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INTENSIONAL POSITIVE SET THEORY

A b s t r a c t. This paper shows that, via a simple kind of forcing, one can construct a pure term model for intensional positive set theory, where sets are defined by positive formulas and identifications are ruled by equivalence of the defining formulas. Further one can also construct a model that "contains ZF".

1. Introduction

Positive set theory is at present a subject on its own, from the initial work of R.J. Malitz [16], to the further developments including topological models [6], [7], and very strong variants, in particular Esser's theory GPK^+_∞ where the class of all well-founded sets interprets ZF [3], [4].
In another direction, namely in languages allowing terms, does positive set theory appear as a weakening of partial set theory, linked to the concept of partial information; this actually follows the line initiated in Gilmore's

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pioneer work [8], [11], [15].
The fact that this presents incompatibilities with extensionality appeared at once [8], [11].
More recently Gilmore himself prefered the λ-calculus approach, with very convincing arguments in favour of **intensionality** in place of **extensionality** [9], [10].
This paper shows that intensional worlds modelizing positive set theory are possible. In the models constructed here, every element (set) has a name (this name is a term, actually a generalized term in those models that "contain ZF"), and this name reveals the "meaning" of that set. Further the identification process of terms is ruled by intensionality rather than by extensionality, i.e. two sets will be equal if they have "reasonably" the same meaning (naturally this will be made precise).
In [14] we suggested that alternative set theories should be modelisable in reasonable extensions of ZF, and also allow models where some transitive class (or even set) interprets ZF. This is indeed the case here : we get a true pure term model for intensional positive set theory; and a (generalized) term model containing ZF (in the sense of [12], [14]), this time modulo the assumption of the existence of an uncountable inaccessible cardinal. Respectively the consistency of the concerned theories can actually be deduced from the known consistency results for GPK and GPK^+_∞ [4], [6], but the difference is that the models here are term models (sometimes generalized term models), and that, in the second case, the price to pay concerns an uncountable inaccessible cardinal instead of an uncountable weakly compact cardinal (as in [4], [6]).
In order to separate the different problems, the paper is structured like this :

- section 2 is devoted to the description of the theories that we want to modelize, and motivates the choice of the kind of involved language,

- section 3 explains the construction of a pure term model, satisfying abstraction and intensionality,

- section 4 shows how one can get an analogue model of generalized terms, where some transitive set interprets ZF (modulo the assumption of the existence of an uncountable inaccessible cardinal),

- section 5 mentions further links with "partial information".

Our metatheory will be ZF, i.e. Zermelo-Fraenkel with the axiom of choice, enriched (in section 4 only) with a large cardinal assumption.

2. Abstraction and intensionality

$\mathcal{L}$ will denote the usual first-order set-theoretical language, with primitive symbols $\in$ and $=$.
$\mathcal{L}_\tau$ is the stronger language, using $\in$ and $=$, but also the abstractor $\{\ |\ \}$; $\mathcal{L}_\tau$ is built up via the following rules :

1. any variable (the variables will be letters $x, y, z, \ldots$) is a **positive term**,
2. if τ, τ' are positive terms (we will use $\tau, \tau', \tau'', \ldots$ for terms), then $\tau \in \tau'$, $\tau = \tau'$ are **positive formulas**
3. if φ, ψ are **positive formulas**, then so are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists x\ \ \varphi$, $\forall x\ \ \varphi$,
4. $\bot$, $\top$ (respectively "false" and "true") are **positive formulas**,
5. if φ is a **positive formula**, then $\{x|\varphi\}$ is a **positive term**,
6. if φ is a **formula**, then so is $\neg\varphi$ ($\neg$ is the negation symbol).

Notice that we allow "general" formulas, but only **positive** terms. Actually is $\mathcal{L}$ the fragment of $\mathcal{L}_\tau$ obtained by just renouncing to the abstraction, i.e. to rule 5. The "natural" behaviour of (positive) sets in a langage L with abstractor is ruled by the "abstraction scheme for L" :

$$\mathrm{Abst}(L) : \forall\vec{y}\ \forall x\ (x \in \{t|\varphi(t, \vec{y})\} \leftrightarrow \varphi(x, \vec{y})),$$

for any positive formula $\varphi(x, \vec{y})$ in L.
While the "natural" behaviour of (positive) sets, in the first-order language $\mathcal{L}$, is ruled by the usual "comprehension scheme" :
Comp($\mathcal{L}$) :

$$\forall\vec{y}\ \ \exists z\ \ \forall x\ (x \in z \leftrightarrow \varphi(x, \vec{y})),$$

for any positive formula $\varphi(x, \vec{y})$ in $\mathcal{L}$ (where z is not free in φ).
Naturally: $\vec{y}$ represents some n-tuple of variables $y_1, y_2, \ldots, y_n$;
$\psi(v_1, v_2, \ldots, v_k)$ indicates that any free variable of ψ is in the list $v_1, v_2, \ldots, v_k$;

the free variables of a term $\{x|\varphi\}$ are those of φ, minus "x"; a sentence is a formula without free variables; a closed term is a term without free variables. Now is it known since [11] that Abst($\mathcal{L}_\tau$) disproves the extensionality axiom

$$\text{EXT} :\equiv \forall x \forall y((\forall t(t \in x \leftrightarrow t \in y)) \rightarrow x = y).$$

The reader can easily check this by reproducing Russell's paradox for the term $R := \{x|\{t|x \in x\} = \{t|\bot\}\}$, in the theory Abst($\mathcal{L}_\tau$)+ EXT.
So it is natural to try to see what happens if one assumes "intensionality" instead of EXT. The idea is that we expect terms, say $\{x|\varphi(x)\}$ and $\{x|\psi(x)\}$, to be equal whenever the formulas $\varphi(x)$ and $\psi(x)$ are "equivalent". But what does "equivalent" mean ? One should reasonably at least expect that **logically equivalent** formulas are "equivalent" in that sense. But one can further imagine arguments in favour of "equivalence" in the style :

$$\Gamma \vdash \forall x(\varphi(x) \leftrightarrow \psi(x)),$$

for some theory Γ.
Now there are surely natural such Γ's, inspired by the "partial information" point of view (see also section 5).
Indeed : if we think about a positive term $\{x|\varphi(x)\}$ as being the possibly incomplete, but reliable list of those objects x that satisfy $\varphi(x)$, we will at least expect the following condition of "pre-abstraction" to be fulfilled :

$$\forall z(z \in \{x|\varphi(x)\} \rightarrow \varphi(z)).$$

So a kind of "minimal" Γ_0 could just be this scheme, and the corresponding intensionality rule would then be :

$$\frac{\Gamma_0 \vdash \forall x(\varphi(x) \leftrightarrow \psi(x))}{\{x|\varphi(x)\} = \{x|\psi(x)\}}.$$

It should be noticed that intensionality, in place of extensionality, is not in itself a guarantee against paradoxes. For example, Abst($\mathcal{L}_\tau$) is incompatible with the intensionality rule where Γ is Abst($\mathcal{L}_\tau$) itself. Indeed : consider the same term R as before. One can easily check that $\Gamma \vdash R \in R \leftrightarrow \bot$, so that via this intensionality rule, one gets $\{x|R \in R\} = \{x|\bot\}$, and then via abstraction : $R \in R$, precisely contradicting $\{x|R \in R\} = \{x|\bot\}$! The problem clearly arises from the possibility of abstracting on free variables in **proper** terms (i.e. terms that are not just variables); also is quantification on such free variables “dangerous", as it allows to pass round the previous interdiction, with “tricks" like :

$$\{x|\exists y(y = x \wedge \{t|y \in y\} = \{t|\bot\})\}.$$

The weaker language $\mathcal{L}^\star$ that we propose precisely takes that in account.

Definition 2.1. $\mathcal{L}^\star$ is the fragment of $\mathcal{L}_\tau$ where one forbids abstraction and quantification on free variables in **proper** terms (in the formulas, as well as in the positive terms).

Comments: While $\mathcal{L}^\star$ is still closed for sub-formulas and sub-terms, one has to be more careful for replacement of free variables by proper terms, that is now only allowed if the free variables in those replacing terms stay free, i.e. don't become abstracted or quantified. Remember also that only **positive** terms are admitted.
$\mathcal{L}^\star$ is obviously “in between" $\mathcal{L}$ and $\mathcal{L}_\tau$, and actually very close to $\mathcal{L}$, in the following sense : whenever we have a model for Comp($\mathcal{L}$), we can see it also as a model for Abst($\mathcal{L}^\star$), just by interpreting (inductively) the terms (via the axiom of choice in our metatheory).
Notice further that, when EXT is assumed, this interpretation already holds in the theory itself.
We are able now to discuss the construction of a pure term model for Abst($\mathcal{L}^\star$), satisfying an adequate intensionality rule.

3. A pure term model

We call Ω the set of all closed positive $\mathcal{L}^\star$-terms. Actually one can as well construct Ω inductively :

$$\Omega_0 := \{\{x|\psi(x)\} \mid \psi(x) \text{ is a positive } \mathcal{L}\text{-formula}\}$$
$$\Omega_{k+1} := \{\{x|\psi(x,\vec{b})\} \mid \psi(x,\vec{y}) \text{ is a positive } \mathcal{L}\text{-formula and each } b_i \text{ is in } \Omega_k\}$$
$$\Omega := \Omega_\omega, \text{ i.e. } \bigcup_{k\in\omega} \Omega_k$$

Obviously is $(\Omega_k)_{k\in\omega}$ an increasing chain for $\subset$. We will use letters $a, b, c, \ldots$ for elements of Ω, except when we want to “look inside" them, in which case we will prefer τ, τ', τ'',... Each τ in Ω is of type $\{x|\psi(x,\vec{b})\}$, with $\vec{b}$ in Ω and

$\psi(x,\vec{y})$ a positive $\mathcal{L}$-formula; further is $\psi(x,\vec{b})$ a positive $\mathcal{L}$-formula that we will denote, for convenience : $\varphi_\tau(x)$.
All the models considered in this section will have Ω as universe, so will be of type : $(\Omega, \in_M, =_M)$, where M is the name of the model and $\in_M$, $=_M$ are binary relations on Ω.
The abstraction scheme that we want to modelize is Abst$(\mathcal{L}^\star)$; as Ω is only made of closed positive terms, it is clear that we just have to modelize "abstraction without parameters" :

$$\text{Abst}_0(\mathcal{L}^\star) : \forall x(x \in \tau \leftrightarrow \varphi_\tau(x)), \text{ for each } \tau \text{ in } \Omega.$$

The same remark applies to "pre-abstraction".
Further we will denote "Congr(=)" the first-order axiom expressing that = is a congruence for $\mathcal{L}$, i.e. that = is an equivalence relation satisfying $\mathcal{L}$-substitutivity (as is well-known, it suffices to ask, for this last part : $(x = x' \wedge y = y' \wedge x \in y) \to x' \in y')$.
We will restrict our attention to **admissible** models, i.e. those that satisfy Preabst$(\mathcal{L}^\star)$ and Congr(=), where Preabst$(\mathcal{L}^\star)$ is the scheme :

$$\forall \vec{y}\, \forall x(x \in \{t|\psi(t,\vec{y})\} \to \psi(x,\vec{y})),$$

for any positive $\mathcal{L}^\star$-formula ψ.
As noticed before is it equivalent here to satisfy Preabst$_0(\mathcal{L}^\star)$, i.e. :

$$\forall x(x \in \tau \to \varphi_\tau(x)), \text{ for any } \tau \text{ in } \Omega.$$

We denote Adm the set of all admissible models and put a "notion of extension" on it, more precisely a partial order $\leq$ defined by :

$$M \leq N \text{ iff } (M \in \text{ Adm } \&\ N \in \text{Adm } \&\ \in_M \subset \in_M\ \&\ =_M \subset =_N).$$

As usually are binary relations seen as collections of ordered pairs.
We say that "N is an extension of M" when $M \leq N$. Further we denote PFP the following "positive formulas preservation" property (easy to check) :
If $\psi(\vec{z})$ is a positive $\mathcal{L}^\star$-formula, $\vec{b}$ is in Ω and $N \geq M \models \psi(\vec{b})$,
Then $N \models \psi(\vec{b})$.
This argument will be often used in this section.
Notice that the argument holds even when N is not supposed to be admissible.
We introduce now some convenient notations and the particular "notion of forcing" that we will use (for $X \subset$ Adm)

- $M \leq_X N$ iff (def.) $M \in X$ & $N \in X$ & $M \leq N$.
- $M \Vdash_X \tau = \tau'$ iff (def.)
  $M \in X$ & $\forall N \geq_X M \quad \exists P \geq_X N \quad P \models \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x))$.
- $X^+ := \{M \in X \mid \forall \tau, \tau' \in \Omega(M \models \tau = \tau' \rightarrow M \Vdash_X \tau = \tau')\}$.
- $A_0 :=$ Adm; $A_{\alpha+1} := (A_\alpha)^+$;
  $A_\gamma := \bigcap_{\beta<\gamma} A_\beta$ (for γ limit ordinal).
  This defines a decreasing chain (for $\subset$) of subsets A_α (of Adm), indexed by Von Neumann ordinals (in our metatheory).
- Any increasing chain (for $\leq$) of elements M_α of Adm (for $\alpha < \gamma$, γ limit ordinal) has an obvious limit, namely the "union" of the M_α's :

$$(\Omega, \bigcup_{\alpha<\gamma} \in_\alpha, \bigcup_{\alpha<\gamma} =_\alpha), \quad \text{where } \in_\alpha \text{ is } \in_{M_\alpha}, \text{ and } =_\alpha \text{ is } =_{M\alpha}.$$

We analyse now the properties of $\Vdash$, finding back several familiar "forcing aspects".
The following initial propositions are very easy to check.

Proposition 3.1. *(Extension Lemma)*
If

$$N \geq_X M \Vdash_X \tau = \tau',$$

Then

$$N \Vdash_X \tau = \tau'.$$

Proposition 3.2. *(Inductivity Lemma) If X is closed under "unions" of chains (as described before; with $X \subset Adm$),*
Then so is X^+.

Proposition 3.3. *Each A_α is non-empty and closed under "unions" of chains.*
(Hint : use proposition 3.2; and the fact that $M = (\Omega, \phi, \equiv)$ belongs to each A_α, where ϕ is the empty set and $\equiv$ is "formal identity").

Proposition 3.4. *(Fixpoint Lemma)*
For some ordinal δ, we have $A_{\delta+1} = A_\delta$.

This is obvious via the classical argument. From here on we suppose that δ is the least ordinal for which $A_{\delta+1} = A_\delta$.

Proposition 3.5. *A_δ admits maximal elements (for $\leq$).*

This is obvious via the classical Zorn-argument. We now choose one such maximal element of A_δ, and call it G. Notice that : $\forall\alpha \;\; G \in A_\alpha$.
Actually is this G the model we are looking for.
As we shall see has G the behaviour of a "generic structure", w.r.t. the adequate notion of "forcing" (for analogies, see f.ex. [1]).

Lemma 3.6. *(Abstraction Lemma) : G satisfies Abst($\mathcal{L}^\star$).*

Proof. As G is admissible, with universe Ω, it suffices to prove :

$$G \models \forall x(\varphi_\tau(x) \to x \in \tau),$$

whenever $\tau \in \Omega$.
So let us fix τ and x, both in Ω, such that $\varphi_\tau(x)$ holds in G. We have to check that : $x \in_G \tau$.
We construct an extension $\overline{G}$ of G, like this :

$$\overline{G} \models a \in b \quad \text{iff} \quad G \models a \in b \vee (a = x \wedge b = \tau).$$

Further have G and $\overline{G}$ the same equality relation.
We show now, by induction on α, that $\overline{G}$ belongs to each A_α :

- For $\alpha = 0$:
  As $A_0 = \text{Adm}$, we have to check that $\overline{G} \in \text{Adm}$. That $=_G$ is also a congruence on $\overline{G}$ is easily seen. For Preabst : suppose $\overline{G} \models a \in \eta$, and distinguish two cases :

  Case 1 : $G \models a \in \eta$; then $G \models \varphi_\eta(a)$, and so $\overline{G} \models \varphi_\eta(a)$ by PFP ("positive formulas preservation").

  Case 2 : $G \models \neg a \in \eta$; then $a =_G x$ and $\eta =_G \tau$; as $G \in A_{\delta+1}$, we have (by our definitions) :

  $$\forall M \geq_\delta G \quad \exists N \geq_\delta M \quad N \models \forall t(\varphi_\eta(t) \leftrightarrow \varphi_\tau(t))$$

  (where $\geq_\delta$ denotes $\geq_{A_\delta}$); so, as G is maximal in A_δ, we get :

  $$G \models \forall t(\varphi_\eta(t) \leftrightarrow \varphi_\tau(t)).$$

We assumed that $\varphi_\tau(x)$ holds in G, so we have also $G \models \varphi_\eta(x)$. And, as $a =_G x$, we get finally :

$$G \models \varphi_\eta(a), \text{ and also } \overline{G} \models \varphi_\eta(a).$$

Conclusion: in both cases we get $\overline{G} \models \varphi_\eta(a)$, and that proves Prabst($\mathcal{L}^\star$) for $\overline{G}$.

- If $\overline{G} \in A_\alpha$, then $\overline{G} \in A_{\alpha+1}$:
  So suppose $\overline{G} \in A_\alpha$ and $\overline{G} \models a = b$; then $G \models a = b$, because G and $\overline{G}$ have the same equality relation. As $G \in A_{\alpha+1}$, this implies $G \Vdash_\alpha a = b$ (where $\Vdash_\alpha$ denotes $\Vdash_{A_\alpha}$). As we showed already that $\overline{G} \in$ Adm, we get by Proposition 3.1 that $\overline{G} \Vdash_\alpha a = b$.
  Conclusion : $\overline{G} \in A_{\alpha+1}$.
- If $\forall \beta < \gamma$ (limit ordinal) $\overline{G} \in A_\beta$, then $\overline{G} \in A_\gamma$: obvious.

This achieves the inductive proof of : $\forall \alpha\ \overline{G} \in A_\alpha$; in particular do we get $\overline{G} \in A_\delta$; and so $G = \overline{G}$, because G is maximal in A_δ and $G \leq \overline{G}$.
Conclusion : as $G = \overline{G} \models x \in \tau$, we get finally $G \models x \in \tau$, which achieves the proof of Lemma 3.6.

Lemma 3.7. *(Generic Lemma) :*

$$G \models \tau = \tau' \quad \textit{iff} \quad G \Vdash_{On} \tau = \tau'$$

(where $G \Vdash_{On} \tau = \tau'$ means : $\forall \alpha\ G \Vdash_\alpha \tau = \tau'$; and $\Vdash_\alpha$ is $\Vdash_{A_\alpha}$).

Remark: This shows that $\Vdash_{On}$ is the "right" notion of forcing, i.e. the one for which G is indeed "generic".

Proof of Lemma 3.7. The left-to-right direction is immediate from the definitions and the fact that $\forall \alpha \quad G \in A_\alpha$.
To prove the other direction : fix some term τ and construct the following extension $G^\star$ (actually depending on τ) of G : $G^\star = (\Omega, \in^\star, =^\star)$, where

- $a =^\star b$ iff $a =_G b \vee G \Vdash_{On} a = b = \tau$.
- $a \in^\star b$ iff $\exists a', b' \quad a =^\star a' \in_G b' =^\star b$.

The following facts are easy to check :
Fact 1 : $=^\star$ is an equivalence relation on $G^\star$.
Fact 2 (already mentioned) : $a =_G b \to G \Vdash_{On} a = b$.
Fact 3 : $a =^\star b \to G \models \forall t(t \in a \leftrightarrow t \in b)$.
(hint : use the definitions and lemma 3.6).
Fact 4 : $a =^\star b \to G^\star \models \forall z(a \in z \leftrightarrow b \in z)$
(hint : use the definitions).

It suffices now to show that $G^\star$ belongs to each A_α, to get $G^\star \in A_\delta$, so that $G = G^\star$ and finally : $G \models \tau = \tau'$ whenever $G \Vdash_{On} \tau = \tau'$.
Again, we prove this by induction on α.

- For $\alpha = 0$:
  We should prove that $G^\star$ is admissible.
  That $=^\star$ is a congruence is easily seen thanks to the above-mentioned facts.
  Further, for Preabst : suppose $a \in^\star \eta$; we have to prove that $G^\star \models \varphi_\eta(a)$.

  We will distinguish 4 cases :

  Case 1 : $\neg a =^\star \eta$ and $\neg \eta =^\star \tau$; here one gets quickly that $a \in_G \eta$, so $G \models \varphi_\eta(a)$, and by PFP : $G^\star \models \varphi_\eta(a)$.

  Case 2 : $a =^\star \tau$, but $\neg \eta =^\star \tau$; so some a' satisfies : $a' =^\star a$ and $a' \in_G \eta$; so we get $G \models \varphi_\eta(a')$, and by PFP : $G^\star \models \varphi_\eta(a')$.
  Consider now the permutation exchanging a and a', and fixing all the other points.
  This is an automorphism for the first-order structure $G^\star = (\Omega, \in^\star, =^\star)$, so that one gets : $G^\star \models \varphi_\eta(a)$.

  Case 3 : $\eta =^\star \tau$, but $\neg a =^\star \tau$; here $a \in_G \eta'$ for some $\eta' =^\star \eta$, and so by fact 3, we get : $a \in_G \eta$; so again $G \models \varphi_\eta(a)$, and also (by PFP) : $G^\star \models \varphi_\eta(a)$.

  Case 4 : $\eta =^\star \tau =^\star a$; here, for some a' and η', we have :
  $a =^\star a' \in_G \eta' =^\star \eta$.
  By fact 3 : $a' \in_G \eta$; so again : $G \models \varphi_\eta(a')$, and (by PFP) : $G^\star \models \varphi_\eta(a')$.
  The automorphism argument from case 2 then allows to get : $G^\star \models \varphi_\eta(a)$.

- If $G^\star \in A_\alpha$, then $G^\star \in A_{\alpha+1}$:
  Suppose $G^\star \models a = b$, i.e. $a =^\star b$. If $a =_G b$, the fact that $G \in A_{\alpha+1}$ immediately gives us $G \Vdash_\alpha a = b$, and so by proposition 3.1 also $G^\star \Vdash_\alpha a = b$. In the other case we have $G \Vdash_{On} a = b = \tau$, so again $G \Vdash_\alpha a = b$ and $G^\star \Vdash_\alpha a = b$. So in any case $G^\star \models a = b$ implies $G^\star \Vdash_\alpha a = b$; and for $G^\star \in A_\alpha$, we can conclude $G^\star \in A_{\alpha+1}$.
- If $\forall\beta < \gamma$ (limit ordinal) $G^\star \in A_\beta$, then $G^\star \in A_\gamma$: trivial.

Lemma 3.8. *(Strong Congruence Lemma) :*
$=_G$ *is a* **strong** *congruence on* G*, i.e. a congruence that is also substitutive in terms (whenever* $\vec{a} =_G \vec{b}$*, one has* $\{x|\psi(x,\vec{a})\} =_G \{x|\psi(x,\vec{b})\}$*, for each positive* $\mathcal{L}^\star$*-formula* $\psi(x,\vec{y})$*).*

Comment: The reader should notice that our admissible models only require substitutivity for first-order formulas, not for terms !
For extensional models (modulo abstraction of course) both substitutivities are equivalent, but extensionality is precisely missing here. So it is somewhat astonishing that we get substitutivity for terms here, while we made no effort at all to obtain it !

Proof of Lemma 3.8. Suppose $\vec{a} =_G \vec{b}$; we should prove that

$$\{x|\psi(x,\vec{a})\} =_G \{x|\psi(x,\vec{b})\}$$

for any $\mathcal{L}^\star$-term $\{x|\psi(x,\vec{y})\}$. The definition of $\mathcal{L}^\star$ however shows that it suffices to prove this for positive $\mathcal{L}$-formulas $\psi(x,\vec{y})$. By PFP we have immediately :

$$\forall N \geq G \quad \vec{a} =_N \vec{b}.$$

As $=_N$ is a first-order congruence we have :

$$\forall N \geq G \quad N \models \forall x(\psi(x,\vec{a}) \leftrightarrow \psi(x,\vec{b})).$$

So (a fortiori) :

$$\forall\alpha \quad G \Vdash_\alpha \{x|\psi(x,\vec{a})\} = \{x|\psi(x,\vec{b})\}.$$

Then by lemma 3.7 :

$$G \models \{x|\psi(x,\vec{a})\} = \{x|\psi(x,\vec{b})\}.$$

Definition. "$\text{Intrule}_L(\Gamma)$" will denote the following intensionality rule (where L is some language, Γ a theory expressed in L, and ψ, ψ' are positive L-formulas):

$$\frac{\Gamma \vdash \forall \vec{y} \quad \forall x(\psi(x,\vec{y}) \leftrightarrow \psi'(x,\vec{y}))}{\forall \vec{y} \quad \{x|\psi(x,\vec{y})\} = \{x|\psi'(x,\vec{y})\}}$$

Here we will consider the language $\mathcal{L}^\star$, and also :

$$\begin{aligned} \Gamma^\star &:= \{\sigma | \forall N \in A^\star \quad N \models \sigma\}, \text{ where} \\ A^\star &:= \{N | N \geq G \;\&\; \exists \alpha \quad N \text{ is maximal in } A_\alpha\}. \end{aligned}$$

Lemma 3.9. *(Intensionality Lemma) :*
G satisfies $\text{Intrule}_{\mathcal{L}^\star}(\Gamma^\star)$.

Proof. As G is a pure term model, it suffices to prove that $\tau = \tau'$ whenever $\Gamma^\star \vdash \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x))$.
As each A_α is closed under "unions" of chains, it is clear (via a Zorn-argument) that $\Vdash_\alpha$ is characterized via:

$$\begin{aligned} &G \Vdash_\alpha \tau = \tau' \text{ iff} \\ &\qquad \forall N \geq_\alpha G \; [N \text{ is maximal in } A_\alpha \rightarrow N \models \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x))]. \end{aligned}$$

And so, once $\Gamma^\star \vdash \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x))$, we have :

$$\forall \alpha \quad G \Vdash_\alpha \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x)),$$

and by lemma 3.7 : $G \models \tau = \tau'$.

The results obtained so far can now be summarized in :

Theorem 3.10. *G satisfies* $\mathit{Abst}(\mathcal{L}^\star) + \mathit{Intrule}_{\mathcal{L}^\star}(\Gamma^\star)$, *and* $=_G$ *is a strong congruence.*

As the theory $\Gamma^\star$ is not very "explicit" (it refers to extensions of G), we propose also at once a more "logically explicit" version Γ_1, defined by :

$$\begin{aligned} \Gamma_1 \quad := \quad &\text{Preabst}(\mathcal{L}^\star) + \text{Congr}(=) + \{\sigma | \sigma \text{ is a positive } \mathcal{L}^\star\text{-sentence} \\ &\& \text{ Abst}(\mathcal{L}^\star) + \text{Congr}(=) \vdash \sigma\}. \end{aligned}$$

Γ_1 just involves "admissibility", plus the positive consequences of "admissibility + abstraction".
One can easily check that $\Gamma_1 \subset \Gamma^\star$, so that we get (from theorem 3.10) :

Theorem 3.11. *G satisfies Abst($\mathcal{L}^\star$)+Intrule$_{\mathcal{L}^\star}(\Gamma_1)$, and $=_G$ is a strong congruence.*

Comment: The natural strategy consists in trying to find a Γ as close as possible to the theory of G, i.e. $\{\sigma|\sigma$ is an $\mathcal{L}^\star$-sentence & $G \models \sigma\}$. Indeed : if one could take for Γ exactly the theory of G, then G would be extensional (via abstraction), and would we have a pure term model for Abst($\mathcal{L}^\star$) + Congr(=) + EXT; while the existence of such a model is still an open question !

Remark: Actually one can imagine other versions of the type of forcing used here. There is even one that allows to obtain the same theorem 3.11[1], but a priori a weaker Theorem 3.10 and so probably less identifications in G. More precisely, if one modifies the definition of $\Vdash$ like this :

$$M \Vdash_X \tau = \tau' \text{ iff } M \in X \ \&\ \forall N \geq_X M \quad N \models \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x)),$$

the fixed-point A_δ is reached very quickly, as $A_2 = A_1$! Everything else stays the same, except lemma 3.9 that becomes : if $\forall N \geq G' \quad N \models \forall x(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x))$, then $\tau =_{G'} \tau'$ (where G' is the "new" G); and theorem 3.10 that gets a "new Γ" : $\{\sigma|\sigma$ is an $\mathcal{L}^\star$-sentence & $\forall N \geq G \quad N \models \sigma\}$.
One can easily show that any G' constructed in that way has an extension G of the "old type" : $G' \leq G$. So that (a priori) G seems better, as it proceeds to more identifications. It is however still not clear whether or not $G' \neq G$...

4. A model "containing ZF"

We present here a modification of the construction in section 3, where we lose the fact that any element is a finite term, but get in exchange a transitive part that interprets ZF; the whole model still satisfies abstraction, intensionality and strong congruence. Our metatheory is ZF with the additional assumption that there exists an uncountable **inaccessible** cardinal κ (i.e. a regular, limit cardinal κ such that $2^\alpha < \kappa$ for any cardinal $\alpha < \kappa$).
The infinitary language $\mathcal{L}_\kappa$ that we will use allows κ variables, uses a new unary predicate "$C(x)$" (intended to distinguish the "classical" sets) and is built up following the rules :

[1] This was noticed too (and independently) by O. Esser.

(1) if x, y are variables, then $x = y$ and $x \in y$ are positive $\mathcal{L}_\kappa$-formulas,

(2) $\perp, \top$ are positive $\mathcal{L}_\kappa$-formulas,

(3) $C(x)$ is a positive $\mathcal{L}_\kappa$-formula,

(4) if $(y_\alpha)_{\alpha<\beta}$ (with $\beta < \kappa$) are variables, then $\bigvee\!\!\!\bigvee_{\alpha<\beta} x = y_\alpha$ is a positive $\mathcal{L}$-formula,

(5) if φ, ψ are positive $\mathcal{L}_\kappa$-formulas, then so are $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists x\ \varphi$, $\forall x\ \varphi$,

(6) if φ is an $\mathcal{L}_\kappa$-formula, then so is $\neg\varphi$.

Notice that only rule (4) (allowing κ-finite disjunctions) distinguishes $\mathcal{L}_\kappa$ from the first-order language $\mathcal{L}_C$, i.e. $\mathcal{L}$ enriched with the predicate $C(x)$. We construct now an adapted universe Δ :

$$\begin{aligned}
\Delta_0 &:= \{\{x|\psi(x)\}|\psi(x) \text{ is a positive } \mathcal{L}_\kappa\text{-formula}\} \\
\Delta_{\alpha+1} &:= \{\{x|\psi(x,\vec{b})\} \bigm| \psi(x,\vec{y}) \text{ is a positive } \mathcal{L}_\kappa\text{-formula,} \\
&\qquad \text{and the components of } \vec{b} \text{ are elements of } \Delta_\alpha\} \\
\Delta_\gamma &:= \bigcup_{\beta<\gamma} \Delta_\beta \text{ (for } \gamma \text{ limit ordinal).}
\end{aligned}$$

Finally : $\Delta := \Delta_\kappa$.

Comments: This is similar to the construction of Ω (in section 3), except that we go beyond ω and use $\mathcal{L}_\kappa$-formulas instead of $\mathcal{L}$-formulas. Naturally are the sequences $\vec{y}$ and $\vec{b}$ here κ-finite sequences, i.e. of type $(y_\alpha)_{\alpha<\beta}$ and $(b_\alpha)_{\alpha<\beta}$ (with $\beta < \kappa$).
As we suppose κ inaccessible, it is clear that each Δ_α (with $\alpha < \kappa$) is κ-finite, and that Δ is exactly of cardinality κ. Each element of Δ is a closed term, but naturally no longer necessarily a **finite** term; we call that a "κ-term", and go on denoting "$\varphi_\tau(x)$" the infinitary formula defining the κ-term τ. We can then use the machinery developed in section 3, modulo the obvious adaptation of the notion of "admissible model". Fundamentally the proofs will stay the same (with the obvious interpretation for $\bigvee\!\!\!\bigvee_{\alpha<\beta} x = y_\alpha$ as an infinite disjunction).
At the end one gets a "generic structure" that we denote $G^\star$, and analogues for theorems 3.10 and 3.11. As however the $G^\star$ obtained here is not an

usual "pure term model" (because most terms are not finite), we prefer to reformulate the most interesting results in theorem 4.1.
Before that, we will show that $G^\star$ contains "naturally" the transitive set V_κ (from the Von Neumann hierarchy), constructed (in the metatheory) as usually, by :

$$\begin{array}{rcl} V_0 & := & \emptyset \\ V_{\alpha+1} & := & \mathcal{P}V_\alpha \ \text{ (where } \mathcal{P} \text{ is the power set operator)} \\ V_\gamma & := & \bigcup_{\beta<\gamma} V_\beta \ \text{ for } \gamma \text{ limit ordinal)} \end{array}$$

So put some well-ordering relation $\leq^\star$ on V_κ (via the axiom of choice) and define inductively the interpretation $z^\star$ of an element z of V_κ :

$$\begin{array}{rcl} \emptyset^\star & := & \{x|\bot\} \\ z^\star & := & \{x|\bigvee\!\!\!\!\bigvee_{\alpha<\beta} x = a_{\alpha^\star}\}, \text{ when } z = \{a_\alpha|\alpha < \beta\} \\ & & \text{and } (a_\alpha)_{\alpha<\beta} \text{ is the unique } \kappa\text{-finite (i.e. } \beta < \kappa) \text{ enumeration} \\ & & \text{of the elements of } z, \text{ increasing for } \leq^\star \text{ (i.e. } \alpha \leq \alpha' \leftrightarrow a_\alpha \leq^\star a_{\alpha'}). \end{array}$$

Obviously, each $z^\star$ belongs to Δ (for $z \in V_\kappa$), and we get a "copy" of V_κ in Δ :

$$\{z^\star|z \in V_\kappa\} \subset \Delta.$$

We can now precise the adequate notion of "admissible model", that we use here : it is a model of type $M = (\Delta, \in_M, =_M, C_M(x))$, that satisfies pre-abstraction (for the elements of Δ), where $=_M$ is a first-order congruence (w.r.t. $\in_M$) and where the interpretation C_M of the unary first-order predicate C satisfies : $C_M(x)$ iff $\exists z \in V_\kappa \ \ x =_M z^\star$. The reader can easily check that $=_M$ is then also automatically a congruence w.r.t. $C_M(x)$. Also has the predicate $C(x)$ the agreeable property of preservation under extension (what justifies naturally that $C(x)$ is considered here as a "positive formula") : for M, N admissible models, with $M \leq N$, we have : $C_M(x) \to C_N(x)$.
Thanks to the fact that V_κ is (in the metatheory) a transitive, well-founded set, can one easily check that the class $C := \{x|C(x)\}$, seen in the "generic" model $G^\star$, is a transitive class, isomorphic to V_κ (up to $=_{G^\star}$); more precisely : $(C/=_{G^\star}, \in_{G^\star})$ and $(V_\kappa, \in)$ are isomorphic; where $C/=_{G^\star} := \{[x]|x \in C\}$ and $[x] := \{y \in \Delta|x =_{G^\star} y\}$.
And so, as V_κ (structured by $\in$) satisfies the axioms of ZF (thanks to the uncountable, inaccessible κ), so does the class C in $G^\star$.

Notice further that C is even a **set** in $G^\star$ (as $\{x|C(x)\}$ is a term, member of Δ).
In order to summarize the most interesting facts about $G^\star$, we introduce the language $\mathcal{L}^\star_C$, that simply extends $\mathcal{L}^\star$ (see section 2) via the new predicate $C(x)$, and is built up like $\mathcal{L}^\star$, by allowing also the extra rule : $C(x)$ is a positive $\mathcal{L}^\star_C$-formula.

Theorem 4.1. *(1) $G^\star$ satisfies Abst($\mathcal{L}^\star_C$),*

(2) $=_{G^\star}$ is a strong congruence w.r.t. $\mathcal{L}^\star_C$,

(3) $G^\star$ satisfies Intrule$_{\mathcal{L}^\star_C}(\Gamma_2)$, where Γ_2 is defined like Γ_1 (see section 3, theorem 3.11), but with $\mathcal{L}^\star_C$ instead of $\mathcal{L}^\star$,

(4) the set $\{x|C(x)\}$ interprets ZF (in $G^\star$), i.e. $G^\star$ satisfies σ_C for any σ axiom of ZF (where σ_C is obtained from σ by bounding each quantification to the set $\{x|C(x)\}$),

(5) $G^\star$ satisfies the following "generalized replacement" principle : if F is a class-function (i.e. definable via some, not necessarily positive, $\mathcal{L}^\star_C$-formula, possibly with parameters), then : $\forall a[C(a) \to \exists b\ \forall t\ (t \in b \leftrightarrow (\exists z \in a\quad F(z) = t))]$.

Hints for the proofs: The proofs of (1), (2), (3) are just adaptations of those in section 3. Point (4) is clear from the previous remark that $\{x|C(x)\}$, seen in $G^\star$, is a "copy" of V_κ. Point (5) results from the fact that the collection (subset of Δ) $\{F(z)|z \in_{G^\star} a\}$ will be κ-finite for each a such that $G^\star \models C(a)$, and so be a set in the sense of $G^\star$, simply by the construction of Δ.

Comments:

- The known models of GPK^+_∞ (the strong "positive set theory", allowing inter alia "generalized positive comprehension") actually also satisfy theorem 4.1 ! This is relatively easy to check, and we refer the interested reader to [4], [6]. Their construction however involves a stronger hypothesis than ours, namely the existence of an uncountable, weakly measurable cardinal, so that the modelized theory is indeed stronger, but at a higher price !
- $G^\star$ modelizes a theory of type "ZF in T", in a sense introduced by Boffa [2], and further explored in [12].

Also is the interpretation of ZF via the set $\{x|C(x)\}$ a very "standard" one, in the sense of [14]; about that should we just precise that [14] assumes extensionality in the theory T that "contains ZF"; as here we have only "intensionality", should one take the expectations in [14] "cum grano salis", i.e. adapt them by replacing here and there the idea that some set v is in C by the one that some v' with the same extension as v (i.e. $\forall t(t \in v \leftrightarrow t \in v')$) is in C (so not necessarily v itself).

5. About partial information

The "partial set" theories initiated by P.C. Gilmore [8] can be seen as formalizations of the following natural concepts concerning "partial information" : for a given "property" (predicate) $P(x)$, we can imagine the "set" $\{x|P(x)\}$ as a "double list" based on "positive" and "negative" information; say the left side of the list mentions the objects x for which we got the information that $P(x)$ is true, what we denote $P^+(x)$; while the right side mentions those x's for which we got the information that $P(x)$ is false (notation : $P^-(x)$). We conceive such lists as (probably) incomplete, but however reliable; and we formalize these expectations by :

- $\forall x \ \ \neg(P^+(x) \wedge P^-(x))$
- $z \in^+ \{x|P(x)\} \rightarrow P^+(z)$
- $z \in^- \{x|P(x)\} \rightarrow P^-(z)$

where $\in^+$, $\in^-$ are the "memberships" corresponding respectively to the left/right sides of the lists.
So we find back the "partial aspect" and also pre-abstraction.
Abstraction itself can then be conceived, for a given set $\{x|P(x)\}$, as the fact that, at that moment, the double list is "complete"; there is indeed a dynamic aspect in this, namely the idea that these lists grow w.r.t. time; and that explains the "positivity" expectations concerning the "formulas" P^+ and P^- (as precisely positive formulas are preserved under the kind of "extensions" considered here).
Now can "positive set theory" be seen as a simplification of "partial set theory"; in terms of partial information is a "set" $\{x|P(x)\}$ then reduced to a simple list (instead of a "double" one), that mentions those x for which we

got the information that $P(x)$ is true.
Again does pre-abstraction then express that the information, however incomplete, is at least reliable ; and abstraction expresses "completeness".
All this is just one possible interpretation, giving an "intuitive" basis for "positive set theory".
Further is there also some hope that adequate adaptations of the techniques in this paper might solve the still open problems concerning the compatibility of intensionality (or even extensionality) with several forms of "partial set theory" (see for example [13], [14]). About that should it be mentioned that the successfull constructions for similar problems in the "positive" and "paradoxical" cases simply don't work in the "partial" case (see for example [5], [13], [14]).
That adaptations are necessary if one wants "forcing" results concerning partial sets, can easily be seen in several proofs of section 3, namely those where one has to construct extensions of G : the problem is that we miss arguments to ensure that situations like "$x \in^+ y \wedge x \in^- y$" won't occure ! So far it is not obvious how to overcome this, but the hope remains...

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