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A NOTE ON TRANSITIVE SETS WITHOUT THE FOUNDATION AXIOM

A b s t r a c t. We construct a model of set theory without the foundation axiom in which there exists a transitive set whose intersection is not transitive.

1. Introduction

We say that a set A is transitive if it satisfies condition $\forall x, y \;\; x \in y \in A \Rightarrow x \in A$. This is obviously equivalent to $\bigcup A \subseteq A$. Transitive sets are a fundamental notion for axiomatic set theory (see [2]).

The Foundation Axiom (sometimes called the Axiom of Regularity) is the following statement

$$\forall x \neq \emptyset \;\; \exists y \in x \;\; x \cap y = \emptyset.$$

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It is immediate from the definition, that for every transitive set A also $\bigcup A$ is transitive. One could think that for a nonempty transitive set A also its intersection, i.e.

$$\bigcap A = \{x \in \bigcup A : \forall y \in A \;\; x \in y\},$$

is transitive. This easily follows from the Foundation Axiom, because this axiom implies that, whenever $A \neq \emptyset$ is transitive, $\emptyset \in A$. Then, for every nonempty transitive set A, $\bigcap A$ is empty, thus transitive. The purpose of this note is to show that the Foundation Axiom is essentially needed in this proof, i.e. that the assertion does not follow from the other axioms of Zermelo–Fraenkel set theory.

The interest in this particular question is motivated by the fact, that in [1] the task of showing that the intersection of every transitive set is transitive is given to the reader as an exercise (ex. 2.4d, ch. 7), without mentioning the Foundation Axiom as one of the axioms of set theory.

Most of the set-theoretic notation used in this paper is standard and can be found in [2]. The symbol ω denotes the set of natural numbers. We identify the natural number n with the set of the natural numbers smaller than n. In particular, $n < m \Leftrightarrow n \in m \Leftrightarrow n \subsetneq m$ and $n \leq m \Leftrightarrow n \subseteq m$. By ZFC^- we denote the Zermelo–Fraenkel set theory with the Axiom of Choice, but without the Foundation Axiom. Whenever we speak about a model of ZFC^-, we mean a set M with a binary relation E satisfying the axioms of ZFC^-. We will deal mainly with non-standard models, which means that E is not a restriction of "true membership relation" $\in$ to M.

Some ambiguities may be caused by the fact that we will consider models of set theory sharing the same universe but with different membership relations. We will use the following convention concerning defined notions. When we deal with structures $\mathbb{M} = \langle M, E\rangle$ and $\mathbb{M}' = \langle M, E'\rangle$ and a defined notion such as ω (i.e. the first non-zero limit ordinal), we use symbol $\omega^{\mathbb{M}}$ to denote such an element $x \in M$ that $\mathbb{M} \models$ "x is the first non-zero limit ordinal". Similarly, $\omega^{\mathbb{M}'}$ denotes the element of M satisfying the definition of ω interpreted with respect to E'. Analogous superscript notation is used for other defined notions, such as pairs, etc.

2. The result

First, let us remind an old technique of consistency proofs concerning the axiom of foundation. The following theorem can be found in [2].

Theorem 2.1. *Suppose that* $\mathbb{M} = \langle M, E \rangle$ *is a model of* ZFC$^-$*. Let* $F : M \longrightarrow M$ *be a bijection definable in* $\mathbb{M}$*, i.e. a bijection from* M *to* M *such that*

$$\forall x, y \in M \ \ (y = f(x) \Leftrightarrow \mathbb{M} \models \varphi(x, y))$$

for some formula φ *of language of set theory. Define* $E' \subseteq M \times M$ *by*

$$xE'y \Leftrightarrow xEf(y).$$

Then

$$\mathbb{M}' = \langle M, E' \rangle \models \mathrm{ZFC}^-.$$

The following theorem is the main result of this paper.

Theorem 2.2. *Assume* Con(ZFC$^-$)*. Then*

$$\mathrm{Con}\big(\mathrm{ZFC}^- + \text{“there exists a transitive set whose intersection is not transitive”}\big).$$

Proof. Assume Con(ZFC$^-$); let $\mathbb{M} = \langle M, E \rangle$ be a model of ZFC$^-$. Let $\varphi(x, y)$ be the formula which says

$$y = \begin{cases} x + 2 & \text{if } x \in \omega, \\ 0 & \text{if } x = \omega + 1, \\ 1 & \text{if } x = \omega + 2, \\ x - 2 & \text{if } \omega + 3 \leq x < \omega + \omega, \\ x & \text{otherwise.} \end{cases}$$

Define $f : M \longrightarrow M$ as follows:

$$y = f(x) \Leftrightarrow \mathbb{M} \models \varphi(x, y). \tag{2.1}$$

Clearly f is a function and it is definable in M, because we have just written its defininition. It is also easy to see that f is a bijection. Thus, working in $\mathbf{V}$, we may define E' by the formula $xE'y \leftrightarrow xEf(x)$. From theorem 2.1 we obtain

$$\mathbb{M}' = \langle M, E' \rangle \models \mathrm{ZFC}^-. \tag{2.2}$$

Obviously, in $\mathbb{M}$ we have the element $\omega^{\mathbb{M}} \in M$ such that $\mathbb{M} \models$ "$\omega^{\mathbb{M}}$ is the first non-zero limit ordinal". We will show that

$$\mathbb{M}' \models \text{"}\omega^{\mathbb{M}} \text{ is transitive"} \tag{2.3}$$

but

$$\mathbb{M}' \models \text{"}(\bigcap \omega^{\mathbb{M}}) \text{ is not transitive"}. \tag{2.4}$$

Observe that $f(\omega^{\mathbb{M}}) = \omega^{\mathbb{M}}$ and for $n \in M$ we have $nE\omega^{\mathbb{M}} \Rightarrow f(n)E\omega^{\mathbb{M}}$.

We will check that $\omega^{\mathbb{M}}$ is transitive in $\mathbb{M}'$. First observe that, as $f(\omega^{\mathbb{M}}) = \omega^{\mathbb{M}}$, E'-elements of $\omega^{\mathbb{M}}$ are precisely E-elements of $\omega^{\mathbb{M}}$. Now, if $\mathbb{M}' \models x \in \bigcup \omega^{\mathbb{M}}$ then there exists $n \in M$ such that $nE'\omega^{\mathbb{M}}$ and $xE'n$. This means that $nE\omega^{\mathbb{M}}$ and $xEf(n)$. But then $f(n)E\omega^{\mathbb{M}}$ and from the transitivity of $\omega^{\mathbb{M}}$ in $\mathbb{M}$ we get that $xE\omega^{\mathbb{M}}$; equivalently, $xE'\omega^{\mathbb{M}}$. Thus

$$\mathbb{M}' \models \bigcup \omega^{\mathbb{M}} \subseteq \omega^{\mathbb{M}}, \tag{2.5}$$

which precisely means that

$$\mathbb{M}' \models \text{"}\omega^{\mathbb{M}} \text{ is transitive"}. \tag{2.6}$$

Now we will compute $\bigcap \omega^{\mathbb{M}}$ in $\mathbb{M}'$. First observe that $\mathbb{M}' \models 0^{\mathbb{M}} \in \omega^{\mathbb{M}}$. This ensures that $\omega^{\mathbb{M}}$ is not equal to $\emptyset^{\mathbb{M}'}$, so our computation makes sense.

We are looking for the elements of M which are in E'-relation with every E'-member of $\omega^{\mathbb{M}}$. We easily get that

$$\mathbb{M}' \models \bigcap \omega^{\mathbb{M}} = \{0^{\mathbb{M}}, 1^{\mathbb{M}}\}. \tag{2.7}$$

Statement 2.7 may require some additional explanation. It may be unclear what we mean by $\{0^{\mathbb{M}}, 1^{\mathbb{M}}\}$ because the meaning of $\{x, y\}$ depends on the membership relation being considered. The formula 2.7 is intended to say: $0^{\mathbb{M}}, 1^{\mathbb{M}}$ are the only elements, which are in relation E' with all E'-elements of $\omega^{\mathbb{M}}$.

We will now check that $\{0^{\mathbb{M}}, 1^{\mathbb{M}}\}^{\mathbb{M}'}$ is not transitive in $\mathbb{M}'$. Observe that we have $2^{\mathbb{M}}E'1^{\mathbb{M}}$, because $f(1^{\mathbb{M}}) = 3^{\mathbb{M}}$ and $2^{\mathbb{M}}E3^{\mathbb{M}}$. Thus

$$\mathbb{M}' \models 2^{\mathbb{M}} \in \bigcup \{0^{\mathbb{M}}, 1^{\mathbb{M}}\}. \tag{2.8}$$

On the other hand,

$$\mathbb{M}' \models 2^{\mathbb{M}} \notin \{0^{\mathbb{M}}, 1^{\mathbb{M}}\} \tag{2.9}$$

because $2^{\mathbb{M}} \neq 0^{\mathbb{M}}$ and $2^{\mathbb{M}} \neq 1^{\mathbb{M}}$. This finishes the proof. $\square$

3. A note added in proof

Anna Wojciechowska pointed to us that the existence of a transitive set whose intersection is not transitive follows also from the existence of a set C such that $C = \{C, \{C\}\}$ and $C \neq \{C\}$. To construct a model with such a set one can employ the same technique of constructing ill-founded models of set theory as we used in our proof.

References

[1] T. Jech and K. Hrbáček, Introduction to Set Theory, 2. ed., Pure and Applied Mathematics, vol. 85, Marcel Dekker, inc./New York and Basel, New York, 1984.

[2] K. Kunen, Set Theory, an Introduction to Independence Proofs, Studies in Logic, vol. 102, North Holland, 1980.

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