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REFERENTIAL SEMANTICS: DUALITY AND APPLICATIONS

In memory of Willem Blok

A b s t r a c t. In this paper, Wójcicki's characterization of selfextensional logics as those logics that are endowed with a complete local referential semantics is extended to a fully fledged duality between atlas-models (i.e. generalized matrix models) and referential models of an arbitrary selfextensional logic $\mathcal{S}$. This duality serves as a general template where a wide range of Stone- and Priestley-style dualities related with concrete logics can fit. The first application of this duality is a characterization of the *fully selfextensional* logics among the selfextensional ones. Fully selfextensional logics form a subclass of particularly well-behaved selfextensional logics, and only recently [1] this inclusion was shown to be proper. In this paper, fully selfextensional logics are characterized as those selfextensional logics $\mathcal{S}$ whose algebraic counterpart $\mathsf{Alg}(\mathcal{S})$ – seen as a category – is dually equivalent to the *reduced* referential models of $\mathcal{S}$. This implies that if $\mathcal{S}$ is fully selfextensional, then every algebra in $\mathsf{Alg}(\mathcal{S})$ is isomorphic to an algebra of sets.

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1 Introduction

Substitution-invariant consequence relations between sets of formulas and formulas are taken as the primary logical objects in Abstract Algebraic Logic (AAL). Logics so defined (see Section 2) may satisfy different replacement properties, the strongest of which says that for any set of formulas Γ, any formulas φ, ψ, δ and any propositional variable p,

$$\text{if } \Gamma, \varphi \vdash_{\mathcal{S}} \psi \text{ and } \Gamma, \psi \vdash_{\mathcal{S}} \varphi,$$

$$\text{then } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi) \text{ and } \Gamma, \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi),$$

$\delta(p/\varphi)$ and $\delta(p/\psi)$ being obtained by substituting φ for p and ψ for p in δ respectively. This property is possessed by classical and intuitionistic propositional logics ($\mathcal{CPC}$ and $\mathcal{IPC}$ respectively) and can be taken as a formalization of Frege's principle of compositionality for truth-values, or extensionality principle. Logics satisfying it are called Fregean, after Suszko [?]. Examples of important non-Fregean logics abound, for instance almost all the logics of the modal family. Nevertheless non-Fregean logics, such as the local consequence relation of the normal modal logic K (see below), often satisfy the following weaker replacement property: for any formulas φ, ψ, δ and any propositional variable p,

$$\text{if } \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi, \text{ then } \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi) \text{ and } \delta(p/\varphi) \vdash_{\mathcal{S}} \delta(p/\psi).$$

A logic is *selfextensional* if it satisfies this weaker replacement property. In algebraic terms, this means that the relation of logical equivalence between formulas, defined as $\varphi \vdash_{\mathcal{S}} \psi$ and $\psi \vdash_{\mathcal{S}} \varphi$, is a congruence relation of the formula algebra. To our knowledge, the term 'selfextensional' was coined by Wójcicki [16].

Typical examples of selfextensional logics besides $\mathcal{CPC}$ and $\mathcal{IPC}$ are determined by the *local* consequence relations associated with any normal modal logic L, i.e. the deducibility relations associated with a Hilbert-style calculus defined by taking the theorems of L as the set of its axioms, and *modus ponens* as its sole rule of inference. It is well-known that consequence

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relations defined in this way are exactly the local consequence relations determined by the class of Kripke models of L.

Selfextensional logics constitute a class of important logics that cuts across the Leibniz hierarchy defined within AAL; see [6, 10] for the definition and properties of the different classes of this hierarchy. Selfextensional logics occur in any class of this hierarchy: $\mathcal{CPC}$ and $\mathcal{IPC}$ are instances of selfextensional logics that are algebraizable; some selfextensional logics, such as the local consequence of the normal modal logic K, are equivalential but not algebraizable, some are protoalgebraic but not equivalential such as the local consequence of the classical modal logic E, and finally some are non-protoalgebraic, for instance positive modal logic [5], Belnap's four-valued logic [2, 8], the conjunction-disjunction fragment of classical propositional logic [11] and Visser's logic [4, 14]. Moreover, non-selfextensional logics occur in all these classes. This strategic position of selfextensional logics adds to the interest that researchers in AAL have in them, because it makes possible to devise proof strategies for results that involve pairs of logics $\mathcal{S}$ and $\mathcal{S}'$ which may belong to different classes of the hierarchy, but are connected by the condition $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}')$: for in this case, if a certain result holds for $\mathsf{Alg}(\mathcal{S})$ which depends on the hypothesis of $\mathcal{S}$ being selfextensional, then it can be extended to $\mathcal{S}'$ even if $\mathcal{S}'$ is not selfextensional.

Relational (Kripke-style) semantics established themselves as a powerful tool of investigation for a much wider range of logics than the intensional ones which they have been originally developed for. Nowadays, when a new logic is proposed it is a standard practice to endow it with this type of semantics. Relational models for a propositional logic on a language[1] $\mathcal{L}$ are based on nonempty sets W of points, endowed with additional structure which usually serves to define an $\mathcal{L}$-algebra on some family $\mathcal{A}$ of subsets of W; frequently this family is the powerset of W. Formulas are then interpreted by assigning a set $v(p) \in \mathcal{A}$ to every propositional variable p, and extending this assignment to every formula as a homomorphism v from the algebra of formulas into the $\mathcal{L}$-algebra carried by $\mathcal{A}$. In this way, given an assignment v to propositional variables, each formula φ obtains a truth-value at every point $w \in W$: `true` if w belongs to the set $v(\varphi)$ assigned to

[1]'Propositional language' and 'algebraic similarity type' are treated as synonymous expressions.

φ and `false` otherwise. Using Frege's insight that the reference of a sentence is its truth-value, we can say that the interpreted formulas obtain a reference at each point in W, and assuming this perspective the elements of W can be called 'reference points'. If we try and set forth what the Kripke style semantics for all the different propositional logics have in common, we find that it simply is what we emphasized: i) every relational structure is based on a nonempty set W, ii) a designated collection of subsets $\mathcal{A}$ of W is endowed with an algebraic structure of the same type as the algebra of formulas, using resources from the structure (a relation, a monoid operation, a partial order, etc.), and iii) formulas are interpreted using homomorphisms from the algebra of formulas into $\mathcal{A}$ just in the way we described. In this setting, the algebra $\mathcal{A}$ is frequently thought of as the algebra of propositions of the structure under consideration. Looking at relational semantics from this abstract perspective, we are thus left with structures of form $\langle W, \mathcal{A} \rangle$, such that W is a nonempty set and $\mathcal{A}$ is an algebra of subsets of W (the algebra of propositions). These structures were studied by Wójcicki in [17] and named *referential algebras* because they mainly consist of algebras built on sets of reference points. Accordingly, a semantics of this type is called a *referential semantics* and can be regarded as an abstract version of the general-frame-style semantics of many intensional logics. For information on referential semantics see also [6].

Wójcicki ([16, 17]) characterized selfextensional logics as the ones that 'admit a referential semantics': Given a propositional language $\mathcal{L}$, let M be a class of referential algebras of this type, i.e. objects in M are tuples $\langle W, \mathcal{A} \rangle$ such that W is a nonempty set, and $\mathcal{A}$ is an $\mathcal{L}$-algebra of subsets of W. Then a consequence relation for formulas of type $\mathcal{L}$, over a given set of variables, can be defined as follows:

$\Gamma \models_{\mathsf{M}} \varphi$ iff for every $\langle W, \mathcal{A} \rangle \in \mathsf{M}$, every valuation v of the formulas on $\mathcal{A}$, and every point $w \in W$, if for every $\psi \in \Gamma$ it holds that $w \in v(\psi)$, then $w \in v(\varphi)$.

The algebra of formulas of type $\mathcal{L}$, endowed with the consequence relation $\models_{\mathsf{M}}$, is a logic in the AAL sense, and we will refer to it as the *local logic* of M. A logic $\mathcal{S}$, however defined, has a *complete local referential semantics* if there is a class M of referential algebras such that the consequence relation of $\mathcal{S}$ coincides with $\models_{\mathsf{M}}$. Wójcicki [17] showed that selfextensional logics are exactly the ones endowed with a complete local referential semantics.

Fully selfextensional logics. Selfextensionality is a property that can be defined by using consequence relation as the only primitive symbol. So it is a metalogical property, and as such it falls within the area of interest of AAL, a theory that was originally developed by Blok and Pigozzi for providing a setting in which metalogical properties of logics could be studied and characterized in terms of algebraic properties of their associated classes of algebras. It turns out that, in such a general setting, some metalogical properties cannot be captured by properties of algebraic structures alone. This is the main reason why several kinds of algebra-based, enhanced structures were put in play. In particular, two more kinds of models for logics were fruitfully used in AAL to develop a general and uniform procedure for associating any logic $\mathcal{S}$ with its class of algebras $\mathsf{Alg}\mathcal{S}$ (see Section 5), and to characterize the level of the Leibniz hierarchy $\mathcal{S}$ belongs to. Models of the first kind are called *logical matrices* and are structures of the form $\langle \boldsymbol{A}, F\rangle$ such that $\boldsymbol{A}$ is an algebra and F a subset of the carrier of $\boldsymbol{A}$. Models of the second kind are structures of the form $\langle \boldsymbol{A}, \mathcal{B}\rangle$ such that $\boldsymbol{A}$ is an algebra and $\mathcal{B}$ a family of subsets of the carrier of $\boldsymbol{A}$, and are called *generalized matrices* by Wójcicki [16] and Czelakowski [6], and *atlases* by Dunn & Hardegree [7]. We will use this latter terminology in this paper. A particularly relevant kind of atlases are the *abstract logics* of Brown & Suszko [3], which are the atlases such that $\mathcal{B}$ is a closure system (a.k.a. topped intersection structure). Abstract logics were used in [9] to develop a general algebraic semantics for propositional logics.

A logical matrix $\langle \boldsymbol{A}, F\rangle$ is a *model* of a logic $\mathcal{S}$ if $\boldsymbol{A}$ is of the type of $\mathcal{S}$ and for every set of formulas $\varGamma$ and every formula φ,

$$\text{if } \varGamma \vdash_{\mathcal{S}} \varphi \text{ then for every } h \in Hom(\boldsymbol{Fm}, \boldsymbol{A}) \text{ if } h[\varGamma] \subseteq F, \text{ then } h(\varphi) \in F \tag{1}$$

In this case F is said to be an $\mathcal{S}$*-filter* of $\boldsymbol{A}$. The set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$. An atlas $\langle \boldsymbol{A}, \mathcal{C}\rangle$ is a *model* of $\mathcal{S}$ if $\boldsymbol{A}$ is of the type of $\mathcal{S}$ and every element of $\mathcal{C}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$. Notice that atlases of form $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ are abstract logics since $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ is a closure system.

Fully selfextensional logics form a subclass of particularly well-behaved selfextensional logics. For every logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$ of the type of $\mathcal{S}$, consider the atlas $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$, which is clearly a model of $\mathcal{S}$. In general, the relation $\varLambda_{\boldsymbol{A}}\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ given by

$$\langle a, b\rangle \in \varLambda_{\boldsymbol{A}}\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \quad \text{iff} \quad \forall F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\ (a \in F \Leftrightarrow b \in F)$$

is not a congruence of $\boldsymbol{A}$. A logic $\mathcal{S}$ is *fully selfextensional* if for every algebra $\boldsymbol{A}$ the relation $\Lambda_{\boldsymbol{A}}\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ is a congruence of $\boldsymbol{A}$. Part of [9] is devoted to the study of fully selfextensional logics. Almost all known selfextensional logics are fully selfextensional: sufficient conditions for a selfextensional logic to be fully selfextensional are to possess a conjunction or an implication that satisfies modus ponens and deduction theorem [9]. The question whether every selfextensional logic is fully selfextensional was raised as an open problems in [9]. Babyonyshev [1] presented an *ad hoc* example of a selfextensional logic that is not fully selfextensional.

In this paper, we characterize fully selfextensional logics among the selfextensional ones by properties of their referential semantics. To this purpose we set a general duality[2] between certain categories of atlases and of referential algebras that will be introduced in 2.2 and 3. This duality can be seen as an abstraction of the well-known dualities between categories of algebras and of general Kripke frames for several propositional logics, and analogously to those dualities, it establishes a formal connection between atlas semantics and referential semantics holding uniformly for every selfextensional logic. This duality is the basic tool for characterizing fully selfextensional logics: they are exactly those selfextensional logics $\mathcal{S}$ such that $\mathsf{Alg}\mathcal{S}$ – seen as a category – is dually equivalent to the *reduced referential algebra $\mathcal{S}$-models.*

As for the structure of this paper, in Section 2, after giving the definition of logic adopted in AAL, we formally introduce referential semantics, prove Wójcicki's theorem and define the category of reduced referential algebras of a given algebraic similarity type. In Section 3 we present the atlas semantics and define the category of Frege-reduced atlases for an arbitrary algebraic similarity type. In Section 4 we establish the dual equivalence between the category of reduced referential algebras of an arbitrary similarity type and the category of Frege-reduced atlases of the same type. Section 5 is the main section of the paper and contains some characterizations of fully selfextensional logics in terms of properties of their (reduced) referential semantics.

Notational convention. Objects that form part of a compound structure that is labelled with a subindex will inherit the subindex without further notice. For any category C, $\mathsf{C}(X_1, X_2)$ is the set of morphisms between

[2]Some first steps in establishing this duality were taken by Czelakowski.

objects X_1, X_2 in C. The notation $Hom(\boldsymbol{A}_1, \boldsymbol{A}_2)$ refers only to algebra homomorphisms, whenever $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are arbitrary algebras. Set is the category of sets and set-maps.

2 Selfextensional Logics and Referential Semantics

Let us fix an algebraic similarity type $\mathcal{L}$ and a denumerable set Var of propositional variables. A *logic* (or deductive system) of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ such that $\boldsymbol{Fm}$ is the $\mathcal{L}$-algebra of formulas over Var and $\vdash_{\mathcal{S}}$ is a substitution-invariant consequence relation on the universe Fm of $\boldsymbol{Fm}$, i.e. $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm) \times Fm$ satisfies the following conditions for all sets of formulas Γ, Δ, every $\varphi \in Fm$ and every *substitution* $\sigma \in Hom(\boldsymbol{Fm}, \boldsymbol{Fm})$:

1. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.

2. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \psi$, then $\Delta \vdash_{\mathcal{S}} \varphi$.

3. If $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$ (invariance under substitutions).

Conditions (1) and (2) yield that:

4. If $\Gamma \vdash_{\mathcal{S}} \varphi$, then for any ψ, $\Gamma \cup \{\psi\} \vdash_{\mathcal{S}} \varphi$.

A logic $\mathcal{S}$ is *finitary* if the consequence relation $\vdash_{\mathcal{S}}$ is finitary, namely if for every set of formulas Γ and every formula φ

5. if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\Delta \vdash_{\mathcal{S}} \varphi$ for some finite $\Delta \subseteq \Gamma$.

A *theory* of a logic $\mathcal{S}$, or $\mathcal{S}$-theory, is a set of formulas that is closed under the relation $\vdash_{\mathcal{S}}$. An $\mathcal{S}$-theory can be equivalently described as an $\mathcal{S}$-filter of the formula algebra. Let Th$\mathcal{S}$ be the set of theories of $\mathcal{S}$. This set is a closure system, i.e. it is closed under intersections of arbitrary subfamilies[3].

[3]If $\mathcal{X}$ is ranging in $\mathcal{P}(\mathcal{P}(X))$ for some set X, then we stipulate $\bigcap \mathcal{X} = X$ whenever $\mathcal{X} = \emptyset$.

2.1 Referential semantics

An $\mathcal{L}$-*referential algebra* is a structure $\mathcal{F} = \langle W, \mathcal{A}\rangle$ such that W is a nonempty set and $\mathcal{A}$ is an $\mathcal{L}$-algebra of subsets of W. The elements w in W are called points, reference points, indices or states, etc. We say that $\langle W, \mathcal{A}\rangle$ is a referential algebra *based on* W. Homomorphisms $h \in Hom(\boldsymbol{Fm}, \mathcal{A})$ are called *interpretations*. For any point $w \in W$, a formula φ is *true at* w under the interpretation h if $w \in h(\varphi)$; otherwise, φ is *false at* w.

We present here Wójcicki's characterization of selfextensional logics with some detail since it is not widely known.

There are two substitution-invariant consequence relations on the $\mathcal{L}$-algebra of formulas $\boldsymbol{Fm}$ that are naturally induced by any $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$: They are the *local* and the *global* consequences induced by $\mathcal{F}$, are denoted by $\vdash_{\mathcal{F}}$ and $\vdash^{g}_{\mathcal{F}}$ respectively, and are defined by the following clauses: for every set of formulas Γ and every formula φ,

$$\Gamma \vdash_{\mathcal{F}} \varphi \ \text{ iff } \ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi);$$

$$\Gamma \vdash^{g}_{\mathcal{F}} \varphi \ \text{ iff } \ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \text{ if } \bigcap_{\psi \in \Gamma} h(\psi) = W, \text{ then } h(\varphi) = W.$$

Similarly, we define the *local* and *global* (substitution-invariant) consequence relations on $\boldsymbol{Fm}$, denoted $\vdash_{\mathsf{F}}$ and $\vdash^{g}_{\mathsf{F}}$ respectively, for any class F of $\mathcal{L}$-referential algebras:

$$\vdash_{\mathsf{F}} \ = \ \bigcap \{ \vdash_{\mathcal{F}} : \mathcal{F} \in \mathsf{F}\} \qquad \text{and} \qquad \vdash^{g}_{\mathsf{F}} \ = \ \bigcap \{ \vdash^{g}_{\mathcal{F}} : \mathcal{F} \in \mathsf{F}\}.$$

We will see that selfextensional logics are exactly those that are determined by the local consequences associated with classes of referential algebras.

Proposition 2.1. *For every class* F *of* $\mathcal{L}$*-referential algebras, the local logic* $\mathcal{S}_{\mathsf{F}} = \langle \boldsymbol{Fm}, \vdash_{\mathsf{F}}\rangle$ *is selfextensional.*

Proof. Let us denote $\vdash_{\mathsf{F}}$ simply by $\vdash$. Assume that $\varphi \dashv\vdash \psi$. Then, for every $\langle W, \mathcal{A}\rangle \in \mathsf{F}$ and every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$, $h(\varphi) = h(\psi)$. Therefore, for every variable p and every formula δ, $h(\delta(p/\varphi)) = h(\delta(p/\psi))$. Hence, $\delta(p/\varphi) \dashv\vdash \delta(p/\psi)$. □

For every logic $\mathcal{S}$ of type $\mathcal{L}$, an $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ is a *local* $\mathcal{S}$*-model* if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{F}}$, i.e, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ imply that $\bigcap_{\psi \in \Gamma} h(\psi) \subseteq h(\varphi)$. Similarly, $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a *global $\mathcal{S}$-model* if $\vdash_{\mathcal{S}} \subseteq \vdash^{g}_{\mathcal{F}}$, i.e. if $\Gamma \vdash_{\mathcal{S}} \varphi$, $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ and $\bigcap_{\psi \in \Gamma} h(\psi) = W$ imply that $h(\varphi) = W$.

A class F of $\mathcal{L}$-referential algebras is a *complete local referential semantics* for a logic $\mathcal{S}$ of type $\mathcal{L}$ if $\vdash_{\mathcal{S}} = \vdash_{\mathsf{F}}$, and in this case we say that $\mathcal{S}$ *admits a complete local referential semantics*. With these concepts at hand we can state Wójcicki's theorem:

Theorem 2.2 (Wójcicki). *A logic $\mathcal{S}$ is selfextensional iff it admits a complete local referential semantics.*

Proof. Proposition 2.1 gives the implication from right to left. As for the converse, let us assume that $\mathcal{S}$ is selfextensional, and base the canonical referential algebra $\mathcal{F}_c$ on the set $\mathrm{Th}\mathcal{S}$ of $\mathcal{S}$-theories. Now notice that the following clause

$$\eta(\varphi) = \{T \in \mathrm{Th}\mathcal{S} : \varphi \in T\}$$

defines a map η from Fm into the powerset of $\mathrm{Th}\mathcal{S}$.

Then $\mathcal{F}_c = \langle \mathrm{Th}\mathcal{S}, \eta(\boldsymbol{Fm}) \rangle$ is obtained by defining the algebra $\eta(\boldsymbol{Fm})$ of subsets of $\mathrm{Th}\mathcal{S}$ as follows:

1. The carrier of $\eta(\boldsymbol{Fm})$ is the set $\eta[Fm] = \{\eta(\varphi) : \varphi \in Fm\}$.

2. For every n-ary operation symbol $\star$ of $\mathcal{L}$, we set

$$\star(\eta(\varphi_1), \ldots, \eta(\varphi_n)) = \eta(\star(\varphi_1 \ldots \varphi_n)).$$

The selfextensionality of $\mathcal{S}$ guarantees that this definition is independent of the choice of representatives, and so we also get that $\eta \in \mathrm{Hom}(\boldsymbol{Fm}, \eta(\boldsymbol{Fm}))$. We claim that $\vdash_{\mathcal{F}_c} = \vdash_{\mathcal{S}}$. If $\Gamma \vdash_{\mathcal{F}_c} \varphi$, then $\bigcap_{\psi \in \Gamma} \eta(\psi) \subseteq \eta(\varphi)$. Let $T = C_{\mathcal{S}}(\Gamma)$. Clearly $T \in \bigcap_{\psi \in \Gamma} \eta(\psi)$, so $T \in \eta(\varphi)$, hence $\Gamma \vdash_{\mathcal{S}} \varphi$. Assume now that $\Gamma \vdash_{\mathcal{S}} \varphi$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \eta(\boldsymbol{Fm}))$ and $T \in \bigcap_{\psi \in \Gamma} h(\psi)$. For every variable p, let $\delta_p \in Fm$ such that $h(p) = \eta(\delta_p)$, and let σ be the substitution defined by $\sigma(p) = \delta_p$ for every propositional variable p. Then a proof by induction shows that, for every formula δ, $h(\delta) = \eta(\sigma(\delta))$, therefore $T \in \bigcap_{\psi \in \Gamma} \eta(\sigma(\psi))$, i.e. $\sigma[\Gamma] \subseteq T$, and as the assumption $\Gamma \vdash_{\mathcal{S}} \varphi$ implies that $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$, we get that $\sigma(\varphi) \in T$, i.e. $T \in \eta(\sigma(\varphi)) = h(\varphi)$. This proves that $\Gamma \vdash_{\mathcal{F}_c} \varphi$. $\square$

Remark 2.3. We could have based the canonical referential algebra of the proof above, instead of on the whole collection $\mathrm{Th}\mathcal{S}$ of $\mathcal{S}$-theories, on one of the *bases* of $\mathrm{Th}\mathcal{S}$, i.e. on a smaller collection of $\mathcal{S}$-theories that generates $\mathrm{Th}\mathcal{S}$ by closure under arbitrary intersections. Thus there are as many canonical referential algebras as there are bases of $\mathrm{Th}\mathcal{S}$.

Since this paper is mainly focused on referential algebras as *local* $\mathcal{S}$-models, we will drop this adjective from now on. Sometimes we will refer to referential algebras that are $\mathcal{S}$-models as *$\mathcal{S}$-referential algebras.*

2.2 Reduced Referential Algebras

For any referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ the following clause

$$\langle u, v\rangle \in R_{\mathcal{F}} \quad \text{iff} \quad (\forall X \in \mathcal{A})(u \in X \Leftrightarrow v \in X)$$

defines the equivalence relation $R_{\mathcal{F}}$ on W that identifies the points of $\mathcal{F}$ which can not be separated by elements of $\mathcal{A}$. Then $\mathcal{F}$ is *reduced* if $R_{\mathcal{F}}$ is the identity relation on W. When this is the case, any two distinct points of W can be separated by some element of $\mathcal{A}$. This is analogous to the separation property T_0 of topological spaces or to the characterizing property of *differentiated* general Kripke frames.

Any referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ can be reduced, i.e. can be associated with a reduced referential algebra that induces the same local and global consequence relations as $\mathcal{F}$, and that is obtained by identifying elements of W by the relation $R_{\mathcal{F}}$: indeed, if π is the projection map from W onto $W/R_{\mathcal{F}}$, we can base a referential algebra $\mathcal{F}/R_{\mathcal{F}}$ on the quotient set $W/R_{\mathcal{F}}$ by endowing the set $\{\pi[X] : X \in \mathcal{A}\}$ with the algebraic structure of type $\mathcal{L}$ inherited from $\boldsymbol{A}$: if $f \in \mathcal{L}$ is n-ary,

$$f^{\mathcal{A}/R_{\mathcal{F}}}(\pi[X_1], \ldots \pi[X_n]) = \pi[f^{\mathcal{A}}(X_1, \ldots, X_n)]$$

for every $X_1, \ldots, X_n \in \mathcal{A}$. This stipulation is sound because the definition of $R_{\mathcal{F}}$ easily implies that for every $X, Y \in \mathcal{A}$, $\pi[X] = \pi[Y]$ iff $X = Y$. Abusing of notation, we denote the algebra just defined by $\mathcal{A}/R_{\mathcal{F}}$. It is easy to see that the referential algebra $\mathcal{F}/R_{\mathcal{F}} = \langle W/R_{\mathcal{F}}, \mathcal{A}/R_{\mathcal{F}}\rangle$ so obtained is reduced. $\mathcal{F}/R_{\mathcal{F}}$ will be called *the reduction* of $\mathcal{F}$ and sometimes will be denoted by $\mathcal{F}^*$.

Referential algebra morphisms

A *morphism* $f : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ between $\mathcal{L}$-referential algebras is a set map $f \in \mathsf{Set}(W_1, W_2)$ such that the assignment $Y \longmapsto f^{-1}[Y]$ defines a homomorphism $f^{-1} \in Hom(\mathcal{A}_2, \mathcal{A}_1)$. A morphism $f : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ is *strict* if $A_1 = \{f^{-1}[Y] : Y \in A_2\}$, i.e if $f^{-1} \in Hom(\mathcal{A}_2, \mathcal{A}_1)$ is surjective.

For every algebraic similarity type $\mathcal{L}$, the composition of composable (strict) referential algebra morphisms is a (strict) referential algebra morphism and the identity map is a strict referential algebra morphism. Hence $\mathcal{L}$-referential algebras and their morphisms form a category $\mathsf{RA}_{\mathcal{L}}$, of which referential algebras and strict morphisms form a subcategory $\mathsf{sRA}_{\mathcal{L}}$. Reduced referential algebras of type $\mathcal{L}$ form a full subcategory $\mathsf{RA}^*_{\mathcal{L}}$ of $\mathsf{RA}_{\mathcal{L}}$.

Remark 2.4. The construction of $\mathcal{F}^*$ can be extended to a functor $(\ )^* : \mathsf{sRA}_{\mathcal{L}} \longrightarrow \mathsf{RA}^*_{\mathcal{L}}$: Indeed if $f \in \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_1, \mathcal{F}_2)$ is strict, then the assignment $f^*([w]) = [f(w)]$ for every $w \in W_1$ defines a morphism $f^* \in \mathsf{RA}^*_{\mathcal{L}}(\mathcal{F}^*_1, \mathcal{F}^*_2)$.

Example 2.5. The notion of referential algebra, abstract as it is, is very powerful and versatile, and can encode information of both algebraic and topological nature: for example let us show how Priestley spaces, Stone spaces and descriptive general frames can be encoded into this setting without loss of information: Let $\mathcal{L}$ be the similarity type of bounded lattices. The category Pri of Priestley spaces and continuous and monotone maps is isomorphic to a full subcategory of $\mathsf{RA}_{\mathcal{L}}$: indeed, for every Priestley space $\mathcal{H} = \langle X, \leq, \tau\rangle$, the collection $\mathsf{CU}(\mathcal{H})$ of the clopen up-sets of $\mathcal{H}$ is a distributive lattice, so we can associate $\mathcal{H}$ with the referential algebra $\mathcal{F}_{\mathcal{H}} = \langle X, \mathsf{CU}(\mathcal{H})\rangle$ which, by total order-disconnectedness, is reduced. Under this assignment we get:

$$\mathsf{Pri}(\mathcal{H}_1, \mathcal{H}_2) \ = \ \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_{\mathcal{H}_1}, \mathcal{F}_{\mathcal{H}_2}).$$

This identity says that the embedding of categories we defined is full. Conversely, for every reduced $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ such that $\mathcal{A}$ is a subalgebra of the powerset lattice $\mathcal{P}(W)$, the triple $\langle W, \leq, \tau\rangle$, such that $\leq$ is the specialization order induced by $\mathcal{A}$ and τ is the topology obtained by taking $\{Y, (W \setminus Y) \mid Y \in \mathcal{A}\}$ as a subbase, is a Priestley space.

Now let $\mathcal{L}$ be the Boolean (BAO) similarity type. Like the previous case, the category Stn (DGF) of Stone spaces and continuous maps (descriptive general frames and p-morphisms) is isomorphic to a full subcategory of $\mathsf{RA}_{\mathcal{L}}$: indeed, for every Stone space $\mathcal{X} = \langle X, \tau\rangle$, (descriptive general frame $\mathcal{X} = \langle X, R, \tau\rangle$) the collection $\mathsf{Cl}(\mathcal{X})$ of the clopen subsets of $\mathcal{X}$ is a Boolean algebra (a BAO), so we can associate $\mathcal{X}$ with the referential algebra $\mathcal{F}_{\mathcal{X}} = \langle X, \mathsf{Cl}(\mathcal{X})\rangle$ which, by total disconnectedness, is reduced. Under this assignment we get:

$$\mathsf{Stn}(\mathcal{X}_1, \mathcal{X}_2) \;=\; \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_{\mathcal{X}_1}, \mathcal{F}_{\mathcal{X}_2}) \quad (\mathsf{DGF}(\mathcal{X}_1, \mathcal{X}_2) \;=\; \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_{\mathcal{X}_1}, \mathcal{F}_{\mathcal{X}_2})).$$

Conversely, for every reduced $\mathcal{L}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ such that $\mathcal{A}$ is a subalgebra of the Boolean algebra (BAO) $\mathcal{P}(W)$, the structure $\langle W, \tau\rangle$, such that τ is the topology obtained by taking $\{Y \mid Y \in \mathcal{A}\}$ as a base, is a Stone space (that becomes a descriptive general frame when augmented with the relation R defined by setting $R[w] = \bigcap\{U \in \mathcal{A} \mid w \in \Box U\}$ for every $w \in W$).

Some facts on morphisms of referential algebras are collected in the propositions below. The proof of the first one is straightforward.

Proposition 2.6. *If $f \in \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_1, \mathcal{F}_2)$ is strict and $\mathcal{F}_1$ is reduced, then f is injective.*

Proposition 2.7. *For every $f \in \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_1, \mathcal{F}_2)$,*

1. *if f is strict, then $\vdash_{\mathcal{F}_2} \subseteq \vdash_{\mathcal{F}_1}$;*
2. *if f is surjective, then $\vdash_{\mathcal{F}_1} \subseteq \vdash_{\mathcal{F}_2}$ and $\vdash^g_{\mathcal{F}_1} \subseteq \vdash^g_{\mathcal{F}_2}$;*
3. *if f is strict and surjective, then $\vdash_{\mathcal{F}_1} = \vdash_{\mathcal{F}_2}$ and $\vdash^g_{\mathcal{F}_1} = \vdash^g_{\mathcal{F}_2}$.*

Proof. (1) Assume $\varGamma \vdash_{\mathcal{F}_2} \varphi$, let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_1)$ and show that $\bigcap_{\psi\in\varGamma} h(\psi) \subseteq h(\varphi)$. As f is strict, then $f^{-1}(\mathcal{A}_2, \mathcal{A}_1)$ is surjective, so there exists $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_2)$ such that $h = f^{-1} \circ h'$. By assumption, we get that $\bigcap_{\psi\in\varGamma} h'(\psi) \subseteq h'(\varphi)$. Therefore, $\bigcap_{\psi\in\varGamma} f^{-1}[h'(\psi)] \subseteq f^{-1}[h'(\varphi)]$, and we obtain the desired result. (2) For every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_2)$, let $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_1)$ be defined by $h' = f^{-1} \circ h$. If $\varGamma \vdash_{\mathcal{F}_1} \varphi$, then $\bigcap_{\psi\in\varGamma} h'(\psi) \subseteq h'(\varphi)$, hence $\bigcap_{\psi\in\varGamma} f^{-1}[h(\psi)] \subseteq f^{-1}[h(\varphi)]$, and since f is onto A_2, this implies that $\bigcap_{\psi\in\varGamma} h(\psi) \subseteq h(\varphi)$, so $\varGamma \vdash_{\mathcal{F}_2} \varphi$. As for the second part, if $\varGamma \vdash^g_{\mathcal{F}_1} \varphi$ and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_2)$ such that $h(\psi) = W_2$ for every $\psi \in \varGamma$, then $h'(\psi) = f^{-1}[h(\psi)] = f^{-1}[W_2] = W_1$, and so by assumption we get that $W_1 = h'(\varphi) = f^{-1}[h(\varphi)]$, which implies, by surjectivity, that $W_2 = f[W_1] = h(\varphi)$.

The first part of (3) follows from (1) and (2). As for the second part, we are only left to show that $\vdash^g_{\mathcal{F}_2} \subseteq \vdash^g_{\mathcal{F}_1}$, so assume that $\Gamma \vdash^g_{\mathcal{F}_2} \varphi$ and and $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_1)$ such that $h(\psi) = W_1$ for every $\psi \in \Gamma$. As f is strict, then $f^{-1}(\mathcal{A}_2, \mathcal{A}_1)$ is surjective, so there exists $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}_2)$ such that $h = f^{-1} \circ h'$. Hence, $W_1 = f^{-1}[h'(\psi)]$, and as f is surjective, then $W_2 = f[W_1] = h'(\psi)$ for every $\psi \in \Gamma$. By assumption this implies that $W_2 = h'(\varphi)$, i.e. $W_1 = f^{-1}[W_2] = f^{-1}[h'(\varphi)] = h(\varphi)$. □

Corollary 2.8. *For every logic $\mathcal{S}$ and every $f \in \mathsf{RA}_{\mathcal{L}}(\mathcal{F}_1, \mathcal{F}_2)$,*

1. *if f is strict and $\mathcal{F}_2$ is an $\mathcal{S}$-model then $\mathcal{F}_1$ is an $\mathcal{S}$-model;*
2. *if f is surjective and $\mathcal{F}_1$ is a (global) $\mathcal{S}$-model then $\mathcal{F}_2$ is a (global) $\mathcal{S}$-model;*
3. *if f is strict and surjective, then $\mathcal{F}_1$ is a (global) $\mathcal{S}$-model iff $\mathcal{F}_2$ is.*

Proposition 2.9. *For every referential algebra $\mathcal{F}$, the local and global consequence relations induced by $\mathcal{F}$ and by $\mathcal{F}/R_{\mathcal{F}}$ coincide, i.e. $\vdash_{\mathcal{F}} = \vdash_{\mathcal{F}/R_{\mathcal{F}}}$ and $\vdash^g_{\mathcal{F}} = \vdash^g_{\mathcal{F}/R_{\mathcal{F}}}$*

Proof. The canonical projection π from $\mathcal{F}$ onto its reduction $\mathcal{F}/R_{\mathcal{F}}$ is a strict and surjective morphism. □

3 Atlases

We are about to introduce the other type of model-structures for propositional logics that will be involved in the duality: For any algebraic similarity type $\mathcal{L}$, *$\mathcal{L}$-atlases* are pairs $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ such that $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $\mathcal{B}$ is a family of subsets of the carrier A of $\boldsymbol{A}$.

An $\mathcal{L}$-atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ induces the following substitution-invariant consequence relation $\vdash_{\mathbb{A}}$ on the $\mathcal{L}$-algebra of formulas $\boldsymbol{Fm}$: for every set of formulas Γ and every formula φ,

$$\Gamma \vdash_{\mathbb{A}} \varphi \ \text{ iff } \ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A}), \forall X \in \mathcal{B} \text{ if } h[\Gamma] \subseteq X, \text{ then } h(\varphi) \in X.$$

Similarly, we can associate the following substitution-invariant consequence relation $\vdash_{\mathsf{K}}$ on $\boldsymbol{Fm}$ with any class K of $\mathcal{L}$-atlases:

$$\vdash_{\mathsf{K}} = \bigcap \{ \vdash_{\mathbb{A}} : \mathbb{A} \in \mathsf{K} \}.$$

For every logic $\mathcal{S}$ of type $\mathcal{L}$, an $\mathcal{L}$-atlas $\mathbb{A}$ is an *$\mathcal{S}$-model* if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathbb{A}}$, i.e. if $\Gamma \vdash_{\mathcal{S}} \varphi$, $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, $X \in \mathcal{B}$ and $h(\Gamma) \subseteq X$ imply that $h(\varphi) \in X$. This is equivalent to saying that every element of $\mathcal{B}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$. So $\mathbb{A}$ is an $\mathcal{S}$-model iff $\mathcal{B} \subseteq \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$. $\mathcal{S}$ is *complete* w.r.t. a class K of $\mathcal{L}$-atlases if $\vdash_{\mathcal{S}} = \vdash_{\mathsf{K}}$.

Remark 3.1. Every logic $\mathcal{S}$ is complete w.r.t. the class of atlases $\mathsf{K} = \{\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S}\rangle\}$. This atlas is usually called the *Lindenbaum generalized matrix* (or Lindenbaum atlas) of $\mathcal{S}$. Any class of atlas-models of $\mathcal{S}$ that includes $\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S}\rangle$ is a complete atlas semantics for $\mathcal{S}$.

We already saw that every referential algebra can be reduced, i.e. can be transformed into a reduced referential algebra that induces the same local and global consequence relations on $\boldsymbol{Fm}$. An analogous construction can be performed on the atlas side: For any atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B}\rangle$, let the *Frege relation* $\varLambda\mathbb{A}$ (or $\varLambda_{\boldsymbol{A}}(\mathcal{B})$) be the following equivalence relation on $\boldsymbol{A}$:

$$\langle a, b\rangle \in \varLambda\mathbb{A} \quad \text{iff} \quad (\forall X \in \mathcal{B})(a \in X \Leftrightarrow b \in X).$$

If $\varLambda\mathbb{A}$ is a congruence, then we say that $\mathbb{A}$ is *congruential*, or that it satisfies the *congruence property*, and if $\varLambda\mathbb{A}$ is the identity relation then $\mathbb{A}$ is *Frege-reduced* (or simply *reduced*, when this is not ambiguous). So an atlas $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is Frege-reduced iff any two distinct elements of the algebra can be separated by an element of $\mathcal{B}$, which is a *separation property* analogous to the one satisfied by reduced referential algebras.

For every congruential atlas $\mathbb{A}$, let $\pi \in Hom(\boldsymbol{A}, \boldsymbol{A}/\varLambda\mathbb{A})$ be the canonical projection. Then the *reduction* of $\mathbb{A}$ is the atlas

$$\mathbb{A}^* = \langle \boldsymbol{A}/\varLambda\mathbb{A}, \mathcal{B}/\varLambda\mathbb{A}\rangle,$$

such that $\mathcal{B}/\varLambda\mathbb{A} = \{\pi[X] : X \in \mathcal{B}\}$, and it is easy to see that it is Frege-reduced.

Remark 3.2. If $\mathcal{S}$ is selfextensional, then the atlas $\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S}\rangle$ is congruential. This implies that every selfextensional logic $\mathcal{S}$ is complete with respect to the class of the reductions of its congruential atlas-models. The reduction of $\langle \boldsymbol{Fm}, \mathrm{Th}\mathcal{S}\rangle$ can be called the *Lindenbaum-Tarski atlas* of $\mathcal{S}$.

Atlas morphisms

A *morphism* $h : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ between atlases is a homomorphism $h \in Hom(\boldsymbol{A}_1, \boldsymbol{A}_2)$ such that

$$\{h^{-1}[Y] \mid Y \in \mathcal{B}_2\} \subseteq \mathcal{B}_1.$$

An atlas morphism $h : \mathbb{A}_1 \longrightarrow \mathbb{A}_2$ is *strict* if $\mathcal{B}_1 = \{h^{-1}[Y] : Y \in \mathcal{B}_2\}$.

For every algebraic similarity type $\mathcal{L}$, the composition of composable (strict) atlas morphisms is a (strict) atlas morphism and the identity map is a strict atlas morphism, hence $\mathcal{L}$-atlases and their morphisms form a category $\mathsf{Atl}_{\mathcal{L}}$, of which atlases and strict morphisms form a subcategory $\mathsf{sAtl}_{\mathcal{L}}$. We will be mainly interested in the category $\mathsf{CA}_{\mathcal{L}}$ of congruential $\mathcal{L}$-atlases, and in its full subcategory $\mathsf{CA}^*_{\mathcal{L}}$ of reduced congruential $\mathcal{L}$-atlases.

Remark 3.3. Let $\mathsf{sCA}_{\mathcal{L}}$ be the category of congruential atlases and strict morphisms. The construction of $\mathbb{A}^*$ can be extended to a functor $(\ )^* : \mathsf{sCA}_{\mathcal{L}} \longrightarrow \mathsf{CA}^*_{\mathcal{L}}$: Indeed if $f \in \mathsf{CA}_{\mathcal{L}}(\mathbb{A}_1, \mathbb{A}_2)$ is strict, then the assignment $f^*([a]) = [f(a)]$ for every $a \in \boldsymbol{A}_1$ defines a morphism $f^* \in \mathsf{CA}^*_{\mathcal{L}}(\mathbb{A}^*_1, \mathbb{A}^*_2)$.

Example 3.4. We saw that $\mathsf{Pri}, \mathsf{Stn}$ and DGF can be embedded into full subcategories of referential algebras: we can perform the analogous move on the atlas side, by associating any object $\boldsymbol{A}$ in any of the categories BDL, Bool and BAO, of bounded distributive lattices, Boolean algebras and Boolean algebras with operators respectively, with the atlas $\mathbb{A}_{\boldsymbol{A}} = \langle \boldsymbol{A}, Pr(\boldsymbol{A})\rangle$ ($Pr(\boldsymbol{A})$ being the collection of the prime filters of the lattice-reduct of $\boldsymbol{A}$), which by Birkhoff-Stone theorem is Frege-reduced. Under this assignment we get:

$$Hom(\boldsymbol{A}_1, \boldsymbol{A}_2) \;=\; \mathsf{CA}^*_{\mathcal{L}}(\mathbb{A}_{\boldsymbol{A}_1}, \mathbb{A}_{\boldsymbol{A}_2}).$$

Below we state some facts on atlas morphisms, that are similar to the ones stated for referential algebra morphisms.

Proposition 3.5. *If $h \in \mathsf{Atl}_{\mathcal{L}}(\mathbb{A}_1, \mathbb{A}_2)$ and $\mathbb{A}_1$ is Frege-reduced, then h is injective.*

Proposition 3.6. *For every $h \in \mathsf{Atl}_{\mathcal{L}}(\mathbb{A}_1, \mathbb{A}_2)$,*

1. *if h is strict, then $\vdash_{\mathbb{A}_2} \subseteq \vdash_{\mathbb{A}_1}$;*

2. *if h is surjective, then* $\vdash_{\mathbb{A}_1} \subseteq \vdash_{\mathbb{A}_2}$;

3. *if h is strict and surjective, then* $\vdash_{\mathbb{A}_1} = \vdash_{\mathbb{A}_2}$.

Proof. (1) Assume $\Gamma \vdash_{\mathbb{A}_2} \varphi$ and let $X \in \mathcal{B}_1$ and $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}_1)$ such that $g[\Gamma] \subseteq X$. As h is strict, then $X = h^{-1}[Y]$ for some $Y \in \mathcal{B}_2$, and so $h[g[\Gamma]] \subseteq Y$. By assumption, this implies that $h(g(\varphi)) \in Y$, i.e. $g(\varphi) \in h^{-1}[Y] = X$. This proves that $\Gamma \vdash_{\mathbb{A}_1} \varphi$. (2) Assume $\Gamma \vdash_{\mathbb{A}_1} \varphi$ and let $Y \in \mathcal{B}_2$, $g \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}_2)$ such that $g[\Gamma] \subseteq Y$. As h is surjective, there exists $g' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}_1)$ such that $g = h \circ g'$. Hence, $g'[\Gamma] \subseteq h^{-1}[Y] \in \mathcal{B}_1$, so by assumption, $g'(\varphi) \in h^{-1}[Y]$, so $g(\varphi) = h(g(\varphi)) \in Y$. This proves that $\Gamma \vdash_{\mathbb{A}_2} \varphi$. (3) follows from (1) and (2). □

Corollary 3.7. *For every logic $\mathcal{S}$ and every $h \in \mathsf{Atl}_{\mathcal{L}}(\mathbb{A}_1, \mathbb{A}_2)$,*

1. *if h is strict and $\mathbb{A}_2$ is an $\mathcal{S}$-model, then $\mathbb{A}_1$ is an $\mathcal{S}$-model;*

2. *if h is surjective and $\mathbb{A}_1$ is an $\mathcal{S}$-model, then $\mathbb{A}_2$ is an $\mathcal{S}$-model;*

3. *if h is strict and surjective, then $\mathbb{A}_1$ is an $\mathcal{S}$-model iff $\mathbb{A}_2$ is an $\mathcal{S}$-model.*

Proposition 3.8. *For every congruential atlas $\mathbb{A}$, $\vdash_{\mathbb{A}} = \vdash_{\mathbb{A}^*}$.*

Proof. The projection of $\mathbb{A}$ onto its reduction is strict and surjective. □

On the preservation of the congruence property by atlas morphisms, we have :

Proposition 3.9. *For every $h \in \mathsf{Atl}_{\mathcal{L}}(\mathbb{A}_1, \mathbb{A}_2)$,*

1. *$\Lambda\mathbb{A}_1 \subseteq h^{-1}[\Lambda\mathbb{A}_2]$ or equivalently, $h[\Lambda\mathbb{A}_1] \subseteq \Lambda\mathbb{A}_2$;*

2. *if h is strict, then $h^{-1}[\Lambda\mathbb{A}_2] = \Lambda\mathbb{A}_1$;*

3. *if h is strict and surjective, then $\mathbb{A}_2$ is congruential iff $\mathbb{A}_1$ is congruential.*

Proof. (1) If $\langle a, b\rangle \in \Lambda\mathbb{A}_1$, then for any $Y \in \mathcal{B}_2$, $h(a) \in Y$ iff $a \in h^{-1}[Y]$ iff $b \in h^{-1}[Y]$ iff $h(b) \in Y$, i.e. $\langle h(a), h(b)\rangle \in \Lambda\mathbb{A}_2$. (2) Let $\langle a, b\rangle \in h^{-1}[\Lambda\mathbb{A}_2]$. As h is strict, it is enough to show that for every $Y \in \mathcal{B}_2$, $a \in h^{-1}[Y]$ iff $b \in h^{-1}[Y]$. By assumption we get that $\langle h(a), h(b)\rangle \in \Lambda\mathbb{A}_2$, hence for every

$Y \in \mathcal{B}_2$, $a \in h^{-1}[Y]$ iff $h(a) \in Y$ iff $h(b) \in Y$ iff $b \in h^{-1}[Y]$. (3) If $\Lambda\mathbb{A}_2$ is a congruence, then $h^{-1}[\Lambda\mathbb{A}_2]$ is a congruence, and as h is strict, then by (2), $\Lambda\mathbb{A}_1$ is a congruence. Assume now that $\Lambda\mathbb{A}_1$ is a congruence. As h is strict, then by (2), $h^{-1}[\Lambda\mathbb{A}_2]$ is a congruence. Using the assumption that h is surjective, one verifies by direct computation that $\Lambda\mathbb{A}_2$ is a congruence. $\square$

4 Duality

Let us fix a similarity type $\mathcal{L}$ throughout the section (and drop the corresponding sub-indices). The aim of this section is to show that CA^* (reduced congruential atlases) is dually equivalent to RA^* (reduced referential algebras), so in the next subsections we define the contravariant functors $(\ )_+ : \mathsf{CA} \longrightarrow \mathsf{RA}$ and $(\ )^+ : \mathsf{RA} \longrightarrow \mathsf{CA}$, and show that the restrictions of these functors to the respective subcategories of reduced objects establish the duality. The back-and-forth correspondence for objects was already noticed in [6].

4.1 The functor $(\ )_+ : \mathsf{CA} \longrightarrow \mathsf{RA}$

Let $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ be a congruential atlas. Then the referential algebra $\mathbb{A}_+$ will be based on $\mathcal{B}$. Now notice that the clause

$$\eta(a) = \{X \in \mathcal{B} : a \in X\}$$

defines a map $\eta \in \mathsf{Set}(A, \mathcal{P}(\mathcal{B}))$. Then $\mathbb{A}_+ = \langle \mathcal{B}, \eta(\boldsymbol{A}) \rangle$ is obtained by defining the algebra $\eta(\boldsymbol{A})$ of subsets of $\mathcal{B}$ as follows:

1. the universe of the algebra $\eta(\boldsymbol{A})$ is the set $\eta[A] = \{\eta(a) : a \in A\}$
2. for every n-ary symbol $f \in \mathcal{L}$ define the following n-ary operation on $\eta[A]$:
$$f^{\eta(\boldsymbol{A})}(\eta(a_1), \ldots, \eta(a_n)) = \eta(f^{\boldsymbol{A}}(a_1, \ldots, a_n))$$
for every $a_1, \ldots, a_n \in A$.

The assumption that $\mathbb{A}$ is congruential guarantees that the operation $f^{\eta(\boldsymbol{A})}$ is well-defined for every $f \in \mathcal{L}$. Moreover, the definition of $\eta(\boldsymbol{A})$ guarantees that the map η defines a surjective homomorphism $\eta \in Hom(\boldsymbol{A}, \eta(\boldsymbol{A}))$.

Proposition 4.1. *For every congruential atlas $\mathbb{A}$,*

1. $\mathbb{A}_+$ *is a reduced referential algebra;*

2. $\vdash_{\mathbb{A}} = \vdash_{\mathbb{A}_+}$*;*

3. *For every logic* $\mathcal{S}$*,* $\mathbb{A}$ *is an* $\mathcal{S}$*-model iff* $\mathbb{A}_+$ *is an* $\mathcal{S}$*-model;*

4. *if* $\mathbb{A}$ *is reduced, then* η *is an isomorphism between* $\boldsymbol{A}$ *and* $\eta(\boldsymbol{A})$*.*

Proof. The proof of (1) is immediate. To prove (2), assume first that $\Gamma \vdash_{\mathbb{A}} \varphi$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \eta(\boldsymbol{A}))$ and $X \in \mathcal{B}$ such that $X \in \bigcap h[\Gamma]$. Since $\eta \in Hom(\boldsymbol{A}, \eta(\boldsymbol{A}))$ is surjective, there exists $h' \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h = \eta \circ h'$, so $X \in \eta(h'(\psi))$, for every $\psi \in \Gamma$. Hence $h'[\Gamma] \subseteq X$, which implies $h'(\varphi) \in X$ and so $X \in \eta(h'(\varphi)) = h(\varphi)$. This proves that $\Gamma \vdash_{\mathbb{A}_+} \varphi$. Assume that $\Gamma \vdash_{\mathbb{A}_+} \varphi$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ and $X \in \mathcal{B}$ be such that $h[\Gamma] \subseteq X$. Then, $X \in \bigcap \eta[h[\Gamma]]$, and as $\eta \circ h \in \mathrm{Hom}(\boldsymbol{Fm}, \eta(\boldsymbol{A}))$, then $X \in \eta(h(\varphi))$; thus $h(\varphi) \in X$. This shows that $\Gamma \vdash_{\mathbb{A}} \varphi$. (3) follows from (2). As for (4), if $\mathbb{A}$ is reduced and a, b are distinct elements of $\boldsymbol{A}$, then they are separated by some element of $\mathcal{B}$. Hence η is injective. $\square$

Let us define $(\ )_+$ on morphisms: by definition, if $h \in \mathsf{Atl}(\mathbb{A}_1, \mathbb{A}_2)$, then $h^{-1}[Y] \in \mathcal{B}_1$ for every $Y \in \mathcal{B}_2$. So the assignment $Y \longmapsto h^{-1}[Y]$ defines a set map $h_+ \in \mathsf{Set}(\mathcal{B}_2, \mathcal{B}_1)$.

Remark 4.2. Clearly, if $h \in \mathsf{Atl}(\mathbb{A}_1, \mathbb{A}_2)$ and $j \in \mathsf{Atl}(\mathbb{A}_2, \mathbb{A}_3)$, then $(j \circ h)_+ = h_+ \circ j_+ \in \mathsf{Set}(\mathcal{B}_3, \mathcal{B}_1)$.

Proposition 4.3. *For every* $h \in \mathsf{CA}(\mathbb{A}_1, \mathbb{A}_2)$*,*

1. $h_+ \in \mathsf{RA}^*((\mathbb{A}_2)_+, (\mathbb{A}_1)_+)$*;*

2. *if* h *is surjective, then* h_+ *is strict.*

3. *if* h *is strict, then* h_+ *is surjective.*

Proof. We claim that for every $a \in A_1$,

$$(h_+)^{-1}[\eta(a)] = \eta(h(a)).$$

Indeed, for every $X \in \mathcal{B}_1$, $X \in (h_+)^{-1}[\eta(a)]$ iff $h_+(X) \in \eta(a)$ iff $a \in h^{-1}[X]$ iff $h(a) \in X$ iff $X \in \eta(h(a))$.

(1) The claim implies that $(h_+)^{-1}[\eta[A_1]] \subseteq \eta[A_2]$, and that for every n-ary function symbol f in $\mathcal{L}$ and $a_1, \ldots, a_n \in A_1$,

$$\begin{aligned}(h_+)^{-1}[f^{\eta(\boldsymbol{A}_1)}(\eta(a_1),\ldots,\eta(a_n))] &= (h_+)^{-1}[\eta(f^{\boldsymbol{A}_1}(a_1,\ldots,a_n))]\\ &= \eta(h(f^{\boldsymbol{A}_1}(a_1,\ldots,a_n))\\ &= \eta(f^{\boldsymbol{A}_2}(h(a_1),\ldots,h(a_n))\\ &= f^{\eta(\boldsymbol{A}_2)}(\eta(h(a_1)),\ldots,\eta(h(a_1)))\\ &= f^{\eta(\boldsymbol{A}_2)}(h_+^{-1}[\eta(a_1)],\ldots,h_+^{-1}[\eta(a_1)]).\end{aligned}$$

Hence $(h_+)^{-1} \in Hom(\eta(\boldsymbol{A}_1), \eta(\boldsymbol{A}_2))$.

(2) Assume that h is surjective, then for every $b \in A_2$, $b = h(a)$ for some $a \in A_1$, hence the claim implies that $\eta(b) = \eta(h(a)) = (h_+)^{-1}[\eta(a)]$. As $\eta(b)$ for $b \in A_2$ is the form of an arbitrary element of $\eta(\boldsymbol{A}_2)$, this shows that h_+ is strict. (3) If h is strict and $X \in \mathcal{B}_1$, then $X = h^{-1}[Y] = h_+(Y)$ for some $Y \in \mathcal{B}_2$. Thus h_+ is surjective. □

Corollary 4.4. *For every $h \in \mathsf{CA}(\mathbb{A}_1, \mathbb{A}_2)$, if h is surjective, then h_+ is injective.*

Proof. By the proposition above, h_+ is strict and $(\mathbb{A}_2)_+$ is a reduced referential algebra. Therefore, by Proposition 2.6, h_+ is injective. □

Recall that, for every congruential atlas $\mathbb{A}$, $\mathbb{A}^* \in \mathsf{CA}^*$ is its Frege-reduction.

Corollary 4.5. *For every congruential atlas $\mathbb{A}$, $\mathbb{A}_+$ and $(\mathbb{A}^*)_+$ are isomorphic.*

Proof. The projection atlas morphism π from $\mathbb{A}$ into $\mathbb{A}^*$ is strict and surjective. Therefore, by the proposition and corollary above, π_+ is surjective and one-to-one. Thus it is an isomorphism. □

So the functor $(\ )_+ : \mathsf{CA} \longrightarrow \mathsf{RA}$ and its restriction to CA^* have the same range (up to isomorphism).

4.2 The functor $(\ )^+ : \mathsf{RA} \longrightarrow \mathsf{CA}$

Let $\mathcal{F} = \langle W, \mathcal{A}\rangle$ be a referential algebra. Then the $\mathcal{F}^+$ is the atlas $\langle \mathcal{A}, W^+\rangle$ such that

$$W^+ = \{\varepsilon(v) : v \in W\} \quad \text{and for every } v \in W,\ \varepsilon(v) = \{Y \in \mathcal{A} : v \in Y\}.$$

Proposition 4.6. *For every referential algebra $\mathcal{F}$,*

1. *$\mathcal{F}^+$ is a Frege-reduced atlas;*

2. $\vdash_{\mathcal{F}} = \vdash_{\mathcal{F}^+}$;

3. *For every logic $\mathcal{S}$, $\mathcal{F}$ is an $\mathcal{S}$-model iff $\mathcal{F}^+$ is an $\mathcal{S}$-model;*

4. *if $\mathcal{F}$ is reduced, then the map $\varepsilon \in \mathsf{Set}(W, W^+)$ is bijective.*

Proof. The proof of (1) is immediate. (2) $\Gamma \vdash_{\mathcal{F}^+} \varphi$ holds iff for every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ and every $w \in W$, if $h[\Gamma] \subseteq \varepsilon(w)$, then $w \in h(\varphi)$. This is equivalent to say that for every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$ and every $w \in W$, if $w \in \bigcap h[\Gamma]$, then $w \in h(\varphi)$, i.e. that for every $h \in \mathrm{Hom}(\boldsymbol{Fm}, \mathcal{A})$, $\bigcap h[\Gamma] \subseteq h(\varphi)$, i.e. that $\Gamma \vdash_{\mathcal{F}} \varphi$. (3) follows from (2). (4) The map ε is clearly surjective by definition. $\mathcal{F}$ is reduced means that any two distinct point v, w are separated by some element of $\mathcal{A}$, hence injectivity follows. □

Remark 4.7. Item (2) of the proposition above implies that any referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$ is a model of a logic $\mathcal{S}$ iff $\varepsilon(w)$ is an $\mathcal{S}$-filter of $\mathcal{A}$ for every $w \in W$.

Let us define $(\ )^+$ on morphisms: by definition, if $f \in \mathsf{RA}(\mathcal{F}_1, \mathcal{F}_2)$, then the assignment $Y \longmapsto f^{-1}[Y]$ defines $f^+ \in Hom(\mathcal{A}_2, \mathcal{A}_1)$.

Remark 4.8. If $f \in \mathsf{RA}(\mathcal{F}_1, \mathcal{F}_2)$ and $g \in \mathsf{RA}(\mathcal{F}_2, \mathcal{F}_3)$, then $(g \circ f)^+ = f^+ \circ g^+ \in Hom(\mathcal{A}_3, \mathcal{A}_1)$.

Proposition 4.9. *For every $f \in \mathsf{RA}(\mathcal{F}_1, \mathcal{F}_2)$,*

1. *$f^+ \in \mathsf{CA}^*((\mathcal{F}_2)^+, (\mathcal{F}_1)^+)$;*

2. *if f is surjective, then f^+ is strict;*

3. *if f is strict, then f^+ is surjective.*

Proof. We claim that, for every $w \in W_1$,

$$(f^+)^{-1}[\varepsilon(w)] = \varepsilon(f(w)) :$$

for every $Z \in \mathcal{A}_1$, $Z \in (f^+)^{-1}[\varepsilon(w)]$ iff $w \in f^+(Z)$ iff $w \in f^{-1}[Z]$ iff $Z \in \varepsilon(f(w))$.

(1) We saw that $f^+ \in Hom(\mathcal{A}_2, \mathcal{A}_1)$. Claim implies that $(f^+)^{-1}[W_1^+] \subseteq W_2^+$, which concludes the proof.

(2) If f is surjective then every element in W_2 is of form $f(w)$ for some $w \in W_1$, and so by claim, every element in W_2^+ is of form $\varepsilon(f(w)) = (f^+)^{-1}[\varepsilon(w)]$ for some $w \in W_1$, which proves that f^+ is strict.

(3) If f is strict, then $\mathcal{A}_1 = \{f^{-1}[X] : X \in \mathcal{A}_2\} = f^+[\mathcal{A}_2]$, i.e. f^+ is surjective. □

Corollary 4.10. *For every $f \in \mathsf{RA}(\mathcal{F}_1, \mathcal{F}_2)$, if f is surjective, then f^+ is injective.*

Proof. $f^+ \in \mathsf{CA}((\mathcal{F}_2)^+, (\mathcal{F}_1)^+)$ is strict and $(\mathcal{F}_2)^+$ is Frege-reduced, hence by Proposition 3.5, f^+ is injective. □

Recall that for every referential algebra $\mathcal{F}$, $\mathcal{F}^* \in \mathsf{RA}^*$ is its reduction.

Corollary 4.11. *For every referential algebra $\mathcal{F}$, $\mathcal{F}^+$ and $(\mathcal{F}^*)^+$ are isomorphic.*

Proof. The canonical projection $\pi \in \mathsf{RA}(\mathcal{F}, \mathcal{F}^*)$ is strict and surjective. Then, by the proposition and corollary above, $\pi^+ \in \mathsf{CA}((\mathcal{F}^*)^+, \mathcal{F}^+)$ is surjective and injective, i.e. it is an iso. □

So, similarly to $(\ )_+$, the functor $(\ )^+ : \mathsf{RA} \longrightarrow \mathsf{CA}$ and its restriction to RA^* have the same range (up to isomorphism).

4.3 The duality theorem

The results in Subsections 4.1 and 4.2 establish, among other things, a back-and-forth correspondence given by the contravariant functors

$$(\ )_+ : \mathsf{CA}^* \longrightarrow \mathsf{RA}^* \quad \text{and} \quad (\ )^+ : \mathsf{RA}^* \longrightarrow \mathsf{CA}^*.$$

The aim of this section is to show that

Theorem 4.12. CA^* *and* RA^* *are dually equivalent categories through* $(\ )_+$ *and* $(\ )^+$.

For every reduced referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$, the map $\varepsilon_{\mathcal{F}} \in \mathsf{Set}(W, W^+)$ is defined by putting, for every $w \in W$,

$$\varepsilon_{\mathcal{F}}(w) = \{Y \in A : w \in Y\}.$$

For every reduced congruential atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$, $\eta_{\mathbb{A}} \in Hom(\boldsymbol{A}, \eta(\boldsymbol{A}))$ is defined by putting, for every $a \in A$,

$$\eta_{\mathbb{A}}(a) = \{X \in \mathcal{B} : a \in X\}.$$

Recall that $(\mathcal{F}^+)_+$ is the reduced referential algebra $\langle W^+, \eta_{\mathcal{F}_+}(\mathcal{A})\rangle$ such that $W^+ = \{\varepsilon_{\mathcal{F}}(w) : w \in W\}$ and the elements of $\eta_{\mathcal{F}_+}(\mathcal{A})$ are sets of the form

$$\eta_{\mathcal{F}^+}(Y) = \varepsilon_{\mathcal{F}}[Y] = \{\varepsilon_{\mathcal{F}}(w) : w \in Y\},$$

for $Y \in \mathcal{A}$.

Proposition 4.13. *For every $\mathcal{F} \in \mathsf{RA}^*$, $\varepsilon_{\mathcal{F}} \in \mathsf{RA}^*(\mathcal{F}, (\mathcal{F}^+)_+)$ is strict and bijective. Hence $\mathcal{F}$ and $(\mathcal{F}^+)_+$ are isomorphic.*

Proof. As $\mathcal{F}$ is reduced, then $\varepsilon_{\mathcal{F}}$ is bijective. Let $\varepsilon = \varepsilon_{\mathcal{F}}$ and $\eta = \eta_{\mathcal{F}^+}$. Then, for every $X \in \mathcal{A}$, $\varepsilon^{-1}[\eta(X)] = \varepsilon^{-1}[\varepsilon[X]] = X$.

So if we show that $\varepsilon^{-1} \in Hom(\eta(\mathcal{A}), \mathcal{A})$, then the claim would imply that ε is strict and the proof will be complete. For every n-ary symbol f and every $X_1, \ldots, X_n \in \mathcal{A}$,

$$\begin{aligned}
&\varepsilon^{-1}[f^{\eta(\mathcal{A})}(\eta(X_1), \ldots, \eta(X_n))] \\
&\quad = \varepsilon^{-1}[\eta(f^{\mathcal{A}}(X_1, \ldots, X_n))] && \text{(def. of } f^{\eta(\mathcal{A})}) \\
&\quad = f^{\mathcal{A}}(X_1, \ldots, X_n) \\
&\quad = f^{\mathcal{A}}(\varepsilon^{-1}[\eta(X_1)], \ldots, \varepsilon^{-1}[\eta(X_n)]) && \text{(claim).}
\end{aligned}$$

□

Proposition 4.14. *For every $\mathbb{A} \in \mathsf{CA}^*$, $\eta_{\mathbb{A}} \in \mathsf{CA}^*(\mathbb{A}, (\mathbb{A}_+)^+)$ is strict and bijective. Hence $\mathbb{A}$ and $(\mathbb{A}_+)^+$ are isomorphic.*

Proof. As $\mathbb{A}$ is Frege-reduced, then $\eta_{\mathbb{A}} \in Hom(\boldsymbol{A}, \eta(\boldsymbol{A}))$ is bijective. Let $\eta_{\mathbb{A}} = \eta$ and $\varepsilon_{\mathbb{A}_+} = \varepsilon$. For every $X \in \mathcal{B}$, $\eta^{-1}[\varepsilon(X)] = X$: indeed, $a \in \eta^{-1}[\varepsilon(X)]$ iff $\eta(a) \in \varepsilon(X)$ iff $X \in \eta(a)$ iff $a \in X$. This implies that $\eta \in \mathsf{CA}^*(\mathbb{A}, (\mathbb{A}_+)^+)$ and it is strict. □

The following facts follow straightforwardly from the definitions involved. They show that ε and η are the requiered natural transformations we need to finish the proof of Theorem 4.14.

Proposition 4.15.

1. *If $f \in \mathsf{RA}^*(\mathcal{F}, \mathcal{F}')$, then for every $w \in W$, $(f^+)_+(\varepsilon(w)) = \varepsilon'(f(w))$.*
2. *If $h \in \mathsf{CA}^*(\mathbb{A}, \mathbb{A}')$, then for every $a \in A$, $(h_+)^+(\eta(a)) = \eta'(h(a))$.*

5 A characterization of fully selfextensional logics

The aim of this section is to obtain a characterization of fully selfextensional logics within the selfextensional ones. To this purpose, we will apply the duality in Section 4: for every logic $\mathcal{S}$, we will consider a designated category of atlases that is isomorphic to $\mathsf{Alg}\mathcal{S}$. Whenever $\mathcal{S}$ is fully selfextensional, we will be able to apply the duality to this category, and obtain the desired characterization.

Let us fix an algebraic similarity type $\mathcal{L}$ throughout this section and drop any reference to it. Recall that $\mathsf{Alg}\mathcal{S}$ is the class of algebras canonically associated with any logic $\mathcal{S}$, and can be defined in several ways. In the context of this paper we opt for one that involves the following remark: Every algebra $\boldsymbol{A}$ can be endowed with an atlas structure $\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}} = \langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \rangle$, so that $\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}$ is an $\mathcal{S}$-model and is called the *basic $\mathcal{S}$-atlas* on $\boldsymbol{A}$[4]. In general, for an arbitrary logic $\mathcal{S}$ and any algebra $\boldsymbol{A}$, the relation $\Lambda\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}$ is not necessarily a congruence of A. However the greatest congruence of A that it is included in it always exists. It is called the *Tarski congruence* of $\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}$ (cf. [9]). Actually, the Tarski congruence can be defined for any atlas $\mathbb{A} = \langle \boldsymbol{A}, \mathcal{B} \rangle$ as the greatest congruence of $\boldsymbol{A}$ that is included in $\Lambda\mathbb{A}$, and as such, it is intrinsic to $\mathbb{A}$. If the Tarski congruence of an atlas $\mathbb{A}$ is the identity, then the atlas is *Tarski-reduced*. Notice that for an arbitrary logic $\mathcal{S}$ there might exist Tarski-reduced atlas-models of $\mathcal{S}$ that are not Frege-reduced. The class of algebras $\mathsf{Alg}\mathcal{S}$ is defined by:

$$\mathsf{Alg}\mathcal{S} = \{\boldsymbol{A} : \langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \rangle \text{ is Tarski-reduced}\}.$$

The elements of $\mathsf{Alg}\mathcal{S}$ are usually called $\mathcal{S}$-algebras. Recall that a logic $\mathcal{S}$ is *fully selfextensional* (or an *fs-logic* for short) if for every algebra $\boldsymbol{A}$, $\Lambda\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}$ is a congruence. This is equivalent to saying that for every $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}$ is Frege-reduced. So the distinction between the Tarski- and Frege- notions drops for fs-logics. Recall also that every fs-logic is selfextensional. Given an fs-logic $\mathcal{S}$, consider the following full subcategories of RA and of CA respectively:

$\mathsf{RA}^*_{\mathcal{S}}$	reduced referential algebra $\mathcal{S}$-models
$\mathsf{CA}^*_{\mathcal{S}}$	reduced atlas $\mathcal{S}$-models
$\mathsf{BA}^*_{\mathcal{S}}$	reduced basic atlas $\mathcal{S}$-models

[4]Structures of this kind are called *basic full models* in [10].

Since being an $\mathcal{S}$-filter is preserved under taking inverse images of algebra homomorphisms, $\mathsf{BA}^*_{\mathcal{S}}$ and $\mathsf{Alg}\mathcal{S}$ (seen as a category) are isomorphic under the assignments $\boldsymbol{A} \longmapsto \langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ and $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle \longmapsto \boldsymbol{A}$. Then, by the Duality Theorem 4.12 restricted to $\mathsf{BA}^*_{\mathcal{S}}$, we get:

$$\mathsf{Alg}\mathcal{S} \ \equiv \ \mathsf{BA}^*_{\mathcal{S}} \ \equiv^{op} \ \mathsf{C}_{\mathcal{S}},$$

for some full subcategory $\mathsf{C}_{\mathcal{S}}$ of $\mathsf{RA}^*_{\mathcal{S}}$. These remarks can be turned into a characterization of fs-logics.

Theorem 5.1. *A logic $\mathcal{S}$ is fully selfextensional iff a dual equivalence can be established between $\mathsf{Alg}\mathcal{S}$ and a full subcategory $\mathsf{C}_{\mathcal{S}}$ of $\mathsf{RA}^*_{\mathcal{S}}$ in such a way that the dual functor $\mathsf{C}_{\mathcal{S}} \longrightarrow \mathsf{Alg}\mathcal{S}$ is given by $(\ )^+$ composed with the assignment $\langle \boldsymbol{A}, \mathcal{B}\rangle \longmapsto \boldsymbol{A}$.*

Proof. The 'only if' direction is given by the discussion above. As for the 'if' direction, the assumptions imply that for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, there exists a family $\mathcal{B}$ of subsets of A such that $\langle \boldsymbol{A}, \mathcal{B}\rangle = \mathcal{F}^+$ (up to isomorphism) for some $\mathcal{F} \in \mathsf{C}_{\mathcal{S}} \subseteq \mathsf{RA}^*_{\mathcal{S}}$. As $\mathcal{F}$ is an $\mathcal{S}$-model, so is $\langle \boldsymbol{A}, \mathcal{B}\rangle$, i.e. $\mathcal{B} \subseteq \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$, and as $\langle \boldsymbol{A}, \mathcal{B}\rangle$ is Frege-reduced, then so is $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$. This shows that $\mathcal{S}$ is an fs-logic. $\square$

The two remaining subsections will be devoted to finding a characterization of $\mathsf{C}_{\mathcal{S}}$, for every fs-logic $\mathcal{S}$. The duality implies that the referential algebras in $\mathsf{C}_{\mathcal{S}}$ are isomorphic to reduced referential algebras of the form $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \eta(\boldsymbol{A})\rangle$, where $\eta = \eta_{\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}}$.

Before focusing on $\mathsf{C}_{\mathcal{S}}$, we present a second characterization of fs-logics among the selfextensional ones, which is given in Theorem 5.4 below. The first part of the equivalence is an abstract *representation theorem* of $\mathsf{Alg}\mathcal{S}$ for every fs-logic $\mathcal{S}$ and will be highlighted in a separate theorem. We will see that fs-logics are exactly those selfextensional logics for which this representation theorem holds.

For every logic $\mathcal{S}$, let us define

$$\mathsf{AlgRef}\mathcal{S} = \{\mathcal{A} : \text{for some } \langle W, \mathcal{A}\rangle \in \mathsf{RA}^*_{\mathcal{S}}\},$$

i.e. $\mathsf{AlgRef}\mathcal{S}$ is the class of the algebraic reducts of the reduced $\mathcal{S}$-referential algebras.

Remark 5.2. For every logic $\mathcal{S}$, $\mathsf{AlgRef}\mathcal{S} \subseteq \mathsf{Alg}\mathcal{S}$.

Proof. If $\langle W, \mathcal{A}\rangle$ is a reduced $\mathcal{S}$-referential algebra, then its dual $\langle \mathcal{A}, W^+\rangle$ is a reduced congruential $\mathcal{S}$-atlas. Hence, $W^+ \subseteq \mathrm{Fi}_{\mathcal{S}}\mathcal{A}$ and so $\langle \mathcal{A}, \mathrm{Fi}_{\mathcal{S}}\mathcal{A}\rangle$ is Frege-reduced, hence $\mathcal{A} \in \mathsf{Alg}\mathcal{S}$. $\square$

We are ready to give the abstract representation theorem for fs-logics:

Theorem 5.3. *For every fs-logic $\mathcal{S}$,* $\mathsf{Alg}\mathcal{S} = \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$.

Proof. One inclusion is implied by the remark above. As for the converse inclusion, if $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, by Theorem 5.1 there exists a family $\mathcal{B}$ of subsets of A such that $\langle \boldsymbol{A}, \mathcal{B}\rangle = \mathcal{F}^+$ (up to isomorphism) for some reduced $\mathcal{S}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}'\rangle$. Since $\boldsymbol{A} \cong \mathcal{A}'$, and $\mathcal{A}' \in \mathsf{AlgRef}\mathcal{S}$, then $\boldsymbol{A} \in \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$. $\square$

The abstract representation theorem above can be strengthened to the following characterization of fs-logics:

Theorem 5.4. *The following statements are equivalent for every logic $\mathcal{S}$:*

1. *$\mathcal{S}$ is fully selfextensional;*
2. $\mathsf{Alg}\mathcal{S} \subseteq \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$*;*
3. $\mathsf{Alg}\mathcal{S} = \mathbf{I}(\mathsf{AlgRef}\mathcal{S})$.

Proof. The abstract representation theorem gives that (1) implies (3), so we are only left to show that (2) implies (1). If $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, then by assumption $\boldsymbol{A} \cong \mathcal{A}$ for some $\langle W, \mathcal{A}\rangle \in \mathsf{RA}^*_{\mathcal{S}}$. Then $\langle \mathcal{A}, W^+\rangle$ is a Frege-reduced $\mathcal{S}$-atlas, which implies that $W^+ \subseteq \mathrm{Fi}_{\mathcal{S}}\mathcal{A}$, hence $Id \subseteq \Lambda_{\mathcal{A}}(\mathrm{Fi}_{\mathcal{S}}\mathcal{A}) \subseteq \Lambda_{\mathcal{A}}(W^+) = Id$, and so, using the isomorphism $\boldsymbol{A} \cong \mathcal{A}$, we conclude that $\Lambda_{\boldsymbol{A}}(\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A})$ is the identity relation. This shows that $\mathcal{S}$ is an fs-logic. $\square$

5.1 Perfect $\mathcal{S}$-referential algebras

The aim of this section is to characterize the category $\mathsf{C}_{\mathcal{S}}$ of Theorem 5.1 for any fs-logic $\mathcal{S}$. Our starting point is that for every fs-logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the atlas $\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}} = \langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ is congruential and so it can be associated with the reduced referential algebra $\mathcal{F}_{\boldsymbol{A}} = \mathcal{F}^{\mathcal{S}}_{\boldsymbol{A}} = (\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}})_+ = \langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \eta(\boldsymbol{A})\rangle$. Our next move will be defining *perfect $\mathcal{S}$-referential algebras* as the abstract versions of $\mathcal{F}_{\boldsymbol{A}}$, and showing that the full subcategory $\mathsf{PRA}_{\mathcal{S}} \subseteq \mathsf{RA}^*_{\mathcal{S}}$ they form is dually equivalent to $\mathsf{Alg}\mathcal{S}$. This will provide the desired characterization of $\mathsf{C}_{\mathcal{S}}$.

For every referential algebra $\langle W, \mathcal{A}\rangle$, let $\leq_{\mathcal{A}}$ be the specialization preorder on W induced by $\mathcal{A}$:

$$w \leq_{\mathcal{A}} w' \quad \text{iff} \quad \text{for every } U \in A, \text{ if } w \in U, \text{ then } w' \in U,$$

Then, for every fs-logic $\mathcal{S}$, an $\mathcal{S}$-referential algebra $\mathcal{F} = \langle W, \mathcal{A}\rangle$ is *perfect* if

1. $\langle W, \leq_{\mathcal{A}}\rangle$ is a complete lattice;
2. $\mathcal{A}_{\mathcal{F}} = \mathcal{A}$ is an algebra of subsets of the form $w\uparrow$ for some $w \in W$;
3. the set $\{w' : w'\uparrow \in \mathcal{A}\}$ is join-dense in W, i.e. every $w \in W$ is the join of elements of $\{w' : w'\uparrow \in \mathcal{A}\}$;
4. for every $\mathcal{X} \subseteq \mathcal{A}$ and every $w\uparrow \in \mathcal{A}$, if $\bigcap \mathcal{X} \subseteq w\uparrow$ then $w\uparrow \in C^{\mathcal{A}}_{\mathcal{S}}(\mathcal{X})$, $C^{\mathcal{A}}_{\mathcal{S}}(\mathcal{X})$ being the least $\mathcal{S}$-filter of $\mathcal{A}$ that includes $\mathcal{X}$.

Condition (1) implies that any perfect $\mathcal{S}$-referential algebra is reduced, so perfect $\mathcal{S}$-referential algebras form a full subcategory $\mathsf{PRA}_{\mathcal{S}}$ of $\mathsf{RA}^*_{\mathcal{S}}$. So, as an immediate consequence of this remark, we get:

Lemma 5.5. *For every perfect $\mathcal{S}$-referential algebra $\mathcal{F}$, $\mathcal{A}_{\mathcal{F}} \in \mathsf{Alg}\mathcal{S}$.*

Notice that an equivalent restatement of condition (3) is that for every $w \in W$,

$$w = \bigvee \{w' : w' \leq w \text{ and } w'\uparrow \in \mathcal{A}\}.$$

It is well-known that for any logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the lattice $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$ is a complete lattice. Recall that $\eta = \eta_{\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}}$ is defined by $\eta(a) = \{F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} : a \in F\}$ for every $a \in A$. For every algebra $\boldsymbol{A}$, let $C^{\boldsymbol{A}}_{\mathcal{S}}$ be the closure operator associated with the closure system $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$. So $C^{\boldsymbol{A}}_{\mathcal{S}}$ is the closure operator of $\mathcal{S}$-filter generation, i.e. for $X \subseteq A$, $C^{\boldsymbol{A}}_{\mathcal{S}}(X)$ is the least $\mathcal{S}$-filter that includes X. Notice that, for every $a \in A$, $\eta(a) = \{F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} : C^{\boldsymbol{A}}_{\mathcal{S}}(a) \subseteq F\}$. Thus $\eta(a)$ is the principal filter of the lattice $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq\rangle$ generated by $C^{\boldsymbol{A}}_{\mathcal{S}}(a)$.

Lemma 5.6. *For every logic $\mathcal{S}$,*

1. *if $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ is Frege-reduced, then for every $X \cup \{a\} \subseteq A$,*

$$\bigcap \eta[X] \subseteq \eta(a) \quad \textit{iff} \quad \eta(a) \in C^{\eta(\boldsymbol{A})}_{\mathcal{S}}(\eta[X]).$$

2. *For every* $F, G \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$,

$$F \subseteq G \quad \textit{iff} \quad (\forall a \in A)(F \in \eta(a) \Rightarrow G \in \eta(a)).$$

Proof. (1) Assume that $\bigcap \eta[X] \subseteq \eta(a)$. If F is an $\mathcal{S}$-filter of $\eta(\boldsymbol{A})$ such that $\eta[X] \subseteq F$, then $\eta^{-1}[F]$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$ such that $X \subseteq \eta^{-1}[F]$, hence $\eta^{-1}[F] \in \bigcap \eta[X]$, which implies by assumption that $\eta^{-1}[F] \in \eta(a)$, i.e. $a \in \eta^{-1}[F]$, i.e. $\eta(a) \in F$. This proves the 'only if'. Conversely, assume that $\eta(a) \in C_{\mathcal{S}}^{\eta(\boldsymbol{A})}(\eta[X])$. If $F \in \bigcap \eta[X]$, then $X \subseteq F$, hence $\eta[X] \subseteq \eta[F]$. Since $\langle \boldsymbol{A}, \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}\rangle$ is Frege-reduced, then $\eta \in Hom(\boldsymbol{A}, \eta(\boldsymbol{A}))$ is an isomorphism, so $\eta[F] \in \mathrm{Fi}_{\mathcal{S}}\eta(\boldsymbol{A})$, which implies by assumption that $\eta(a) \in \eta[F]$, so by injectivity of η we get $a \in F$, i.e. $F \in \eta(a)$. (2) follows immediately from the definition of η. □

So the lemma above readily implies the following

Proposition 5.7. *For every fs-logic* $\mathcal{S}$ *and every* $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, $\mathcal{F}_{\boldsymbol{A}} = \langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \eta(\boldsymbol{A})\rangle \in \mathsf{PRA}_{\mathcal{S}}$.

Proposition 5.8. *For every* $\mathcal{F} \in \mathsf{PRA}_{\mathcal{S}}$, $\varepsilon \in \mathsf{PRA}_{\mathcal{S}}(\mathcal{F}, \mathcal{F}_{\mathcal{A}_{\mathcal{F}}})$ *is an isomorphism.*

Proof. Lemma 5.5 implies that $\varepsilon \in \mathsf{RA}^*_{\mathcal{S}}(\mathcal{F}, (\mathcal{F}^+)_+)$ is an isomorphism, i.e. it is strict and bijective, so let us show that $W^+ = \mathrm{Fi}_{\mathcal{S}}\mathcal{A}$. By definition, $\mathcal{F}$ is an $\mathcal{S}$-model and this implies that W^+ is a set of $\mathcal{S}$-filters of $\mathcal{A}$. As for the converse inclusion, let us show that, if $F \in \mathrm{Fi}_{\mathcal{S}}\mathcal{A}$, then $F = \varepsilon(w)$ for $w = \bigvee\{w' \in W : w'\!\uparrow\, \in F\}$. Clearly $F \subseteq \varepsilon(w)$. If $v\!\uparrow\, \in \varepsilon(w)$, then $v \leq \bigvee\{w' \in W : w'\!\uparrow\, \in F\}$, and this implies that $\bigcap F \subseteq v\!\uparrow$. Then condition (4) implies that $v\!\uparrow\, \in C_{\mathcal{S}}^{\mathcal{A}}(F) = F$. □

Theorem 5.9. *For every fs-logic* $\mathcal{S}$, $\mathsf{Alg}\mathcal{S}$ *and* $\mathsf{PRA}_{\mathcal{S}}$ *are dually equivalent.*

Proof. The results above together with propositions 4.14 and 4.15 imply that the assignments $\langle \boldsymbol{A} \mapsto \mathcal{F}_{\boldsymbol{A}}; h \mapsto h_+\rangle$ and $\langle \mathcal{F} \mapsto \mathcal{A}_{\mathcal{F}}; f \mapsto f^+\rangle$ establish a dual equivalence. □

There is a family of well-known substructural and substructural-related logics which includes the relevance logic R and Łukasiewicz's infinite-valued logic, whose Hilbert-style formulations are not selfextensional, and so the theorem above is not directly applicable to their associated classes of algebras: residuated lattices, MV-algebras, and so on. But every logic $\mathcal{S}$ in this family is endowed with a *fully selfextensional logic companion*, that is a fully selfextensional logic $\mathcal{S}'$ for which it holds in particular that $\mathsf{Alg}\mathcal{S}' = \mathsf{Alg}\mathcal{S}$ (see [13]), to which the theorem above applies, and so we obtain a duality between $\mathsf{Alg}\mathcal{S}$ and the class of perfect referential algebras of its fully selfextensional companion. If $\mathcal{S}$ is one of these logics, it has a conjunction $\wedge$. Then the results in [13] imply that for every algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$ the set of points of the corresponding perfect referential algebra is the set of all $\wedge$-semilattice filters of $\boldsymbol{A}$.

5.2 Topologizing $\mathcal{S}$-filters

The discussion in the previous section can be specialized to the case in which $\mathcal{S}$ is a finitary fs-logic. For any partially ordered set (poset for short) $\langle X, \leq \rangle$, a subset Y of X is an *up-set* if for every $a, b \in X$, $a \in Y$ and $a \leq b$ imply that $b \in Y$.

An *ordered topological space* is a triple $\langle X, \leq, \mathcal{O} \rangle$ such that $\langle X, \leq \rangle$ is a poset and $\langle X, \mathcal{O} \rangle$ is a topological space. Let OT be the category of ordered topological spaces and continuous and order-preserving maps. Recall that a *Priestley space* is an ordered topological space $\langle X, \leq, \mathcal{O} \rangle$ such that $\langle X, \mathcal{O} \rangle$ is a Boolean (or Stone) space and the following *total order-disconnectedness* condition holds: if $x, y \in X$ and $x \not\leq y$, then there exists a clopen up-set V such that $x \in V$ and $y \notin V$.

For an arbitrary logic $\mathcal{S}$ and any algebra $\boldsymbol{A} \in \mathsf{Alg}\mathcal{S}$, recall that $\eta = \eta_{\mathbb{A}^{\mathcal{S}}_{\boldsymbol{A}}}$ is defined by $\eta(a) = \{F \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} : a \in F\}$ for every $a \in A$. Then let $\eta(a)^c = \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A} \setminus \eta(a)$, and define $\mathcal{T}$ as the topology on $\mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$ generated by taking the following family as subbasic open subsets:

$$\{\eta(a) : a \in A\} \cup \{\eta(a)^c : a \in A\}.$$

Hence $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq, \mathcal{T} \rangle$ is an ordered topological space.

It is well-known that for any finitary logic $\mathcal{S}$ and every algebra $\boldsymbol{A}$, the lattice $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq \rangle$ is a complete algebraic lattice.

The following theorem is due to Czelakowski.

Theorem 5.10. *If $\mathcal{S}$ is finitary, then for every algebra $\boldsymbol{A}$ the space $\langle \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}, \subseteq, \mathcal{T} \rangle$ is a Priestley space.*

Proof. Clearly, every $\eta(a)$ is a clopen up-set, hence total order-disconnectedness holds, for if $F \not\subseteq G$, there exists $a \in F \setminus G$ and so $F \in \eta(a)$ and $G \notin \eta(a)$. Let us show compactness. By Alexander's subbase theorem it is enough to show that any cover by elements of the subbase has a finite subcover, so suppose for contradiction that

$$\mathcal{C} = \{\eta(a) : a \in I\} \cup \{\eta(b)^c : b \in J\}$$

covers the space but has no finite subcover. This implies in particular that $I \cap J = \emptyset$. Let $F = C_{\mathcal{S}}(J) \in \mathrm{Fi}_{\mathcal{S}}\boldsymbol{A}$; in order to complete the proof it is enough to show that $F \cap I = \emptyset$, for this implies that $F \notin \bigcup \mathcal{C}$, contradicting the assumption that $\mathcal{C}$ is a covering. Suppose that $a \in F \cap I$ and recall that the finitarity of $\mathcal{S}$ implies that the consequence operation of $\mathcal{S}$-filter generation on A is finitary. So $a \in C_{\mathcal{S}}(J)$ implies that $a \in C_{\mathcal{S}}(J')$ for some finite $J' \subseteq J$. The family $\{\eta(b)^c : a \in J'\} \cup \{\eta(a)\}$ does not cover the space because it is finite, so there exists $G \in \bigcap_{b \in J'} \eta(b) \cap \eta(a)^c$, which means that $J' \subseteq G$ and $a \notin G$, contradicting $a \in C_{\mathcal{S}}(J') \subseteq G$. $\square$

For every reduced referential algebra $\mathcal{F} = \langle W, \mathcal{A} \rangle$, consider the ordered topological space $\mathbb{X}_{\mathcal{F}} = \langle W, \leq_{\mathcal{A}}, \mathcal{T}_{\mathcal{A}} \rangle$ such that $\leq_{\mathcal{A}}$ is the specialization preorder on W induced by $\mathcal{A}$ and $\mathcal{T}_{\mathcal{A}}$ is the topology defined by taking the family $\mathcal{A} \cup \{W \setminus U : U \in \mathcal{A}\}$ as a subbase. By definition, $\mathbb{X}_{\mathcal{F}}$ is totally order-disconnected, and moreover,

$$\mathsf{RA}(\mathcal{F}_1, \mathcal{F}_2) = \mathsf{OT}(\mathbb{X}_{\mathcal{F}_1}, \mathbb{X}_{\mathcal{F}_2}).$$

Putting together all the results in this and the previous section, we obtain:

Theorem 5.11. *The finitary fs-logics are exactly those logics $\mathcal{S}$ such that the referential algebras in $\mathsf{PRA}_{\mathcal{S}}$ satisfy the following conditions:*

1. $\mathbb{X}_{\mathcal{F}} = \langle W, \leq_{\mathcal{A}}, \mathcal{T}_{\mathcal{A}} \rangle$ *is a Priestley space;*
2. $\langle W, \leq_{\mathcal{A}} \rangle$ *is an algebraic lattice;*
3. $\mathcal{A}_{\mathcal{F}} = \mathcal{A}$ *is an algebra of clopen subsets of form $w{\uparrow}$ for some compact $w \in W$;*
4. *the set $\{w' : w'{\uparrow} \in \mathcal{A}\}$ is join-dense in W, i.e. every $w \in W$ is the join of elements of $\{w' : w'{\uparrow} \in \mathcal{A}\}$;*

5. *for every* $w_1\!\uparrow, \ldots, w_n\!\uparrow, w\!\uparrow \in \mathcal{A}$, *if* $\bigcap_{i=1}^{n} w_i\!\uparrow \subseteq w\!\uparrow$ *then* $w\!\uparrow \in C_{\mathcal{S}}^{\mathcal{A}}(w_1\!\uparrow, \ldots, w_n\!\uparrow)$.

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