$$\begin{cases} G \Vdash \tau = \tau' \text{ iff } (\tau, \tau' \in \mathcal{R} \ \& \ G \Vdash_{On} \tau = \tau') \\ G \Vdash \tau \neq \tau' \text{ iff } (\tau, \tau' \in \mathcal{R} \ \& \ G \Vdash_{On} \tau \neq \tau') \end{cases}$$

Examples. (for k a natural number)

- $\mathcal{R} = \{\tau \in U_\omega | \tau \text{ is of length} \leq k\}$,
- $\mathcal{R} = \{\tau \in U_\omega | \tau \text{ is of rank} \leq k\}$.

Final remarks. Many non-classical set theories have been proposed and studied (an interesting attempt to list them is Randall Holmes' "Bibliography of Set Theory with a Universal Set"), and one may roughly divide them into "strong" ones and "weak" ones, these concepts being obviously vague!

In the strong ones, one can develop Peano arithmetic and "reasonable" fragments of ordinary set theory (Zermelo-Fraenkel). This is the case for NF (Quine's New Foundations) and GPK^+_∞ (the strong positive set theory developped by O. Esser: see [2], [4], [5]). But besides, these strong theories can also present bad aspects: both preceding examples disprove AC (the axiom of choice)! So, those weaker theories may present the advantage of being more flexible. In the case of NF, for example, it was shown that the weaker version NFU has many agreeable compatibility properties (namely with respect to infinity axioms, anti-foundation axioms, axioms of choice, etc; see e.g. Holmes [12], [14], [15], [16]), to the point that NFU might appear as "the right version" of NF (see [13])!

The intensional partial set theories discussed in this paper belong obviously to the "weak" group. So, in the light of the preceding remarks, it is surely interesting to explore their compatibility with Peano arithmetic, with the presence of a transitive set that would be a model of ZF, etc (somewhat in the line of [11]): a subject for future research...

References

[1] M. Boffa, *Forcing et négation de l'axiome de fondement*, Académie Royale de Belgique, Vol. **40** (1972).

[2] O. Esser, *On the consistency of a positive set theory*, Mathematical Logic Quarterly **45** (1999), pp. 105–116.

[3] O. Esser, *A model of a strong paraconsistent set theory*, Notre Dame Journal of Formal Logic **44** (2003).

[4] O. Esser, *An interpretation of the Zermelo-Fraenkel Set Theory and the Kelley-Morse Set Theory in a Positive Theory*, Mathematical Logic Quarterly **43** (1999), pp. 369–377.

[5] O. Esser, *Inconsistency of the axiom of choice with the positive theory* GPK^+_∞, Journal of Symbolic Logic **65** (2000), 1911–1916.

[6] M. Forti & R. Hinnion, *The consistency problem for positive comprehension principles*, The Journal of Symbolic Logic **54** (1989), pp. 1401–1418.

[7] P.C. Gilmore, *The consistency of partial set theory without extensionality*, Proceedings of Symposia in Pure Mathematics **13** (1974), pp. 147–153.

[8] R. Hinnion, *Le paradoxe de Russell dans des versions positives de la théorie naïve des ensembles*, Comptes Rendus de l'Académie des Sciences de Paris **304** (1987), pp. 307–310.

[9] R. Hinnion, *Naive set theory with extensionality in partial logic and in paradoxical logic*, Notre Dame Journal of Formal Logic **35** (1994), pp. 15–40.

[10] R. Hinnion, *Intensional positive set theory*, Reports on Mathematical Logic **40** (2006).

[11] R. Hinnion, *About the coexistence of classical sets with non-classical ones: a survey*, Logic and Logical Philosophy **11** (2003), pp. 79–90.

[12] M.R. Holmes, *The axiom of antifoundation in Jensen's "New Foundations with Ur-Elements"*, Bulletin de la Société Mathématique de Belgique **43** (1991), série B, pp. 167–179.

[13] M.R. Holmes, *The set theoretical program of Quine succeeded (but nobody noticed)*, Modern Logic **4** (1994), pp. 1–47.

[14] M.R. Holmes, *Elementary set theory with a universal set*, Cahiers du Centre de Logique **10** (1998), Academia, Louvain-la-Neuve (Belgium), pp. 1–241.

[15] M.R. Holmes, *Subsystems of Quine's "New Foundations" with predicativity restrictions*, Notre Dame Journal of Formal Logic **40**, n^{o}2, pp. 183–196.

[16] M.R. Holmes, *Strong axioms of infinity in* NFU, Journal of Symbolic Logic **66** (2001), n^{o}1, pp. 87–116.

[17] A. Rigo, Uniformly continuous model theory, Ph.D. (2002), Université Libre de Bruxelles.

Université Libre de Bruxelles,
CP.211, Bd du Triomphe,
B1050 Brussels, Belgium

`rhinnion@ulb.ac.be`