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Alexandre A. M. RODRIGUES,
Ricardo C. MIRANDA FILHO and Edelcio G. de SOUZA

INVARIANCE AND SET-THEORETICAL OPERATIONS IN FIRST ORDER STRUCTURES

A b s t r a c t. We present a generalization of a theorem of Krasner showing how to construct relations invariant by automorphisms of a first order structure, by means of suitable set-theoretical operations.

1. Introduction

M. Krasner [2], rightly considered a forerunner of the introduction of infinitary languages in model theory (Karp [1], Introduction), showed, in 1938, how to generate all relations invariant under the action of the group of automorphisms of a first order structure by means of set-theoretical operations applied to the primitive elements of the structure.

Although Krasner, in his paper [2], makes no reference to infinitary languages, his set-theoretical operations are the interpretations of the syntactic

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rules of definition of formulas of a suitable infinitary language, as himself recognized much latter (see [3] and the other papers in the references). However, his operations are neither easily described nor very manageable.

A significant feature of this work is the introduction of two natural operators (operators ξ^* and $(\xi^*)^{-1}$, section 2) on relations of a first order structure. These operators are not the immediate interpretation of the syntactic rules of definition of formulas but they behave very nicely with respect to composition and extensions to relations of bijections of the domain of the structure. Relying on these nice algebraic properties we substantially simplified and conceptually improved Krasner's arguments. We consider, in the universe $\mathcal{U}_\mu(D)$ (section 2), relations whose arities are infinite ordinals. This is not an empty generalization; for instance, we are able to give a smaller bound than Krasner's to the arity μ of the universe $\mathcal{U}_\mu(D)$ where one has to operate in order to construct all invariant relations. If d is the power of the domain D, for Krasner μ is the first cardinal greater than d whereas for us it is the ordinal $d+2$ (theorem 3.3). On the other hand, proves are exactly the same whether arities are cardinals or ordinals numbers.

2. The closure of first order structures

Given a nonempty set D and an ordinal γ, a γ-tuple of elements of D is a map from γ into D. The set of all γ-tuples of elements of D will be denoted by D^γ. A *γ-ary relation* of elements of D is a subset of D^γ. Let μ be an infinite ordinal and let $\mathcal{U}_\mu(D)$ be the set of all γ-ary relations of D for all $\gamma < \mu$.

A *first order structure of arity* μ is a triple $E = \langle D, \mu, \mathcal{R}\rangle$ where $\mathcal{R}$ is a subset of $\mathcal{U}_\mu(D)$ which contains the diagonal Δ of D^2. The elements of $\mathcal{R}$ are the *primitive relations* of the structure and $\mathcal{U}_\mu(D)$ is the *universe* of E. We shall refer to elements of D^γ as *points of arity* γ of E.

Let A, B and D be sets and let $\xi : A \to B$ be any map. ξ induces in a natural way a map $\xi^* : D^B \to D^A$. By definition, if $f \in D^B$, then $\xi^*(f) = f \circ \xi$. Let $\wp(D^A)$ and $\wp(D^B)$ be the power sets of D^A and D^B, respectively. We shall denote with the same notation ξ^* the extension of the map $\xi^* : D^B \to D^A$ to the power sets $\wp(D^B)$ and $\wp(D^A)$, and we shall write $\xi^* : \wp(D^B) \to \wp(D^A)$. Moreover, $(\xi^*)^{-1} : \wp(D^A) \to \wp(D^B)$

denotes the usual inverse map of $\xi^* : D^B \to D^A$. If η maps B into C, then $(\eta \circ \xi)^* = \xi^* \circ \eta^*$ and if ξ is a bijection, then ξ^* is also a bijection and $(\xi^*)^{-1} = (\xi^{-1})^*$. In most cases, A and B will be ordinal numbers δ and γ and ξ^* will map points of D^γ into points of D^δ.

For each ordinal $\gamma < \mu$, let $\mathcal{C}_\gamma : \wp(D^\gamma) \to \wp(D^\gamma)$ be the map which maps $R \in \wp(D^\gamma)$ into its complement with respect to D^γ, i.e., $D^\gamma \setminus R$.

Corresponding to all possible choices of $\delta, \gamma < \mu$, let $\mathcal{K}$ be the set of all maps $\xi^*, (\xi^*)^{-1}$ and $\mathcal{C}_\gamma$, where ξ is any map from δ into γ.

Definition 2.1. We say that a set of relations $\mathcal{S} \subseteq \mathcal{U}_\mu(D)$ is E-**closed** if:

1) $\mathcal{R} \subseteq \mathcal{S}$;

2) For all $R \in \mathcal{S}$ and $f \in \mathcal{K}$ such that $f(R)$ is defined, $f(R) \in \mathcal{S}$;

3) If $\mathcal{S}'$ is a subset of $\mathcal{S}$, then $\bigcap \mathcal{S}' \in \mathcal{S}$.

The set of subsets of $\mathcal{U}_\mu(D)$ which are E-closed is not empty because $\mathcal{U}_\mu(D)$ is E-closed. Hence, we may define the **closure** of E, denoted by $\hat{E}$, to be the intersection of all E-closed subsets of $\mathcal{U}_\mu(D)$.

If $\mathcal{S}$ is E-closed and $\mathcal{S}' \subseteq \mathcal{S}$ is a subset of relations of same arity γ, then the conditions $\mathcal{C}_\gamma \in \mathcal{K}$ and (3) above implie that $\bigcup \mathcal{S}' \in \mathcal{S}$.

Let $\gamma < \mu$ be an ordinal and let $\mathcal{P} = (\mathcal{P}_k)_{k \in I}$ be a partition of γ, that is, $\bigcup_{k \in I} \mathcal{P}_k = \gamma$ and $\mathcal{P}_k \cap \mathcal{P}_{k'} = \emptyset$, if $k \neq k'$. We denote by $\mathcal{D}(\mathcal{P})$ the set of points $p \in D^\gamma$ such that $p(i) = p(j)$ for all $i, j \in \mathcal{P}_k$ and for all $k \in I$. We say that $\mathcal{D}(\mathcal{P})$ is the *diagonal* of D^γ defined by the partition $\mathcal{P}$. Similarly, we define $\bar{\mathcal{D}}(\mathcal{P})$ to be the set of points $p \in D^\gamma$ such that $p(i) \neq p(j)$ for all $i \in \mathcal{P}_k$ and $j \in \mathcal{P}_{k'}$ and all $k, k' \in I, k \neq k'$.

Proposition 2.2. $\mathcal{D}(\mathcal{P}), \bar{\mathcal{D}}(\mathcal{P}) \in \hat{E}$.

Proof. Let Δ be the diagonal of D^2 and let $\bar{\Delta} = \mathcal{C}_2(\Delta)$. For $i, j \in \gamma$, $i < j$ let $\xi_{ij} : 2 = \{0, 1\} \to \gamma$ be the map $\xi_{ij}(0) = i$, $\xi_{ij}(1) = j$. Then, $\mathcal{D}(\mathcal{P})$ is the intersection of the sets $(\xi_{ij}^*)^{-1}(\Delta)$ for all $i, j \in \mathcal{P}_k$ and for all $k \in I$. Similarly, $\bar{\mathcal{D}}(\mathcal{P})$ is the intersection of all sets $(\xi_{ij}^*)^{-1}(\bar{\Delta})$ for all $i \in \mathcal{P}_k$ and $j \in \mathcal{P}_{k'}$ and all $k, k' \in I, k \neq k'$. Since, by definition of E, $\Delta \in \mathcal{R}$, it follows from the definition of $\hat{E}$ that $\mathcal{D}(\mathcal{P}), \bar{\mathcal{D}}(\mathcal{P}) \in \hat{E}$. □

3. Invariant relations and the closure of E

Any map $g : D \to D$ extends naturally to a map $g^\gamma : D^\gamma \to D^\gamma$, $p \mapsto g^\gamma(p) = g \circ p$. This last map extends itself to a map from $\wp(D^\gamma)$ into $\wp(D^\gamma)$. We shall use the same notation to denote the extended map. If $h : D \to D$ is another map then, $(g \circ h)^\gamma = g^\gamma \circ h^\gamma$ and, if g is a bijection, $(g^{-1})^\gamma = (g^\gamma)^{-1}$.

An *automorphism* of E is a bijection $g : D \to D$ such that for any primitive relation $R \in \mathcal{R}$ of arity γ, $g^\gamma(R) = R$. Let G be the group of automorphisms of E. An *invariant relation* of E is a relation $R \in \mathcal{U}_\mu(D)$ which is kept fixed by G, that is, $g^\gamma(R) = R$ for all $g \in G$, γ being the arity of R. By definition, the primitive relations are invariant. We denote by $\mathcal{I}(E)$ the set of invariant relations of E.

Proposition 3.1. *For any maps $\xi : \delta \to \gamma$ and $g : D \to D$, we have that*

$$g^\delta \circ \xi^* = \xi^* \circ g^\gamma.$$

Proof. The proposition is an immediate consequence of definitions. □

Proposition 3.2. $\hat{E} \subseteq \mathcal{I}(E)$.

Proof. It suffices to show that $\mathcal{I}(E)$ is E-closed. By definition, $\mathcal{R} \subseteq \mathcal{I}(E)$, and it is trivial to verify that $\mathcal{I}(E)$ is closed under action of all maps $\mathcal{C}_\gamma$, $\gamma < \mu$, and also that it is closed under the intersection of subsets of $\mathcal{I}(E)$. That $\mathcal{I}(E)$ is closed under the actions of ξ^* and $(\xi^*)^{-1}$ follows from proposition 3.1. □

Our next theorem shows that, for sufficiently large μ, all invariant relations belong to the closure $\hat{E}$.

Theorem 3.3. *Let δ be the cardinal of D and assume $\mu \geq \delta + 2$. Then, $\hat{E} = \mathcal{I}(E)$.*

The proof of theorem 3.3 is made clearer introducing previously 3 lemmas.

Lemma 3.4. *Assume $\mu \geq \delta + 2$ and let N_1, N_2, N be the sets of points $p \in D^\delta$ such that the map $p : \delta \to D$ is respectively injective, surjective and bijective. Then, $N_1, N_2, N \in \hat{E}$ and $N \neq \emptyset$.*

Proof. Let $\mathcal{P} = (\mathcal{P}_i)_{i\in\delta}$ be a partition of δ such that $\mathcal{P}_i = \{i\}$ for all $i < \delta$. Then, $N_1 = \bar{\mathcal{D}}(\mathcal{P})$. Hence, by proposition 2.2, $N_1 \in \hat{E}$.

Consider now the partition $\mathcal{P}' = (\mathcal{P}'_i)_{i<2}$ of $\delta+1$ where $\mathcal{P}'_0 = \delta, \mathcal{P}'_1 = \{\delta\}$ and let ξ be the map $\xi : i \in \delta \mapsto i \in \delta + 1$. We shall prove that:

$$N_2 = \mathcal{C}_\delta \xi^*(\bar{\mathcal{D}}(\mathcal{P}')).$$

In fact, $p \in \bar{\mathcal{D}}(\mathcal{P}')$ if and only if $p(i) \neq p(\delta + 1)$ for all $i \in \delta$. Hence, $\xi^*(\bar{\mathcal{D}}(\mathcal{P}'))$ is the set of $p \in D^\delta$ for which there exists $a \in D$ and $i \in \delta$ with $p(i) \neq a$. Therefore, $\mathcal{C}_\delta \xi^*(\bar{\mathcal{D}}(\mathcal{P}'))$ is the set of $p \in D^\delta$ for which, for all $a \in D$, there exists $i \in \delta$ such that $p(i) = a$. This proves our assertion. It follows that $N_2 \in \hat{E}$. Finally, $N = N_1 \cap N_2 \in \hat{E}$. Since δ is the cardinal of D, there exists a bijection from δ onto D. Hence, $N \neq \emptyset$. □

Lemma 3.5. *Let $q \in N$ be a point and let $R \in \hat{E}$ be a relation of arity γ. There exists a relation $M_R \in \hat{E}$ of arity δ such that for every bijection $g : D \to D$, $g^\delta(q) \in M_R \Leftrightarrow g^\gamma(R) \subseteq R$.*

Proof. For every point $p \in R$ there exists a map $\xi : \gamma \to \delta$ such that $\xi^*(q) \in R$. In fact, it is enough to take $\xi = q^{-1} \circ p$. Let Θ_R be the set of all maps $\xi : \gamma \to \delta$ such that $\xi^*(q) \in R$ and define:

$$M_R = \bigcap_{\xi\in\Theta_R} (\xi^*)^{-1}(R).$$

By definition, $M_R \in \hat{E}$ and, by proposition 3.1,

$$\begin{aligned} g^\delta(q) \in M_R &\Leftrightarrow \forall\xi \in \Theta_R(g^\delta(q) \in (\xi^*)^{-1}(R)) \\ &\Leftrightarrow \forall\xi \in \Theta_R(\xi^*(g^\delta(q)) \in R) \\ &\Leftrightarrow \forall\xi \in \Theta_R(g^\gamma(\xi^*(q)) \in R) \\ &\Leftrightarrow g^\gamma(R) \subseteq R. \end{aligned}$$

This completes the proof. □

Lemma 3.6. *For every point $q \in N$, the orbit $\mathcal{O}_q$ of q in D^δ, with respect to the group of automorphisms G, belongs to $\hat{E}$.*

Proof. If R is a relation of arity γ, let us denote by R' its complement in D^γ and consider the relation:

$$M = \bigcap_{R\in\mathcal{R}} M_R \cap \bigcap_{R\in\mathcal{R}} M_{R'} \cap N.$$

By lemmas 3.4 and 3.5, $M \in \hat{E}$. We shall show that $M = \mathcal{O}_q$. Let $g : D \to D$ be a bijection. Since $g^\delta(q)$ clearly belongs to N, by lemma 3.5, $g^\delta(q) \in M$ if and only if for all $R \in \mathcal{R}$ of arity γ, $g^\gamma(R) \subseteq R$ and $g^\gamma(R') \subseteq R'$. Since $g^\gamma : D^\gamma \to D^\gamma$ is a bijection, the last assertion is equivalent to $g^\gamma(R) = R$ for all $R \in \mathcal{R}$ of arity γ. Hence, $g^\delta(q) \in M$ if and only if $g \in G$. Moreover, for every $p \in M$, the bijection $g = p \circ q^{-1} : D \to D$ is such that $g \in G$ and $g^\delta(q) = p$. Therefore, M is the orbit of q in D^γ. □

Proof of theorem 3.3. Since every invariant relation R is the union of orbits of G, it is enough to prove the theorem when R is an orbit of G. Let R be an orbit of arity γ and let $q \in N$. Then, for $\xi \in \Theta_R$ and for $g \in G$, $\xi^*(g^\delta(q)) = g^\gamma(\xi^*(q))$. Hence, $\xi^*(\mathcal{O}_q) = R$. Theorem 3.3 follows now from lemma 3.6 and the definition of $\hat{E}$. □

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University of São Paulo
Department of Mathematics
aamrod@terra.com.br

Federal University of Bahia
Department of Physics
University of São Paulo
Department of Philosophy
ricmir@ufba.br

Pontifical Catholic University of São Paulo
Program of Graduated-Studies in Philosophy
Department of Philosophy
edelcio@pucsp.br