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ON TRUTH-SCHEMES FOR INTENSIONAL LOGICS

In memory of Willem Blok

A b s t r a c t. The paper is concerned with the question of definability of truth-conditions for the connectives of intensional logics. A certain general solution of the problem is proposed for the class of self-extensional logics. The paper develops some ideas initiated by Suszko and Wójcicki in the seventies.

0 Introduction

In this paper we present a certain conception of truth for intensional logics. This conception is a natural generalization of the well-known relational semantics for intensional systems. It is based on some ideas set forth by Suszko and Wójcicki in the seventies.

Let $\boldsymbol{S} = (S, f_1, f_2, ...)$ be a *sentential language*, i.e., $\boldsymbol{S}$ is an absolutely free algebra generated by an infinite set of *sentential variables* $Var(\boldsymbol{S})$ =

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$\{p_n : n \in \omega\}$ and endowed with a set $Con(\boldsymbol{S})$ of logical connectives $f_1, f_2, \ldots$, each of a finite arity. The members of S, the universe of $\boldsymbol{S}$, are called *sentential formulas*, or simply *sentences*.

Now let C be any logic in $\boldsymbol{S}$, i.e., C is a structural and finitary consequence operation on S. This means that C is a mapping from the power set $P(S)$ of S to $P(S)$ which satisfies the following conditions, for any sets $X, Y \subseteq S$ and any $\alpha \in S$:

(reflexivity) $X \subseteq C(X)$;

(monotonicity) $X \subseteq Y$ implies that $C(X) \subseteq C(Y)$;

(idempotency) $C(C(X)) \subseteq C(X)$;

(structurality) $e[C(X)] \subseteq C(e[X])$ for every endomorphism $e : \boldsymbol{S} \to \boldsymbol{S}$;

(finitariness) $\alpha \in C(X)$ implies that $\alpha \in C(X_f)$ for some finite set $X_f \subseteq X$.

$e[X]$ stands here for the e-image of the set X, i.e., $e[X] := \{e\alpha : \alpha \in X\}$. Every endomorphism $e : \boldsymbol{S} \to \boldsymbol{S}$ is unambiguously determined by its values on the variables of $\boldsymbol{S}$. The terms "endomorphism of $\boldsymbol{S}$" and "substitution" are used interchangeably.

Any set $T \subseteq S$ such that $C(T) = T$ is called a *theory* of C. By $Th(C)$ we denote the class of theories of C. $Th(C)$ is obviously a closure system on S.

A set $\boldsymbol{B} \subseteq Th(C)$ is called a *base* for $Th(C)$ if every theory of C is the intersection of some subfamily of $\boldsymbol{B}$. We assume that S itself is the intersection of the empty subfamily. $Th(C)$ is clearly its own base. We mention here two other important bases:

the family of all relatively maximal theories of C;

the family of all *prime* theories of C.

We recall that T is *prime* (relative to C) if it cannot be represented as the meet of two theories of C distinct from T, i.e., $T = T_1 \cap T_2$ implies $T = T_1$ or $T = T_2$.

A logic C is *inconsistent* if $C(\emptyset) = S$, or equivalently, if $C(X) = S$, for all $X \subseteq S$. C is inconsistent if and only if $Th(C)$ has the empty base.

According to the semantic interpretation of consequence operations, the fact that $\alpha \in C(T)$ means that α is true whenever the sentences of T are true. The semantic interpretation of a logic C is effected by:

(I). selecting suitable model conditions,

(II). choosing a truth-definition.

Model conditions define intended models of a sentential logic (e.g., matrices, algebras, frames, truth-valuations, etc.). The class of intended models, associated with a given logic C, is defined through selecting a list of properties (axioms) characteristic for the class. E.g. by assuming that a modal logic is determined by a class of frames of the form $(W, \boldsymbol{R})$, where $\boldsymbol{R}$ is a binary relation on W with a prescribed list of properties (reflexivity, transitivity, etc.). In many cases the class of intended model collapses to a singleton. This means that the selected axioms characterize the intended models categorically, up to isomorphism. E.g. in case of Łukasiewicz logics the construction of a single intended model is provided, which is an appropriate Łukasiewicz matrix - the intended model of the logic.

Needless to say that the language of model conditions has usually a much more complex structure than the sentential language $\boldsymbol{S}$ in which the logic C is defined.

The basic function of model conditions is to define the *semantic correlate* of sentences. (We use after Suszko this suggestive term.) E.g. when a matrix $M = (\boldsymbol{A}, D)$ is a model of a logic, then homomorphisms h of $\boldsymbol{S}$ into A are called *reference assignments*; the semantic correlate of a sentence α (determined by h) is just the value $h(\alpha)$.

The component (II) defines a recursive scheme of truth-conditions for the sentences of $\boldsymbol{S}$ or, in the terminology we prefer, for the connectives of $\boldsymbol{S}$.

The semantics we discuss here is formed by *truth-valuations*. A truth-valuation for $\boldsymbol{S}$ is any function $h : S \to \{0, 1\}$ from the sentential language to the set of two logical values: "truth" 1 and "falsity" 0. If H is a set of truth-valuations, the function $Cn_H : P(S) \to P(S)$ defined by

$\alpha \in Cn_H(T)$ if and only if, for all $h \in H$, $h(\alpha) = 1$
whenever $h(T) \subseteq \{1\}$,

is a consequence operation on S. Cn_H need be neither finitary nor structural. But if for each $h \in H$ and each substitution $e : \boldsymbol{S} \to \boldsymbol{S}$, the composition $h \circ e$ is in H, then the consequence Cn_H is structural (see Wójcicki [1988] for more information).

Every sentential logic is complete for a set of truth-valuations, i.e., for any C there exists a set H of truth-valuations such that $C = Cn_H$. A justification for this is straightforward: let, for every $T \in \mathit{Th}(C)$, the truth-valuation h_T be defined by the stipulation: $h_T(\alpha) = 1$ iff $\alpha \in T$. h_T is the characteristic function of the theory T. The class H of all characteristic functions of theories of C defines the consequence C. [The elements of H are called *admissible valuations* for C.] This, somewhat trivial fact, was observed by many logicians, we mention here Scott [1971], Routley [1976], Suszko [1977a], van Fraassen [1973], see also da Costa and Béziau [1994]). This observation gave rise to discussions on two-valuedness and the scope of the principle of bivalence. Suszko seems to be the one who has drawn the most far-reaching conclusions from this observation. His views can be summed up in the slogan:

Every logic is two-valued.

While selecting the class of characteristic functions of the theories of C (or, more widely, the characteristic functions of the theories from an arbitrary base $\boldsymbol{B}$ for $\mathit{Th}(C)$) is a good choice of a possible model class for C, some doubts however remain. What seems to be a source of difficulty is that Suszko's thesis does not provide truth-schemes for the connectives of $\boldsymbol{S}$ on the basis of the logic C. In other words, Suszko is not here concerned with the question of finding plausible, general definability conditions which would enable one to evaluate uniformly, for a given connective of $\boldsymbol{S}$, the truth of any compound formula headed by this connective in terms of the truths of the constituents of this formula and some other fixed intensional factors.

In our opinion, for a large class of sentential logics C, the problem of definability of truth-schemes for the connectives for C is adequately formulated as the question concerning the existence of a class H of truth-valuations, a suitably defined two-sorted language L of order higher that 0, and a model $\underline{\boldsymbol{M}}$ for L, defined on the two-sorted universe (S, H), where $\boldsymbol{S}$ is the language of C, such that the following conditions are satisfied:

(i) for every connective f of $\boldsymbol{S}$, say n-ary, there exists a formula $\varphi_f(x_1, ..., x_n; X)$ of L in n variables $x_1, ..., x_n$, ranging over elements of S, and in one variable X, ranging over truth-valuations in H, such that the following holds: for any $\alpha_1, ..., \alpha_n \in S$ and any $h \in H$,

$$h(f(\alpha_1, ..., \alpha_n)) = 1 \text{ iff } \underline{\boldsymbol{M}} \vDash \varphi_f(x_1, ..., x_n; X)[\alpha_1, ..., \alpha_n; h]$$

i.e., $f(\alpha_1, ..., \alpha_n)$ is true under h iff $\varphi_f(x_1, ..., x_n; X)$ holds in $\underline{\boldsymbol{M}}$ on $\alpha_1, ..., \alpha_n, h$.

(ii) H determines C.

We will discuss these issues more formally in the next section. We note here that according to (i), the formula $\varphi_f(x_1, ..., x_n; X)$ defines a truth-scheme for all sentences of S headed by the connective f, and not for an individual sentence.

We also note that there exists a one-to-one correspondence between bases for $Th(C)$ and families H of truth-valuations which determine C. Indeed, we have:

Proposition 0.1. *Let C be a logic defined in $\boldsymbol{S}$. For each set H of truth-valuations on S define $\boldsymbol{B}(H) := \{h^{-1}\{1\} : h \in H\}$. Then the mapping $H \to \boldsymbol{B}(H)$ establishes a bijection between the families H that determine C and the bases for $Th(C)$.*

We omit the easy proof.

The above proposition shows that instead of working with truth-valuations, one can choose another option and work with bases as suggested by Suszko. This option is preferred in this paper.

1 Languages of truth-schemes

To each sentential language $\boldsymbol{S}$ a certain class of second-order languages L is assigned. The elements of this class are called *languages of possible truth-schemes* for sentential logics defined on S. Each such a language L is defined as follows.

L has variables of two types:

$x_0, x_1, ...$ - individual variables;

$X_0, X_1, ...$ - second-order monadic variables.

We assume that the above set of individual variables coincides with *Var*($\boldsymbol{S}$), the set of sentential variables of $\boldsymbol{S}$. [We do not make use of this assumption in this section. It will be used in the construction of canonical models for sentential logics in Section 2.] Furthermore, L is furnished with a (possibly infinite) set of relation symbols $R_1, R_2, ...$ We assume that L contains the symbol "$\in$" of the membership relation. This symbol will play a special role in the formalism we shall outline. Furthermore L contains the equality symbol "$\approx$".

The variables $x_0, x_1, ...$ are assumed to range over sentences of $\boldsymbol{S}$ while the intended interpretation of $X_0, X_1, ...$ is to range over theories in $\boldsymbol{S}$, i.e., subsets of S. The intended interpretations of relation symbols are certain relations holding between sentences and theories in $\boldsymbol{S}$. Thus to each symbol R a certain pair (m, n) of natural numbers is assigned which is called the *arity* of R. This means that R is interpreted as a relation holding between m-tuples of sentences of $\boldsymbol{S}$ and n-tuples of theories in S. We always assume that $n \geq 1$. The case when $m = 0$ is allowed. Thus $(0, n)$-ary relation symbols are interpreted as n-ary relations between theories in $\boldsymbol{S}$. Such relation symbols are called *pure*; otherwise R is called *mixed*. In particular, $\in$ is a mixed relation symbol of arity $(1, 1)$.

The set *Fo*(L) of formulas of L is defined recursively as follows:

(i) If X and Y are theory variables, then $X \approx Y$ is a formula,

(ii) If x is an individual variable and X is a theory variable, then $x \in X$ is a formula,

(iii) If R is an (m, n)-ary relation symbol other than $\in$, $x_1, ..., x_m$ are individual variables and $X_1, ..., X_n$ are theory variables, then $R(x_1, ..., x_m; X_1, ..., X_n)$ is a formula,

(iv) If φ and ψ are formulas, then $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$ are formulas,

(v) If φ is a formula, then $(\neg\varphi)$ is a formula,

(vii) If φ is a formula and x is an individual variable, then $((\forall x)\varphi)$ is a formula,

(viii) If φ is a formula and X is a theory variable, then $((\forall X)\varphi)$ is a formula,

(ix) Nothing else is a formula of L.

We adopt the usual conventions of suppressing outer parentheses in formulas. We also assume that $\varphi \to \psi$ and $\varphi \leftrightarrow \psi$ are abbreviations for

$\neg\varphi \vee \psi$ and $(\varphi \to \psi) \wedge (\psi \to \varphi)$, respectively. Similarly, $(\exists x)\varphi$ and $(\exists X)\varphi$ are abbreviations for $\neg(\forall x)\neg\varphi$ and $\neg(\forall X)\neg\varphi$, respectively.

We note that the identity predicate does not connect individual variables, i.e., there are no formulas of the shape $x \approx y$, where x and y are individual variables. The absence of formulas is justified by the fact that in the models we shall consider, individual variables will range over formulas of sentential languages. Sentential formulas α and β are identical if and only if they are identical as strings of symbols. Let us abbreviate the formula $(\forall X)(x \in X \leftrightarrow y \in X)$ as $x \equiv y$. In the models we shall consider, the second-order predicate $\equiv$ is not interpreted as identity because the variable X ranges over a proper subclass of the power set of the universe of the model. This subclass is too small to determine the identity of two sentential formulas α and β via the formula $\alpha \equiv \beta$.

L is called the language of *truth-schemes for the connectives of* $\boldsymbol{S}$.

The notation $\varphi(x_1, ..., x_m; X_1, ..., X_n)$ means that φ contains at most the individual variables $x_1, ..., x_m$ and second-order variables $X_1, ..., X_n$ as free variables.

A sentence of L is any formula in *Fo*(L) which does not contain free occurrences of variables.

Let $f \in Con(\boldsymbol{S})$ be an n-ary connective of $\boldsymbol{S}$. By a possible *truth-scheme* for f we understand any formula $\varphi_f(x_1, ..., x_n; X)$ of L with n individual variables $x_1, ..., x_n$ and only one free monadic second-order variable X.

Let C be any logic in $\boldsymbol{S}$. Let T be a set of sentences of L. The language L together with T are jointly called (*possible*) *model conditions* for the logic C.

Now let $\boldsymbol{B}$ be any base for $Th(C)$. Suppose that the relation symbols of L have interpretations as relations on the two-sorted domain $(S, \boldsymbol{B})$ so that the resulting structure

(1) $\quad \underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$

is a model for the language L. [Of course, if L involves only pure relation symbols (except of $\in$), then these symbols are interpreted as relations on $\boldsymbol{B}$ only.] The symbol $\in$ is interpreted as the ordinary membership relation.

The pair $(S, \boldsymbol{B})$ is thus thought of as a universe on which appropriate truth schemes for the connectives of $\boldsymbol{S}$ are defined.

We say that (1) is an intended *canonical frame for the logic* C if the following conditions hold:

(a) $\underline{\boldsymbol{M}}$ is a model of T,

(b) For any relation symbol R, say (m,n)-ary, for any two m-tuples $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_m$ of elements of S and any n-tuple $\boldsymbol{X}_1, ..., \boldsymbol{X}_n$ of members of $\boldsymbol{B}$,

if $C(\alpha_1) = C(\alpha_1), ..., C(\beta_m) = C(\beta_m)$, then

$$\boldsymbol{R}(\alpha_1, ..., \alpha_m, \boldsymbol{X}_1, ..., \boldsymbol{X}_n) \text{ iff } \boldsymbol{R}(\beta_1, ..., \beta_m, \boldsymbol{X}_1, ..., \boldsymbol{X}_n).$$

(b) thus assumes the invariance of the relation $\boldsymbol{R}$ under the deductive equivalence of formulas of S with respect to C. We note that $\in$ always satisfies this invariance condition. Indeed, if $C(\alpha) = C(\beta)$, then for any theory $\boldsymbol{X}$ of C, $\alpha \in \boldsymbol{X}$ if and only if $\beta \in \boldsymbol{X}$.

Let f be an n-ary connective of $\boldsymbol{S}$. We say that a possible truth-scheme φ_f for f is *adequate* (for f) in the *intended frame* (1) if the following equivalence holds:

(2) For any $\boldsymbol{X} \in \boldsymbol{B}$ and any $\alpha_1, ..., \alpha_n$ in S,

$f(\alpha_1, ..., \alpha_n)$ belongs to $\boldsymbol{X}$ iff $\underline{\boldsymbol{M}} \models \varphi_f(x_1, ..., x_n; X)[\alpha_1, ..., \alpha_n, \boldsymbol{X}]$.

$\models$ stands here for the ordinary satisfaction in $\underline{\boldsymbol{M}}$.

Following the terminology adopted in modal and tense logics, $\underline{\boldsymbol{M}}$ is called a *canonical frame for the logic* C if it is intended and (2) holds for any connective f of $\boldsymbol{S}$.

Examples.

1. Let C_{CPC} be the consequence operation of the classical propositional logic in the language $\boldsymbol{S} = \langle S, \wedge, \vee, \Rightarrow, \neg \rangle$.

The language L of truth-schemes for the connectives of $\boldsymbol{S}$ has no relation symbols different from $\in$. The above connectives have the following truth-schemes:

$\varphi_\wedge(x, y; X)$ is $x \in X \wedge y \in X$,

$\varphi_\vee(x, y; X)$ is $x \in X \vee y \in X$,

$\varphi_\Rightarrow(x, y; X)$ is $\neg(x \in X) \vee y \in X$,

$\varphi_\neg(x, y; X)$ is $\neg(x \in X)$.

The model conditions for C_{CPC} reduce to the selection of the language L and the empty set T of sentences in L.

Let $\boldsymbol{B}$ be the base of $\mathrm{Th}(C_{\mathrm{CPC}})$ consisting of all prime theories of C_{CPC}. The system $\underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in)$ is evidently a canonical frame for C_{CPC} since the above truth-schemes are adequate for the connectives in this model.

2. Let the language $\boldsymbol{S}$ be as in 1 and let C_{INT} be the consequence operation of the intuitionistic propositional calculus.

The language L of truth-schemes for the above connectives has the epsilon $\in$ and one $(0,2)$-ary predicate $\leq$. This predicate is interpreted as the inclusion between theories in S.

We have the following truth-schemes for the connectives:

$$\varphi_\wedge(x, y; X) \quad \text{is} \quad x \in X \wedge y \in X,$$

$$\varphi_\vee(x, y; X) \quad \text{is} \quad x \in X \vee y \in X,$$

$$\varphi_\Rightarrow(x, y; X) \quad \text{is} \quad (\forall Y)(X \leq Y \rightarrow \neg(x \in Y) \vee y \in Y).$$

$$\varphi_\neg(x, y; X) \quad \text{is} \quad (\forall Y)(X \leq Y \rightarrow \neg(x \in Y)),$$

The pair (L, T), where T is the set of axioms of partial order, defines model conditions for C_{INT}.

Let $\boldsymbol{B}$ be the base consisting of all prime theories of C_{INT}. Then clearly

$$\underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in, \subseteq) \tag{3}$$

is a model of T. (3) is a canonical frame for C_{INT} since the above truth-schemes are obviously adequate for the connectives of $\boldsymbol{S}$ in the model (3). E.g. the adequacy of (3) for $\Rightarrow$ simply means that for any $\alpha, \beta \in S$ and for any theory $\boldsymbol{X} \in \boldsymbol{B}$, $\alpha \Rightarrow \beta$ belongs to $\boldsymbol{X}$ iff $(\forall \boldsymbol{Y} \in \boldsymbol{B})\boldsymbol{X} \subseteq \boldsymbol{Y}$ implies that $\alpha \notin \boldsymbol{Y}$ or $\beta \in \boldsymbol{Y}$.

3. Let C_{KR} be the Kripke's modal logic. C_{KR} is thus the consequence in the modal language

$$\boldsymbol{S}_\Box = (S, \wedge, \vee, \neg, \Box)$$

determined by the Kripke's system $\boldsymbol{K}$ (as a set of logical axioms) and Modus Ponens (MP) for the material implication as the only proper rule of inference. C_{KR} is obviously finitary and structural.

The language L of truth-schemes for $\boldsymbol{S}_\Box$ has $\in$ and only one $(0,2)$-ary predicate R. The truth-schemes for the connectives of $\boldsymbol{S}_\Box$ are standard ones:

$$\varphi_\wedge(x,y;X) \quad \text{is} \quad x \in X \wedge y \in X,$$

$$\varphi_\vee(x,y;X) \quad \text{is} \quad x \in X \vee y \in X,$$

$$\varphi_\neg(x;X) \quad \text{is} \quad \neg(x \in X),$$

$$\varphi_\Box(x;X) \quad \text{is} \quad (\forall Y)(X\,R\,Y \to x \in Y),$$

The model conditions for C_{KR} are formed by the pair: the language L, the empty set T of sentences of L.

Let $\boldsymbol{B}$ be the base of $Th(C_{\mathrm{KR}})$ consisting of all prime (= maximal) theories of C_{KR}. Then clearly

(4) $\quad \underline{\boldsymbol{M}} = (S_\Box, \boldsymbol{B}, \in, \boldsymbol{R})$

is a model of T, where $\boldsymbol{R}$ is defined as follows

$$\boldsymbol{X}\,\boldsymbol{R}\,\boldsymbol{Y} \text{ iff, for every } \alpha \in S_\Box,\ \Box\alpha \in \boldsymbol{X} \text{ implies } \alpha \in \boldsymbol{Y}.$$

The model (4) is a canonical frame for C_{KR} since the above truth-schemes are obviously adequate for the connectives in this model. E.g. the adequacy of (4) for $\Box$ is a consequence of the equivalence

$$\Box\alpha \in \boldsymbol{X} \text{ iff, for every prime theory } \boldsymbol{Y},\ \boldsymbol{X}\,\boldsymbol{R}\,\boldsymbol{Y} \text{ implies } \alpha \in \boldsymbol{Y}. \ \Box$$

A sentential logic $(\boldsymbol{S}, C)$ is *self-extensional* if, for every connective f of $\boldsymbol{S}$ and two n-tuples $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n$ of elements of S, where n is the arity of f, $C(\alpha_1) = C(\beta_1), ..., C(\alpha_n) = C(\beta_n)$ imply that $C(f(\alpha_1, ..., \alpha_n)) = C(f(\beta_1, ..., \beta_n))$.

The class of self-extensional logics was defined and studied by Wójcicki [1979], [1988]. He established the existence of closed links between self-extensional logics and the so called *referential semantics.* We only note that the axiomatic extensions of the Kripke consequence C_{KR} or of the intuitionistic calculus C_{INT} are self-extensional. In turn, many-valued logics are not self-extensional.

Theorem 1.1. *Let C be a logic in a sentential language $\boldsymbol{S}$. Suppose that in the language* L*, for each connective f of $\boldsymbol{S}$ some truth-scheme*

$\varphi_f(x_1,...,x_n,X)$ *has been selected. Let T be a set of sentences in* L *so that the pair* (L, T) *forms model-conditions for C. Suppose furthermore that for some base $\boldsymbol{B} \subseteq Th(C)$,*

$$\underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$$

is a an intended frame for C. If the above truth-schemes are adequate for the connectives in this model, i.e., $\underline{\boldsymbol{M}}$ is a canonical frame for C, then the logic C is self-extensional.

Proof. Under the hypotheses of the theorem we have the following

Lemma 1.2. *For every formula $\psi(x_1,...,x_m;X_1,...,X_n)$ of* L*, for any theories $\boldsymbol{X}_1,...,\boldsymbol{X}_n$ of $\boldsymbol{B}$, and for any formulas $\alpha_1,...,\alpha_m,\beta_1,...,\beta_m \in S$, if $C(\alpha_1) = C(\beta_1),...,C(\alpha_m) = C(\beta_m)$, then*

(*) $\underline{\boldsymbol{M}} \models \psi(x_1,...,x_m;X_1,...,X_n)[\alpha_1,...,\alpha_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n]$ *iff*
$\underline{\boldsymbol{M}} \models \psi(x_1,...,x_m;X_1,...,X_n)[\beta_1,...,\beta_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n]$.

The lemma is proved by induction on the degree of complexity of the formula ψ.

If ψ is $x \in X$, then for any $\boldsymbol{X} \in \boldsymbol{B}$ and any $\alpha, \beta \in S$ such that $C(\alpha) = C(\beta)$ we have:

$$\underline{\boldsymbol{M}} \models (x;X)[\alpha,\boldsymbol{X}] \quad \text{iff} \quad \alpha \in \boldsymbol{X} \quad \text{iff} \quad C(\alpha) \subseteq \boldsymbol{X} \quad \text{iff}$$
$$C(\beta) \in \boldsymbol{X} \quad \text{iff} \quad \underline{\boldsymbol{M}} \models (x;X)[\beta,\boldsymbol{X}].$$

If ψ is $X \approx Y$, then for any theories $\boldsymbol{X}_1, \boldsymbol{X}_2 \in \boldsymbol{B}$ the equivalence (*) reduces to

$$\boldsymbol{X}_1 = \boldsymbol{X}_2 \quad \text{iff} \quad \boldsymbol{X}_1 = \boldsymbol{X}_2.$$

Similarly, if ψ is an atomic formula $R(x_1,...,x_n;X_1,...,X_n)$ and $\alpha_1,...,\alpha_m,\beta_1,...,\beta_m \in S$ so that $C(\alpha_1) = C(\beta_1),...,C(\alpha_m) = C(\beta_m)$, then for any theories $\boldsymbol{X}_1,...,\boldsymbol{X}_n \in \boldsymbol{B}$,

$$\underline{\boldsymbol{M}} \models R(x_1,...,x_n;X_1,...,X_n)[\alpha_1,...,\alpha_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n]$$

iff $R(\alpha_1,...,\alpha_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n)$

iff (by the invariance property of $\boldsymbol{R}$) $\boldsymbol{R}(\beta_1,...,\beta_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n)$

iff $\underline{\boldsymbol{M}} \models R(x_1,...,x_n;X_1,...,X_n)[\beta_1,...,\beta_m;\boldsymbol{X}_1,...,\boldsymbol{X}_n]$.

The cases when ψ is $\psi_1 \wedge \psi_2$ or $\psi_1 \vee \psi_2$, or $\neg_\varphi$ are easy to handle. The case when ψ is $(\forall X)\varphi$ and $\varphi = \varphi(x_1, ..., x_n; X, X_1, ..., X_n)$ is also easy. For let $C(\alpha_1) = C(\beta_1), ..., C(\alpha_m) = C(\beta_m)$ and $\boldsymbol{X}_1, ..., \boldsymbol{X}_n \in \boldsymbol{B}$. Then

$$\underline{\boldsymbol{M}} \models \psi(x_1, ..., x_n; X_1, ..., X_n)[\alpha_1, ..., \alpha_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff for all $\boldsymbol{X} \in \boldsymbol{B}$,

$$\underline{\boldsymbol{M}} \models \varphi(x_1, ..., x_n; X, X_1, ..., X_n)[\alpha_1, ..., \alpha_m; \boldsymbol{X}, \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff (by the induction hypothesis) for all $\boldsymbol{X} \in \boldsymbol{B}$

$$\underline{\boldsymbol{M}} \models \varphi(x_1, ..., x_n; X, X_1, ..., X_n)[\beta_1, ..., \beta_m; \boldsymbol{X}, \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff $\underline{\boldsymbol{M}} \models \psi(x_1, ..., x_n; , X_1, ..., X_n)[\beta_1, ..., \beta_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$.

Now let ψ be $(\forall x)\varphi$ and $\varphi = \varphi(x, x_1, ..., x_n; X_1, ..., X_n)$.
Assume $C(\alpha_1) = C(\beta_1), ..., C(\alpha_m) = C(\beta_m)$ and $\boldsymbol{X}_1, ..., \boldsymbol{X}_n \in \boldsymbol{B}$. Then:

$$\underline{\boldsymbol{M}} \models \psi(x_1, ..., x_n; X_1, ..., X_n)[\alpha_1, ..., \alpha_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff for all $\gamma \in S$,

$$\underline{\boldsymbol{M}} \models \varphi(x_1, ..., x_n; X_1, ..., X_n)[\gamma, \alpha_1, ..., \alpha_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff (by the induction hypothesis) for all $\gamma \in S$

$$\underline{\boldsymbol{M}} \models \varphi(x_1, ..., x_n; X_1, ..., X_n)[\gamma, \beta_1, ..., \beta_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$$

iff $\underline{\boldsymbol{M}} \models \psi(x_1, ..., x_n; , X_1, ..., X_n)[\beta_1, ..., \beta_m; \boldsymbol{X}_1, ..., \boldsymbol{X}_n]$.

To show that C is self-extensional, assume f is an n-ary connective of $\boldsymbol{S}$, $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n \in S$, and $C(\alpha_1) = C(\alpha_1), ..., C(\alpha_n) = C(\beta_n)$. Then, for the truth-scheme $\varphi_f(x_1, ..., x_n, X)$ for f in C and any $\boldsymbol{X} \in \boldsymbol{B}$, we have:

$$f(\alpha_1, ..., \alpha_n) \in \boldsymbol{X},$$

iff $\underline{\boldsymbol{M}} \models \varphi_f(x_1, ..., x_n; X)[(\alpha_1, ..., \alpha_n, \boldsymbol{X}]$

iff (by Lemma 1.2) $\underline{\boldsymbol{M}} \models \varphi_f(x_1, ..., x_n; X)[\beta_1, ..., \beta_n, \boldsymbol{X}]$

iff $f(\beta_1, ..., \beta_n) \in \boldsymbol{X}$.

Thus, for every $\boldsymbol{X} \in \boldsymbol{B}$, $f(\alpha_1, ..., \alpha_n) \in \boldsymbol{X}$ if and only if $f(\beta_1, ..., \beta_n) \in \boldsymbol{X}$. Since $\boldsymbol{B}$ is a base for $Th(C)$, this shows that $C(f(\alpha_1, ..., \alpha_n)) = C(f(\beta_1, ..., \beta_n))$. □

The converse of Theorem 1.1 is also true:

Theorem 1.3. *Suppose C is a self-extensional and finitary logic in $\boldsymbol{S}$ and let $\boldsymbol{B}$ be any base for $Th(C)$. Then, for some language* L *and for the pair* $(\mathrm{L}, \emptyset)$ *selected as model conditions for C, there exists an interpretation of the relation symbols of* L *on $\boldsymbol{B}$ such that the model*

$$\underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$$

is an intended frame for C. Furthermore, there exist truth-schemes which make $\underline{\boldsymbol{M}}$ a canonical frame for C.

Proof. For each $n \in \omega$ and each n-ary connective f of S we define the following $(n, 1)$-ary relation $\boldsymbol{R}_f$ on $\boldsymbol{S}^n \times \boldsymbol{B}$:

$$\boldsymbol{R}_f(\alpha_1, ..., \alpha_n, \boldsymbol{X}) \text{ iff there exist } \gamma_1, ..., \gamma_n \in S \text{ such that}$$
$$C(\alpha_1) = C(\gamma_1), ..., C(\alpha_n) = C(\gamma_n) \text{ and } f(\gamma_1, ..., \gamma_n) \in \boldsymbol{X}.$$

$\boldsymbol{R}_f$ is invariant under logical equivalence with respect to C, i.e., if $C(\alpha_1) = C(\gamma_1), ..., C(\alpha_n) = C(\gamma_n)$, then $\boldsymbol{R}_f(\alpha_1, ..., \alpha_n, \boldsymbol{X})$ if and only if $\boldsymbol{R}_f(\beta_1, ..., \beta_n, \boldsymbol{X})$.

Now let L be the language of truth-schemes which apart from the $(1, 1)$-ary epsilon predicate $\in$ has, for each n-ary connective f, an $(n, 1)$-ary predicate R_f in its vocabulary. The relation $\boldsymbol{R}_f$, defined as above, is then the interpretation of R_f. The model $\underline{\boldsymbol{M}}$ is thus well-defined and it is an intended frame for C.

We claim that the atomic formula $R_f(x_1, ..., x_n; X)$ is adequate for the connective f in this model. Indeed, assume

$$\underline{\boldsymbol{M}} \vDash R_f(x_1, ..., x_n; X)[\alpha_1, ..., \alpha_n, \boldsymbol{X}]$$

for $\alpha_1, ..., \alpha_n \in S$ and $\boldsymbol{X} \in \boldsymbol{B}$. So $\boldsymbol{R}_f(\alpha_1, ..., \alpha_n, X)$ which means that for some $\gamma_1, ..., \gamma_n \in S$,

(*) $\quad C(\alpha_1) = C(\gamma_1), ..., C(\alpha_n) = C(\gamma_n)$ and $f(\gamma_1, ..., \gamma_n) \in \boldsymbol{X}$.

Since C is self-extensional, (*) implies that $C(f(\alpha_1, ..., \alpha_n)) = C(f(\gamma_1, ..., \gamma_n))$. Hence $f(\alpha_1, ..., \alpha_n) \in \boldsymbol{X}$, by the last conjunct of (*).

Conversely, if $f(\alpha_1, ..., \alpha_n) \in \boldsymbol{X}$, then trivially $\boldsymbol{R}_f(\alpha_1, ..., \alpha_n, \boldsymbol{X})$. [Take $\gamma_j := \alpha_j$, for $i = 1, ..., n$.]

The theorem has been proved. □

Notes. 1. The above results do not contradict the well-known incompleteness results in modal logic. The truth-scheme $\varphi_\Box$ for the necessity connective $\Box$, as provided by the above theorem for a modal consequence C, does not agree with the standard one, given in Example 3. E.g. for the consequence C_{KR}, the relation $\boldsymbol{R}_\Box$ is defined by

$$\boldsymbol{R}_\Box(\alpha;\boldsymbol{X}) \text{ iff } (\exists\gamma\in S_\Box)C_{\mathrm{KR}}(\alpha)=C_{\mathrm{KR}}(\gamma)\ \&\ \Box_\gamma\in\boldsymbol{X}.$$

2. The above syntax of languages L allows for mixed predicates. We may take a more restrictive course and assume that the epsilon predicate is the only mixed predicate (of arity $(1,1)$). We then modify point (ii) of the definition of *Fo*(L) by admitting that the remaining predicate letters of L, if there are any, are always pure, i.e., they have arities $(0,n)$ for some natural numbers n. These predicates are thus interpreted as relations holding only between theories of a given sentential logic, and not elements of S. After such a modification of the syntax of the languages of possible truth-schemes, Theorem 1.1 obviously remains true. It is an open problem if its converse, i.e., the modified Theorem 1.3, is true. In this situation, the relation $\boldsymbol{R}_f$, defined as above, does not determine a truth-scheme in the above, restricted sense since this relation is not pure. In the "standard" formulation of truth-schemes for the intuitionistic calculus or for normal modal logics, as given in Examples 2 and 3, only *pure* binary relations are taken into account. $\Box$

2 Canonical models

Let C be a sentential logic in $\boldsymbol{S}$ and let the pair (L,T) define model conditions for C. Furthermore, suppose that for some base $\boldsymbol{B}\subseteq Th(C)$

$$\underline{\boldsymbol{M}}=(S,\boldsymbol{B},\in,\boldsymbol{R}_1,\boldsymbol{R}_2,...)$$

is a canonical frame for C. The logic C is thus self-extensional.

We define a mapping V from the set $Var(\boldsymbol{S})$ of sentential variables to the power set $P(\boldsymbol{B})$. [We recall that $Var(\boldsymbol{S})$ coincides with the set of individual variables of L.] We put:

$$V(x):=\{\boldsymbol{X}\in\boldsymbol{B}:x\in\boldsymbol{X}\}.$$

We then extend V to a mapping from $\boldsymbol{S}$ to $P(\boldsymbol{B})$ as follows. If f is an n-ary connective of $\boldsymbol{S}$ and $\alpha_1,...,\alpha_n\in S$, then:

$$V(f(\alpha_1,...,\alpha_n)):=\{\boldsymbol{X}\in\boldsymbol{B}:\underline{\boldsymbol{M}}\vDash f(x_1,...,x_n;X)[\alpha_1,...,\alpha_n,\boldsymbol{X}]\}.$$

V is called the *canonical valuation* of S in $\underline{\boldsymbol{M}}$ and the pair

$$\langle \underline{\boldsymbol{M}}, V\rangle$$

is called a *canonical model for* C. If $\boldsymbol{X} \in V(\alpha)$, then we say that α *holds* (is *true*) at $\boldsymbol{X}$ in the model $\langle \underline{\boldsymbol{M}}, V\rangle$.

Truth Lemma 2.1. *For every* $\alpha \in S$, $V(\alpha) = \{\boldsymbol{X} \in \boldsymbol{B} : \alpha \in \boldsymbol{X}\}$.

Proof. The lemma trivially holds if α is a sentential variable. Now assume α is a compound sentential formula. So α is of the form $f(\alpha_1, ..., \alpha_n)$ for some sentences $\alpha_1, ..., \alpha_n$ and a connective f. Since $\underline{\boldsymbol{M}}$ is a canonical frame for C, we have that

$$\{\boldsymbol{X} \in \boldsymbol{B} : \alpha \in X\} = \{\boldsymbol{X} \in \boldsymbol{B} : f(\alpha_1, ..., \alpha_n) \in \boldsymbol{X}\} =$$

$$\{\boldsymbol{X} \in \boldsymbol{B} : \underline{\boldsymbol{M}} \models \varphi_f(x_1, ..., x_n; X)[\alpha_1, ..., \alpha_n, \boldsymbol{X}]\} = V(\alpha). \qquad \square$$

Corollary 2.2. *For any* $\alpha, \beta \in S$, $C(\alpha) = C(\beta)$ *iff* $V(\alpha) = V(\beta)$.

The canonical model $\langle \underline{\boldsymbol{M}}, V\rangle$ determines a mapping from $P(S)$ to $P(S)$, denoted by

$$\langle \underline{\boldsymbol{M}}, V\rangle^{\models},$$

where for any set $T \subseteq S$,

$$\alpha \in \langle \underline{\boldsymbol{M}}, V\rangle^{\models}(T) \text{ iff } \bigcap\{V(\gamma) : \gamma \in T\} \subseteq V(\alpha).$$

In fact, $\langle \underline{\boldsymbol{M}}, V\rangle^{\models}$ is a consequence operation on $\underline{S}$. But we have more:

Theorem 2.3. (Strong Completeness Theorem). $C = \langle \underline{\boldsymbol{M}}, V\rangle^{\models}$.

In particular, the above theorem implies that the consequence $\langle \underline{\boldsymbol{M}}, V\rangle^{\models}$ is structural and finitary.

Proof. Let $\alpha \in S$ and $T \subseteq S$. Then:

$$\alpha \in \langle \underline{\boldsymbol{M}}, V\rangle^{\models}(T) \quad \text{iff}$$

$$\bigcap\{V(\gamma) : \gamma \in T\} \subseteq V(\alpha) \quad \text{iff}$$

$$(\forall \boldsymbol{X} \in \boldsymbol{B})(\boldsymbol{X} \in \bigcap\{V(\gamma) : \gamma \in T\} \text{ implies } \boldsymbol{X} \in V(\alpha)) \quad \text{iff}$$

$$(\forall \boldsymbol{X} \in \boldsymbol{B})(T \subseteq \boldsymbol{X} \text{ implies } \alpha \in \boldsymbol{X} \quad \text{iff}$$

$$\alpha \in C(T). \qquad \square$$

3 Relational semantics - open problems

The above considerations leave open the question of defining general relational semantics for self-extensional logics. In Section 2 we only defined canonical models formed from bases. We present here some preliminary remarks on arbitrary frames without the intention of solving completely the problem.

L is a fixed language of truth-schemes for $\boldsymbol{S}$ and C is a self-extensional logic in $\boldsymbol{S}$. Let T be a set of sentences of L so that (L, T) forms model conditions for C. Suppose furthermore that for some base $\boldsymbol{B} \subseteq Th(C)$,

$$\underline{\boldsymbol{M}} = (S, \boldsymbol{B}, \in, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$$

is a canonical frame for C with a list $\varphi_f, f \in Con(S)$, of adequate truth-schemes. Now let L* be the language obtained from L by deleting the epsilon predicate " $\in$ " and adjoining a new $(1,1)$-predicate " $\Vdash$ " called the *satisfaction predicate* (for the sentences of $\boldsymbol{S}$). Thus, instead of " $x \in X$ " we uniformly write " $x \Vdash X$ " and read: " x is satisfied (holds) at X " . The variable X now ranges over possible worlds and x, as before, ranges over elements of S. φ_f^* is the formula obtained from φ_f by replacing uniformly each subformula of the shape " $x \in X$ " by " $X \Vdash x$ " .

By a *frame* we understand any two-sorted model $\underline{\boldsymbol{F}}$ for L* with the universe (S, W), where W is a non-empty set called the set of *worlds* (*states*, etc.). Since L* may involve mixed predicates, these predicates are represented in $\underline{\boldsymbol{F}}$ as relations holding between tuples of elements of S and tuples of elements of W. In particular, " $\Vdash$ " is interpreted as a subset of $S \times W$, and denoted by the same symbol $\Vdash$. As is customary, when $\underline{\boldsymbol{F}}$ is clear from context, the fact that a pair (α, w) is in the relation $\Vdash$ is written as $w \Vdash \alpha$.

We say that

$$\underline{\boldsymbol{F}} = (S, W, \Vdash, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$$

is a frame for C if the following conditions are satisfied:

(1) $\underline{\boldsymbol{F}}$ is a model of T,

(2) $\underline{\boldsymbol{F}}$ has the invariance property, that is, if $\boldsymbol{R}$ is an (m, n)-ary relation of $\underline{\boldsymbol{F}}$ and $C(\alpha_1) = C(\beta_1), ..., C(\alpha_m) = C(\beta_m)$, then for any $w_1, ..., w_n \in W$,

$$\boldsymbol{R}(\alpha_1, ..., \alpha_m; w_1, ..., w_n) \text{ iff } \boldsymbol{R}(\beta_1, ..., \beta_m; w_1, ..., w_n).$$

[In particular, it is assumed that if $C(\alpha) = C(\beta)$, then

$$w \Vdash \alpha \text{ iff } w \Vdash \beta,$$

for every world w.]

(3) For any connective f of $\boldsymbol{S}$, say n-ary, any n-tuple $\alpha_1, ..., \alpha_n \in S$, and any $w \in W$,

$$w \Vdash f(\alpha_1, ..., \alpha_n) \text{ iff } \underline{\boldsymbol{F}} \vDash \varphi_f^*(x_1, ..., x_n; X)[\alpha_1, ..., \alpha_n, w],$$

where, on the right side, $\vDash$ stands for the usual satisfaction of the formulas of L* in the frame $\underline{\boldsymbol{F}}$.

Each frame $\underline{\boldsymbol{F}}$ for C determines a consequence operation on S, denoted by $\underline{\boldsymbol{F}}^{\vDash}$. For $\alpha \in S$ and $X \subseteq S$ we put:

$\alpha \in \underline{\boldsymbol{F}}^{\vDash}(X)$ iff for every $w \in W$, $w \Vdash \alpha$ whenever $w \Vdash \gamma$, for all $\gamma \in X$.

The above definitions give rise to two open problems:

Problem 1. Formulate sufficient conditions imposed on C, on model conditions (L∗, T), and truth-schemes φ_f^*, $f \in \mathit{Con}(\boldsymbol{S})$, under which, for every frame $\underline{\boldsymbol{F}}$ for C, the consequence $\underline{\boldsymbol{F}}^{\vDash}$ is equal or stronger than C.

The second problem concerns extendability of subsets of $\mathit{Var}(\boldsymbol{S}) \times W$ to a satisfaction relation. Before formulating the problem we need one more definition. A logic C in $\boldsymbol{S}$ is *almost-inconsistent* if $C(\emptyset) = \emptyset$ and $C(X) = S$ for every non-empty set X. We observe that C is inconsistent or almost inconsistent iff $y \in C(x)$ for some (equivalently, for any) distinct variables x and y.

Problem 2. Let C be a self-extensional logic in $\boldsymbol{S}$ such that C is neither inconsistent nor almost inconsistent. Let $\underline{\boldsymbol{F}}_0 = (S, W, \Vdash_0, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$ be a frame with the following properties:

(i) the relations $\boldsymbol{R}_1, \boldsymbol{R}_2, ...$ are invariant with respect to the deductive equivalence on the basis of C;

(ii) $\Vdash_0$ is a subset of $\mathit{Var}(\boldsymbol{S}) \times W$.

When can $\Vdash_0$ be extended to a set $\Vdash \subseteq S \times W$ such that:

(a) the intersection of $\Vdash$ with $Var(\boldsymbol{S}) \times W$ coincides with $\Vdash_0$, i.e., $\Vdash$ is an extension of $\Vdash_0$,

(b) $\underline{\boldsymbol{F}} := (S, W, \Vdash, \boldsymbol{R}_1, \boldsymbol{R}_2, ...)$ is a frame for C ?

We see that

(*) for any variables x, y, $C(x) = C(y)$ implies that for every world w, $w \Vdash x$ iff $w \Vdash y$,

is a necessary condition for (a) and (b) to hold. But (*) is true because the antecedent of it is false when $x \neq y$, by the assumptions made about C.

We may use of course the formulas φ_{f^*}, $f \in Con(\boldsymbol{S})$, to extend $\Vdash_0$ to a relation $\Vdash$ on $S \times W$. But the crucial issue here is the invariance of $\Vdash$ under the deductive equivalence with respect to C.

4 Final remarks

It is often said that logical constants, and connectives in particular, acquire their meaning through clauses of a recursive definition which, after Carnap [1942], are called *truth-conditions*. In the case of classical sentential logic, the simplest known procedure of this kind consists in assigning to each connective the corresponding Boolean operation on {0,1}. This approach obviously fails for first-order languages because it is not possible to define the meaning of quantifiers by a simple reference to two objects representing truth and falsity. For some sentential consequence operations, not necessarily self-extensional, it is possible to provide a list of simple conditions which characterize a sufficiently large subset of admissible truth-valuations so that it determines a given consequence. For instance, the consequence of the three-valued Łukasiewicz logic C_3, originally determined by the three-element Łukasiewicz matrix in the language $\boldsymbol{S}$ with implication $\to$ and negation $\neg$, is characterized by the set of truth-valuations h which satisfy the following conditions, for all $\alpha, \beta \in S$ (Suszko [1975]:

(0) either $h(\alpha) = 0$ or $h(\neg\alpha) = 0$

(1) $h(\beta) = 1$ implies $h(\alpha \to \beta) = 1$

(2) if $h(\alpha) = 1$ and $h(\beta) = 0$ then $h(\alpha \to \beta) = 0$

(3) if $h(\alpha) = h(\beta)$ and $h(\neg\alpha) = h(\neg\beta)$ then $h(\alpha \to \beta) = 1$

(4) if $h(\alpha) = h(\beta) = 0$ and $h(\neg\alpha) \neq h(\neg\beta)$ then $h(\alpha \to \beta) = h(\neg\alpha)$

(5) if $h(\neg\alpha) = 0$ then $h(\neg\neg\alpha) = h(\alpha)$

(6) if $h(\alpha) = 1$ and $h(\beta) = 0$ then $h(\neg(\alpha \to \beta)) = h(\neg\beta)$

(7) if $h(\alpha) = h(\neg\alpha) = h(\beta)$ and $h(\neg\beta) = 1$ then $h(\neg(\alpha \to \beta)) = 0$.

Conditions (0) - (7) fully characterize admissible truth-valuations for C_3 determined by reference assignments, i.e., homomorphisms from $\boldsymbol{S}$ to the three-element Łukasiewicz matrix. Further examples can be found e.g. in Malinowski [1977], Scott [1973], Suszko [1974], Urquhart [1973].

The essential property of the truth-defining clauses (0) - (7) is their recursiveness: they enable to compute the truth value of any compound sentence α provided that the values of all atomic sentences appearing in α are already established.

The above example opens a possibility of building a uniform conception of truth- conditions for sentential logics treated as finite lists of Boolean combinations of expressions of the form

$$h(x) = 0,$$

$$h(x) = 1,$$

$$h(f(x_1, ..., x_n)) = 0,$$

$$h(f(x_1, ..., x_n)) = 1,$$

where f ranges over the connectives of $\boldsymbol{S}$ and $x, x_1, x_2, ...$ represent sentences of $\boldsymbol{S}$, and h represents truth-valuations. (The analysis of conditions (6) and (7) shows that the syntax of such a language is even more involved.) Each such list should determine the value of a truth valuation on a complex sentence by means of the values of the truth-valuation on the constituents of the sentence. This conception differs from the one discussed in this paper since the language which reflects the properties of so defined truth-conditions does not resemble the languages of truth schemes, discussed in Section 1. It is clear that such expressions defining truth-conditions for a given logic C cannot be provided without having a prior knowledge of the structure of the matrix models of C. At the same time, even for relatively simple consequences, the expressions defining admissible truth-valuations are rather obscure.

References

[1] R. Carnap, Introduction to Semantics, Harvard University Press, Harvard 1942.

[2] J. Czelakowski, Protoalgebraic Logics, Kluwer, Dordrecht 2001.

[3] J. Czelakowski and G. Malinowski, *Key notions of Tarski's methodology of deductive systems*, Studia Logica **44** (1895), pp. 321–351.

[4] N.C.A. da Costa and J.-Y. Béziau, *Théorie de la valuation*, Logique et Analyse **146** (1994), pp. 95–117.

[5] B.C. van Fraassen, Formal Semantics and Logics, The Macmillan Co., New York 1973.

[6] G. Frege, Über Sinn und Bedeutung, Zeitschrift für Philosophie und philosophische Kritik, pp. 25–50. English translation as 'On Sense and Reference' by M. Black in P.T. Geach and M. Black (eds.), Translations from the Philosphical Writings of Gottlob Frege, Oxford 1952, pp. 56–78.

[7] G. Malinowski, *Classical characterization of m-valued Łukasiewicz calculi*, Reports on Mathematical Logic **9** (1977), pp. 41–45.

[8] G. Malinowski, Many-Valued Logics, Oxford Logic Guides 25, Clarendon Press, Oxford 1993.

[9] R. Routley, *Every sentential logic has a two-valued worlds semantics*, Logique et Analyse 74-75-76 (1976), pp. 353–365.

[10] D. Scott, *On engendering an illusion of understanding*, The Journal of Philosophy **68** (1971), pp. 143–173.

[11] D. Scott, *Background to formalization*, in: Truth, Syntax and Modality (ed. H. Leblanc), North-Holland, Amsterdam-London 1973.

[12] R. Suszko, *A note on the intuitionistic sentential calculus (ISC)*, Bulletin of the Section of Logic **3** (1974), pp. 20–21.

[13] R. Suszko, *The Fregean Axiom and Polish Mathematical Logic in the 1920's*, Studia Logica **36** (1977), pp. 377–380.

[14] R. Suszko, *Remarks on Łukasiewicz's three-valued logic*, Bulletin of the Section of Logic **4** (1977), pp. 87–90.

[15] A. Urquhart, *An interpretation of many-valued logic*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **19** (1973), pp. 111–114.

[16] R. Wójcicki, *Referential matrix semantics for propositional calculi*, Bulletin of the Secion of Logic **8** (1979), pp. 170–176.

[17] R. Wójcicki, Theory of Logical Calculi. Basic Theory of Consequence Operations, Kluwer, Dordrecht 1988.

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