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EXTENSIONS OF INTUITIONISTIC LOGIC WITHOUT THE DEDUCTION THEOREM: SOME SIMPLE EXAMPLES

A b s t r a c t. We provide some illustrations of consequence relations extending that associated with intuitionistic propositional logic but lacking the Deduction Theorem, together with a discussion of issues—of some interest in their own right—raised by these examples. There are two main examples, with some minor variations: one in which the language of intuitionistic logic is retained, and one in which this language is expanded.

1. Introduction

A consequence relation $\vdash$ over a language one of whose connectives is the binary $\rightarrow$ is said to satisfy the Deduction Theorem when the following condition is met for all sets of formulas Γ (of that language) and formulas A, B:

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(DT) *If* $\Gamma, A \vdash B$ *then* $\Gamma \vdash A \to B$.[1]

The use of the phrase 'Deduction Theorem' to describe this behaviour is based on a special case (or special range of cases), namely when $\vdash$ has been defined on the basis of a Hilbert-style axiom system, and understood to mean that the formula on the right can be obtained by successive applications of the system's rules from those on the left together with the axioms. When $\Gamma \vdash C$ for a $\vdash$ so understood, a sequence of formulas terminating in C each of whose earlier members is an axiom, an element of Γ or the result of applying a rule to still earlier members is called a *deduction* of C from Γ (on the basis of the given axiom system). The latter parenthesis we omit from this formulation of the procedure:

(*) $\Gamma \vdash C$ *iff there is a deduction of* C *from* Γ.

Now, although we use the phrase 'Deduction Theorem' in our title, it is not quite appropriate to what is really the more general condition (DT), a condition satisfied by consequence relations that have been defined syntactically in quite different ways, or consequence relations specified entirely semantically – 'not quite appropriate' because there is no appeal in these cases to a deduction as a witness *à la* (*) for the claim that $\Gamma \vdash C$. (For convenience we will help ourselves to a number of locutions with the same origin, generalized to apply in the case of consequence relations other than as provided by (*), such as talk of the 'theorems' of $\vdash$ for formulas A such that $\vdash A$.)

Among the 'different ways' we have in mind here are proof systems which manipulate sequents using sequent-to-sequent rules (by contrast with the formula-to-formula rules of the Hilbert approach). For present purposes we take sequents to be pairs $\langle \Gamma, C \rangle$ in which Γ is a finite set of formulas and C a formula, though of course many alternative conceptions are possible,[2] and for greater suggestiveness we write such a pair in the notation '$\Gamma \succ C$'

[1] We presume familiarity with the general notion of a consequence relation (see for example [6], p.5, conditions (1), (2) and (3)), and avail ourselves of the liberties usually taken with the "$\vdash$" notation, writing "$\Gamma, A \vdash B$" for "$\Gamma \cup \{A\} \vdash B$", "$\vdash C$" for "$\varnothing \vdash C$", "$A \dashv\vdash B$" for "$A \vdash B$ and $B \vdash A$", etc. The object-languages under consideration in what follows differ only in respect of primitive connectives, all sharing a common basis in a countable supply of propositional variables (sentence letters) $p_1, \ldots, p_n, \ldots$, three of which we shall write as p, q, r. (The letter s is reserved for use as a variable ranging over substitutions.)

[2] We could, for example, allow Γ to be an infinite set of formulas, or – see the following

following the example set by Blamey in, e.g., [5], Section 7. If amongst the rules under which the set of provable sequents is closed are the structural rules of weakening (also called thinning or monotonicity), cut (or transitivity), and identity (or reflexivity, the zero-premiss sequent-to-sequent rule: $A \succ A$), then the relation $\vdash$ defined by (**), in which 'provable' means 'provable in the given proof system':

(**) $\Gamma \vdash C$ *iff for some finite* $\Gamma_0 \subseteq \Gamma$, *the sequent* $\Gamma_0 \succ C$ *is provable*

is a (finitary) consequence relation.[3] If the proof system is a sequent calculus proper (or 'Gentzen system'), in other words, with all the operational rules inserting logical vocabulary either on the left or on the right of the $\succ$, then the consequence relation induced by the above definition will typically satisfy (DT) in view of the usual rule for inserting $\to$ on the right. If it is, instead, a natural deduction system, i.e., has operational rules introducing or eliminating logical vocabulary specifically on the right of the $\succ$, possibly with the concomitant disappearance of formulas from the left ('discharge of assumptions'), then (DT) reflects the typical workings of an $\to$-introduction or 'Conditional Proof' rule.[4] And so on.

In the non-axiomatic contexts just sampled it is, as already remarked, somewhat misleading to unpack the abbreviation (DT) to 'Deduction Theorem', but the terminology has stuck, and indeed been extended (e.g. in [7] and references therein) to cover the two-way form, the 'Deduction Detachment Theorem' or (DDT), which itself is sometimes called simply the Deduction Theorem (e.g. in [10] and earlier work by Czelakowski and many others):

(DDT) $\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \to B$.

note – not a set of formulas at all; we could also allow in place of a single formula a right, multiple or empty right-hand sides. (The corresponding variation on the notion of a consequence relation goes under the name "generalized consequence relation" in our discussion at the end of this section.)

[3]We are working with the notion of sequent introduced above, with sets of formulas on the left of the separator $\succ$ rather than, sequences or again multisets of formulas, as is more common especially in sequent calculus proof systems; for those formulations, "$\Gamma_0 \subseteq \Gamma$" in (**) would be replaced by "Γ_0, all formulas appearing in which belong to Γ".

[4]As we are envisaging things here, the sequent calculus rule ("$\to$ Right") for inserting an impliction on the right is the rule as the natural deduction $\to$-introduction rule; this rule appears as (††) at the start of Section 3 below.

It is a familiar fact that the 'if' direction of (DDT) is satisfied by a consequence relation $\vdash$ just in case for all formulas A, B, in the language of $\vdash$, we have $A, A \to B \vdash B$, while the 'only if' direction of (DDT), i.e., (DT) itself, does not admit of any such unconditional reformulation. By an *extension* of a consequence relation $\vdash$ we mean any $\vdash'$, not necessarily over the same language as $\vdash$, for which $\vdash \subseteq \vdash'$.[5] No less familar than the fact just cited is the following corollary: if a consequence relation satisfies the 'if' direction of (DDT) then so does any extension *in the same language* of that consequence relation, while the corresponding claim for (DT) itself fails. In more detail: let us call a consequence relation $\vdash$ *substitution-invariant* when $\Gamma \vdash A$ implies $s(\Gamma) \vdash s(A)$ for all formulas $\Gamma \cup \{A\}$ of the language of $\vdash$ and all substitutions s over that language.[6] We can drop the italicized "in the same language" proviso from the previous formulation by restricting attention to substitution-invariant consequence relations, which have in any case always been regarded as the main candidates to deserve the title of 'logics' when logics are thought of as consequence relations: (1) if the right-to-left (i.e., "detachment", or *Modus Ponens*) direction of (DDT) is satisfied by a consequence relation $\vdash$ then it is also satisfied by any

[5]In the usage of some authors, e.g., Blok and Pigozzi (see [6], p.6), the term 'extension' is reserved for the case in which $\vdash'$ has the same language as $\vdash$. The present understanding of 'extension', however, has considerable currency: consider, for example, the distinction between conservative and non-conservative extensions, a distinction arising only when the 'extending' logic is cast in a language which (considered as an algebra) is an expansion of the language of the original logic. Of course, when the two languages differ, it can only be because the language of the extending logic has some vocabulary additional to that in the language of the extended logic.

[6]$s(\Gamma)$ is $\{s(C) \mid C \in \Gamma\}$. Substitution-invariant consequence relations are often called structural consequence relations. However, we wish to use the same terminology here as in the case of rules—see Section 2—where the terminology of substitution-invariance is greatly preferable to that of structurality because of a conflict in the latter case with Gentzen's notion of a structural (as opposed to operational) rule, already deployed in Section 1 *à propos* of weakening, etc. (A substitution-invariant rule is a rule with the feature that any substitution instance of an application of the rule is itself an application of the rule. The terminology of invariance under substitutions is taken from Mints [33]. A susbtitution-invariant consequence relation is not at all the same as what several Polish writers – see, e.g., [14] – call an invariant consequence relation, which is one for which $s(A)$ is a consequence of A, for all formulas A of the relevant language and all substitutions s.) We retain, however, the phrase 'structural completeness' and the like, below, which echoes this connection with substitutions, since it is so well established and is unlikely to give rise to any confusions.

substitution-invariant extension of $\vdash$, while (2) the same cannot be said of the left-to-right direction, (DT), of (DDT). In particular, Rautenberg [40] stressed that *intermediate* consequence relations, by which is meant those substitution-invariant $\vdash$ such that $\vdash_{IL} \subseteq \vdash \subseteq \vdash_{CL}$, need not satisfy (DT).[7] (Here $\vdash_{IL}$ and $\vdash_{CL}$ are the consequence relations of intuitionistic and classical propositional logic respectively.) In what follows we will illustrate this possibility with two reasonably simply described examples. The first, which will occupy us in Section 2 (where it appears in Corollary 2.3), concerns the usual (shared) language of $\vdash_{IL}$ and $\vdash_{CL}$.[8] The second example, presented in Section 3 (where it appears as Example 3.3, with a variation on the theme at Example 3.8), involves an expansion of the implicational fragment of this language by one additional connective and so does not strictly count as an intermediate consequence relation by the above definition. It does, however, indicate in a rather concrete way how (DT) can fail even when $\to$ behaves as $\vdash_{IL}$ demands. Further, the substitution-invariant consequence relation we use to illustrate this possibility has a straightforward description in terms of Kripke models. Each of Sections 2 and 3 ends with

[7]Our usage follows that of [40] and contrasts, e.g., with that of [11], which does not require intermediate consequence relations to be substitution-invariant, but does require them to satisfy (DT). Taking a *superintuitionistic* consequence relation to be any substitution-invariant extension of $\vdash_{IL}$ in the same language or a fragment thereof, and calling $\vdash$ *consistent* when for some A, $\nvdash A$ (equivalently, when $\vdash \neq \wp(L) \times L$, L being the language of $\vdash$) the classes of consistent superintuitionistic consequence relations and intermediate consequence relations coincide for any fragment containing $\to$, which will be the case for all fragments under consideration here—with the exception of those mentioned in the Digression following Corollary 2.3 below or alluded to in note 7—in view of our interest in (DT). Such observations go back to work by Tarski in the 1930s in which he showed that the implicational prerequisites for having classical logic as the only Post-complete extension can be reduced from those for intuitionistic implication to those for *BCK* logic. See [12] for a pleasant exposition in the pure implicational case and [4] for discussion and references to some other fragments. (The notion of a fragment will be formally defined below, after Proposition 2.1.) Because we are considering logics as consequence relations, the claim that we fall into inconsistency as soon as we fail to be included in classical logic has to be modified for certain atheorematic fragments – i.e., fragments in which the empty set has no (intuitionistic) consequences (these being the 'purely inferential' consequence relations of Wójcicki [54]) – where we can instead arrive at the largest atheorematic consequence relation $\vdash$, for which $\Gamma \vdash A$ for all *non-empty* Γ): not quite the inconsistent consequence relation because of the italicized proviso.

[8]To avoid impoverishing the language of $\vdash_{IL}$ here we presume the language of $\vdash_{CL}$ to have all of conjunction ($\wedge$), disjunction ($\vee$), implication ($\to$) and either negation ($\neg$) or the Falsum ($\bot$) as primitive.

a postscript remarking on the extent to which an example similar to the main case there given for a $\vdash \supseteq \vdash_{IL}$ without (DT) can be furnished for some $\vdash \supseteq \vdash_{CL}$.

Let us recall another example of the phenomenon with which we are concerned here. Like (DDT), the biconditional constraint governing disjunction in intuitionistic (as well as classical) logic, namely that for all Γ, A, B, we have $\Gamma, A \vee B \vdash C$ if and only if $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$. Again one direction is guaranteed to be inherited by extensions (of $\vdash$, in the same language), here the "only if" direction, since this is equivalent (as a condition on consequence relations) to the *unconditional* requirement(s) that $A \vdash A \vee B$ and $B \vdash A \vee B$, while the "if" direction is irreducibly conditional and has no such guaranteee of inheritance under extensions.[9] As is well known, if one uses generalized ("multiple-conclusion") consequence relations,[10] with the corresponding biconditional condition $\Gamma, A \vee B \vdash \Delta$ if and only if $\Gamma, A \vdash \Delta$ and $\Gamma, B \vdash \Delta$, then there is a reduction to an equivalent unconditional form for the "if" direction (namely $A \vee B \vdash A, B$) and inheritance to extensions – now generalized consequence relations extending the given $\vdash$, that is – is secured. Indeed it is also well known and easy to see that the (DT) half of (DDT), reformulated so as to support an arbitrary set (corresponding to the above "Δ") of side-formulas on the right, can be similarly reduced to a combination of the unconditional forms (namely $B \vdash A \to B$ and $\vdash A,\ A \to B$) just as the detachment half reduces to $A, A \to B \vdash B$ and so is inherited by extensions. We are concerned with consequence relations, however, and not with generalized consequence relations – especially since this generalized version of (DT) fails for intuitionistic logic when the "commas on the right" are, as for the above foray into multiple succedents for the treatment of disjunction, interpreted in such a way as to make them equivalent to (iterated) disjunctions.

[9]Rautenberg [40] considers the satisfaction amongst intermediate logics of this conditional form, or a closely related requirement he calls monotonicity ($B \vdash C$ implies $A \vee B \vdash A \vee C$: no relation to monotonicity in the weakening/thinning sense) Note that for superintuitionistic $\vdash$, (DT) implies the satisfaction of the above condition, since if $\Gamma, A \vdash C$ and $\Gamma, B \vdash C$ (DT) delivers $\Gamma \vdash A \to C$ and $\Gamma \vdash B \to C$, which we can (thin and) "cut" with $A \to C, B \to C \vdash (A \vee B) \to C$ to conclude that $\Gamma \vdash (A \vee B) \to C$, and then by appeal to Detachment, $\Gamma, A \vee B \vdash C$.

[10]See, e.g., Segerberg [46] for details.

2. The Structural Completion Example

The idea of structural completeness, i.e., of the derivability of all admissible substitution-invariant rules, has two incarnations: as a property of consequence relations and as a property of proof systems – the latter being that most directly suggested by the 'rules' formulation just given. This contrast cross-cuts another distinction which we shall be expressing with the terminology of "weak" *vs.* "strong". Taking the former (i.e., consequence relations) case first, we use the following terminology.[11] A consequence relation $\vdash$ is *strongly structurally complete* (or "structurally complete in the infinitary sense") just in case for every set $\Gamma \cup \{B\}$ of formulas of the language of $\vdash$, (1), below, implies (2), and *weakly structurally complete* (or "structurally complete in the finitary sense") when the same implication holds for all finite Γ:

(1) For every substitution s for which we have $\vdash s(A)$, for each $A \in \Gamma$, we have $\vdash s(B)$;

(2) $\Gamma \vdash B$.

We say "weakly structurally incomplete" ("strongly structurally incomplete") to mean "not weakly structurally complete" ("not strongly structurally complete"). Thus while strong structural completeness implies weak structural completeness (and not in general conversely), it is weak structural incompleteness that implies strong structural incompleteness.

Although the notion of strong structural completeness as applied to consequence relations is what we need for our example, which appears as Corollary 2.3 below, we pause to distinguish the other notion, structural completeness as applied to proof systems. Take for example proof systems in which the objects proved are sequents in the sense of Section 1

[11] The finitary/infinitary terminology mentioned parenthetically in what follows is taken from Makinson [29]; Rautenberg [39] uses "strong structural completeness" for (what amounts to) the infinitary sense, somewhat by analogy with talk of strong semantic completeness, and we have introduced "weak" for the contrasting case. Tsitkin [52] refers to these as "modus completeness" and "structural completeness", the former based on the use of the phrase "modus rule" for what [28] calls a *sequential* rule. (*Warning*: sequential rules in this sense are those whose applications are precisely the transitions from $s(A_1), \ldots, s(A_n)$ to $s(B)$ for arbitrary substitutions s, and some fixed sequent $A_1, \ldots, A_n \succ B$ – the latter explaining the terminology, which should accordingly not be confused with what we are calling sequent-to-sequent rules.)

above. Certain primitive sequent-to-sequent rules are laid down[12] and we assume familiarity with what it takes for another such rule to be derivable from any given set of such rules, as well as with the fact that not every rule under which the set of provable sequents is closed – every admissible rule for the proof system, that is – need be derivable. When a substitution-invariant rule has this "admissible but not derivable" status for a particular proof system, the proof system is structurally incomplete. As in the consequence relations case, there is a distinction between strong and weak structural completeness to be made, the strong version meaning that every admissible substitution-invariant rule, even including infinitary rules (rules with infinitely many premisses, that is), is derivable, and the weak version requiring this only of the finitary admissible rules. The cut rule for the cut-free version of the Gentzen system LJ was an example cited in the early literature on rules (namely in [53]) of a rule that was admissible though not derivable; since this is a substitution-invariant finitary rule (extending the idea of a substitution instance from formulas to sequents in the obvious way), the cut-free version of LJ is a weakly structurally incomplete proof system.[13] In this case, the consequence relation obtained by the definition (**) from Section 1 is $\vdash_{IL}$ and this is (likewise) weakly structurally incomplete. Such a coincidence in respect of structural completeness between a proof system and structural completeness of the associated consequence relation is not to be expected in general, however, as we see if we restrict attention to the implicational subsystem of cut-free LJ. In this case, the proof system remains weakly structurally incomplete (since the cut rule is still a substitution-invariant admissible rule which is not derivable), while the induced consequence relation in this case is weakly structurally complete. (The weak structural incompleteness of the full $\vdash_{IL}$ and weak structural completeness of its $\rightarrow$ fragment[14] are well known; see Proposition 2.2 and Theorem 2.6(ii) below. On the other hand, the application of the structural (in)completeness terminology to proof systems

[12]Here we subsume any 'initial sequents' under this rubric, as 0-premiss sequent-to-sequent rules, as in the case of the schema $A \succ A$ in Section 1.

[13]Strictly speaking, Wang [53] is concerned (in Theorem 2 and the paragraph preceding it on p.194 of [53]) with formula-to-formula rules which resemble the sequent-calculus rules in the respects under consideration here. Wang, like many of the Polish writers we cite, uses the word "permissible" in place of "admissible".

[14]An explanation of this talk of fragments, in case one is needed, follows Proposition 2.1 below.

with substitution-invariant admissible non-derivable sequent-to-sequent – as opposed to formula-to-formula – rules is not widespread.)

The connection between structural completeness as applied to proof systems and structural completeness as applied to consequence relations arises from the special case in which the proof system under consideration is an axiom system and so its rules are formula-to-formula rules, in which case saying that the proof system is (e.g., weakly) structurally complete is equivalent to saying that a certain consequence relation is (weakly) structurally complete, namely the consequence relation defined on the basis of the given axiomatization by the procedure summarized in (*) in Section 1 above.[15]

We return to the concept of structural completeness as it applies to consequence relations. The *structural completion* $\vdash^{\mathsf{sc}}$ of a consequence relation $\vdash$ is defined thus:

(†) $\Gamma \vdash^{\mathsf{sc}} B$ iff for every substitution s with $\vdash s(A)$, for each $A \in \Gamma$, we have $\vdash s(B)$.

(More explicitly, given the quantification over arbitrary Γ we might call this, as is done in [39], the "strong structural completion" of $\vdash$.) Note that, so defined, $\vdash^{\mathsf{sc}}$ is always substitution-invariant, even if $\vdash$ is not.[16] Focussing on the case of substitution-invariant $\vdash$, we have the following (see [29] and [39] for (ii), parts (i) and (iii) being even more immediate consequences of (†)):

[15]Here we ignore for simplicity the distinction amongst formula-to-formula rules between—to use Smiley's terminology—*rules of proof* and *rules of inference* (provability rules and consequence rules, respectively, in the terminology of Gabbay [16], p.9), certainly necessary to do full justice to issues surrounding the Deduction Theorem: See [49], note 3 and also p.123.

[16]Rybakov ([41], p.89) calls the structural completion of a consequence relation its "admissible closure" and writes "$\vdash^{Ad}$" where we write "$\vdash^{\mathsf{sc}}$" (More accurately, Rybakov does this for consequence relations defined on the basis of axiom systems via the definition (*) of Section 1. Most of [41] conducts the discussion of these issues in terms of formula-to-formula rules—admissible or derivable—for proof systems, rather than in terms of consequence relations, the latter only entering the picture briefly in §1.7.) Wojtylak [55], as with most in the Polish tradition, works with consequence operations rather than consequence relations, where an analogous notion of structural completion is available, denoted in [55] by C^{σ} for any given consequence operation C.

Proposition 2.1. *For any substitution-invariant consequence relation* $\vdash$*:*

(i) $\vdash A$ *if and only if* $\vdash^{\mathsf{sc}} A$*, for all formulas* A*.*

(ii) $\vdash^{\mathsf{sc}}$ *is the largest substitution-invariant consequence relation in the language of* $\vdash$ *meeting the condition mentioned in (i) (i.e., agreeing with* $\vdash$ *on the consequences of the empty set).*

(iii) $\vdash$ *is strongly structurally complete if and only if* $\vdash = \vdash^{\mathsf{sc}}$*, and weakly structurally complete if and only if for all* $A_1, \ldots, A_n, B$ *we have* $A_1, \ldots, A_n \vdash B$ *just in case* $A_1, \ldots, A_n \vdash^{\mathsf{sc}} B$*.*

To proceed further we need the notion of a fragment. The X_0-*fragment* of a (propositional) language L having for its set of primitive connectives $X \supseteq X_0$ comprises those formulas that can be constructed using only the connectives X_0. Call the set of such formulas L_0. If, in such a case, $\vdash$ is a consequence relation over L, then the X_0-*fragment* of $\vdash$ is defined to be $\vdash \cap (\wp(L_0) \times L_0)$.[17]

The weak structural incompleteness of $\vdash_{IL}$ was known from examples supplied by Harrop before the terminology of structural (in)completeness had even been introduced.[18] Our example in the proof below is taken from Mints [33], as is the proof itself.[19]

[17]It is sometimes also convenient to speak of the X_0-fragment of a language or consequence relation when X_0 contains connectives defined rather than primitive, in which case they are taken as new primitives in the language L_0. Thus one may speak (as for example below, in the paragraph following the proof of Theorem 2.6) of the $\{\leftrightarrow, \rightarrow\}$ fragment of $\vdash_{IL}$ even when (as here) the language of the latter consequence relation has not been formulated with $\leftrightarrow$ as a primitive connective.

[18]As far as the author is aware the first occurrence in English of this terminology is in Pogorzelski [35]. Dummett, whose [13] appeared two years later, though in preparation over the previous decade, had had to invent his own terminology ("smooth" for structurally complete: see [13], p.436). The present author does not find congenial Dummett's way of putting his formal definition in informal English, however, according to which the smoothness consists in having every rule of proof be a rule of inference, because the derivable/admissible distinction arises for rules of proof themselves, though we are ignoring these refinements for present purposes, as remarked in note 15.

[19]Further examples of intuitionistically admissible though not derivable (formula-to-formula) rules may be found in [41], esp. Examples 3.5.19 (Harrop's Rule), 3.5.21 (Scott's Rule), and 3.51 (a strengthening of Scott's Rule), with Mints's Rule appearing as Example 3.5.29. Rybakov also has some neat techniques for establishing admissibility, and [41] further provides a general discussion of the topic, as well as detailed referencess not included here.

Proposition 2.2. *Any fragment of* $\vdash_{IL}$ *containing disjunction and implication (and thus the usual full language with* $\rightarrow$, $\vee$, $\wedge$, *and* $\neg$ *or* $\bot$) *is weakly structurally incomplete.*

Proof. By Proposition 2.1(iii) it suffices to exhibit any divergence between $\vdash_{IL}$ and $\vdash^{\mathsf{sc}}_{IL}$ in respect of the consequences of a finite set, which we do thus:

(1) $(p \rightarrow r) \rightarrow (p \vee q) \vdash^{\mathsf{sc}}_{IL} ((p \rightarrow r) \rightarrow p) \vee ((p \rightarrow r) \rightarrow q)$, but
(2) $(p \rightarrow r) \rightarrow (p \vee q) \nvdash_{IL} ((p \rightarrow r) \rightarrow p) \vee ((p \rightarrow r) \rightarrow q)$.

(1) may be verified by a consideration of any possible cut-free LJ proof of a sequent of the form $\succ (A \rightarrow C) \rightarrow (A \vee B)$, checking that such a proof is available only if there is also a proof of $\succ ((A \rightarrow C) \rightarrow A) \vee ((A \rightarrow C) \rightarrow B)$. (2) can be checked using the Kripke semantics for intuitionistic logic, or alternatively by noting the absence of a cut-free LJ proof (for the corresponding sequent). □

Let us remark that, by a frequently employed device (*cf.* note 37, esp. formulation (2)), we can eliminate the disjunction in the conclusion—though not that in the premiss—of Mints's Rule, replacing the latter, i.e.:

$$\frac{(A \rightarrow C) \rightarrow (A \vee B)}{((A \rightarrow C) \rightarrow A) \vee ((A \rightarrow C) \rightarrow B)}$$

with:

$$\frac{(A \rightarrow C) \rightarrow (A \vee B)}{((A \rightarrow C. \rightarrow A) \rightarrow D) \rightarrow [((A \rightarrow C. \rightarrow B) \rightarrow D) \rightarrow D]}$$

where we have used dots to cut down on parentheses. Next, the reason for taking all this interest in $\vdash^{\mathsf{sc}}_{IL}$:

Corollary 2.3. $\vdash^{\mathsf{sc}}_{IL}$ *is an intermediate consequence relation not satisfying (DT).*

Proof. We leave the reader to verify that $\vdash^{\mathsf{sc}}_{IL}$ is an intermediate consequence relation (i.e., is a substitution-invariant $\vdash$ such that $\vdash_{IL} \subseteq \vdash \subseteq \vdash_{CL}$). For the failure of (DT), recall (from (1) of the proof of Proposition 2.2) that

$$(p \to r) \to (p \vee q) \vdash^{\mathsf{sc}}_{IL} ((p \to r) \to p) \vee ((p \to r) \to q).$$

So if (DT) were satisfied we should have:

$$\vdash^{\mathsf{sc}}_{IL} ((p \to r) \to (p \vee q)) \to [((p \to r) \to p) \vee ((p \to r) \to q)].$$

But this, via Proposition 2.1(i), would mean we had:

$$\vdash_{IL} ((p \to r) \to (p \vee q)) \to [((p \to r) \to p) \vee ((p \to r) \to q)],$$

and this would contradict (2) of the proof of Proposition 2.2, since $\vdash_{IL}$ satisfies the 'detachment' half of (DDT), as do – we recalled in Section 1 – all of its substitution-invariant extensions. □

As a further corollary, we may conclude that $\vdash^{\mathsf{sc}}_{IL}$ is not an 'axiomatic extension' of $\vdash_{IL}$, in the sense that there is no set of formulas Θ such that for all Γ, A we have $\Gamma \vdash^{\mathsf{sc}}_{IL} A$ just in case $\Gamma, \Theta \vdash_{IL} A$. It is not hard to see that any axiomatic extension of a consequence relation satisfying (DT) will itself have to satisfy (DT). (For a much stronger result in this vein see Theorem 3.4 in Wojtylak [55].)

Digression. Mints [33] also has something of a converse to Proposition 2.2, which we can put like this: whenever, for the full consequence relation $\vdash_{IL}$ of intuitionistic propositional logic, we have $A_1, \ldots, A_n \vdash^{\mathsf{sc}}_{IL} B$ without having $A_1, \ldots, A_n \vdash_{IL} B$, then both $\to$ and $\vee$ occur in the construction of the formulas $A_1, \ldots, A_n, B$. This does not imply that the $\vee$-free and $\to$-free fragments of $\vdash_{IL}$ are weakly structurally complete, contrary to the following formulation of Theorem 1, addressing the case of $\to$ in Mints [33]: "Every (finitary) rule admissible in a corresponding fragment of intuitionistic logic not including $\to$ is derivable in intuitionistic logic."[20] Interpreted literally, this is not correct. Since the intuitionistic and classical $\{\wedge, \neg\}$-consequences of the empty set coincide (by Gödel [17]), we have, where $\vdash$ is the fragment of $\vdash_{IL}$ in these connectives, $\neg\neg p \vdash^{\mathsf{sc}} p$ while $\neg\neg p \nvdash p$. In terms of rules: the (formula-to-formula) rule of double negation elimination is admissible but not derivable for the current ($\to$-free) fragment. What Mints evidently

[20]Here we have adjusted in an inessential way aspects of Mints's wording to match our terminology. The author is grateful to Mints for explaining—in the face of such apparent counterexamples as that from the negation–conjunction fragment, above—what he had in mind with the talk of structurally (in)complete fragments of intuitionistic logic in [33].

has in mind is not so much the admissibility of a rule in a fragment of $\vdash_{IL}$ as the admissibility *for full* $\vdash_{IL}$ of a rule which can be formulated in the fragment in question.[21] An even more dramatic example is provided by the pure $\neg$ fragment of $\vdash_{IL}$, which, like that of $\vdash_{CL}$, is atheorematic (in the sense of note 7); taking $\vdash$ as this consequence relation, we accordingly have $A \vdash^{\mathsf{sc}} B$ for all formulas A, B, though this is not so for $\vdash$ itself – attesting to weak structural incompleteness for this fragment of $\vdash_{IL}$ (as well as $\vdash_{CL}$). To put the point in terms of $(\cdot)^{\mathsf{sc}}$: one must distinguish, e.g., the structural completion of the $\{\wedge, \neg\}$-fragment of $\vdash_{IL}$, from the $\{\wedge, \neg\}$-fragment of the structural completion of $\vdash_{IL}$. According to the former consequence relation – but not the latter – p is a consequence of $\neg\neg p$. ***End of Digression***.

If we had wanted to illustrate the failure of (DT) in an extension of minimal logic rather than, as our title promises, intuitionistic logic, using the structural completion idea, we could have done so more simply than Corollary 2.3 does. Since minimal logic, conceived as a consequence relation – and denoted here by $\vdash_{ML}$ – has the same $\{\rightarrow, \wedge, \vee\}$ fragment as $\vdash_{IL}$, the same example, namely that provided by Mints's Rule, could be used here to show that $\vdash_{ML}$ is not structurally complete and its structural completion fails to satisfy (DT). But there is a simpler example that could be used to make the same point(s); we presume for definiteness that $\neg$ is taken as a primitive connective.

Example 2.4. $p, \neg p \vdash^{\mathsf{sc}}_{ML} q$ since no formula and its negation are provable in minimal logic. Thus since (by contast with $\vdash_{IL}$) $p, \neg p \nvdash_{ML} q$, the consequence relation $\vdash_{ML}$ is not weakly structurally complete (and nor is its negation–implication fragment). (*Cf.* Theorem T4 in Tokarz [51], and the final paragraph of Perzanowski [34].) Also, and more to the point given our interest in failures of (DT), since we do not have $p \vdash^{\mathsf{sc}}_{ML} \neg p \rightarrow q$ (for instance consider the substitution of $p \rightarrow p$ for p), (DT) fails for even the negation–implication fragment of $\vdash^{\mathsf{sc}}_{ML}$.

[21] Results that look similar to Mints's appear in Latocha [27], though comparison is complicated by the fact that, unlike Mints, Latocha requires all fragments to contain $\rightarrow$, rendering it impossible even to consider the negation–conjunction case mentioned above. For the fragments he does consider, however, when Latocha claims (weak) structural completeness, he means them in the present sense, rather than in Mints's sense – despite his crediting some results involving this notion – e.g., Theorem 2 on p.20 of [27]) – to Mints.

Putting the structural incompleteness aspect of Example 2.4 in terms of rules, the substitution-invariant rule licensing the transition from formulas A and $\neg A$ as premisses to an arbitrary B as conclusion is not just admissible for minimal logic, though not derivable (in, say, any axiomatization of minimal logic with *Modus Ponens* as the sole primitive rule), attesting to structural incompleteness – the rule is *vacuously* admissible, in the sense that there is no substitution s which takes both premisses of the rule to theorems of the logic. This phenomenon cannot arise for intuitionistic logic. That is, and now reverting to the consequence relation formulation, whenever for $A_1 \dots A_n$ there is no s with $\vdash_{IL} s(A_i)$ for all i ($1 \leq i \leq n$), we have $A_1, \dots, A_n \vdash_{IL} B$ for every formula B.[22] The $n = 1$ version of this claim (equivalent to the general form in the presence of $\wedge$) is, in the terminology of Porte [36] the claim that every s-antithesis is a d-antithesis (for $\vdash_{IL}$) – i.e., every formula with no provable substitution instance has every formula as a consequence; in Perzanowski [34] this is put in terms of linguistic gaps and omitted formulas, while Tokarz [51] uses the term "counter-thesis". (The present $A_1, \dots, A_n$ version, essential for application in the absence of conjunction, might be thought of as a linguistic gap distributed across the several formulas involved.)

Having supplied, with Corollary 2.3, the promised example of an intermediate consequence relation not satisfying (DT) we close this section by tidying up one loose end (in Theorem 2.6(ii)) and motivating one aspect of the example to be pursued in the next section (in Theorem 2.6(i)). Both issues concern the implicational fragment of $\vdash_{IL}$, and we cover them together in Theorem 2.6, part (i) of which is adapted from Prucnal [37] (though stated more generally) and part (ii) of which is, as proved here,

[22]To check this, we show the contrapositive: if $A_1, \dots, A_n \vdash_{IL} B$ for all B, then a substitution can be found that maps all of the A_i to intuitionistic theorems ($\vdash_{IL}$-consequences of $\varnothing$, that is). For this purpose, it is most convenient to imagine the language of $\vdash_{IL}$ to come with constants $\top$ and $\bot$, though what follows could be reworked with these replaced by $p \to p$ and its negation, respectively. We should then have to reformulate the appeal to the fact that when $\Gamma \cup \{C\}$ contains only 0-variable formulas, $\Gamma \vdash_{IL} C$ iff $\Gamma \vdash_{CL} C$. For the proof, then, suppose that for the given $A_1, \dots, A_n$, we have $A_1, \dots, A_n \nvdash_{IL} B$ for some B, so in particular $A_1, \dots, A_n \nvdash_{IL} \bot$. By (a version of) Glivenko's Theorem, $A_1, \dots, A_n \nvdash_{CL} \bot$. Thus, for some Boolean valuation v, $v(A_i) = T$ for each i. Defining $s_v(p_i) = \top$ for p_i with $v(p_i) = T$ and $s_v(p_i) = \bot$ when $v(p_i) = F$, we have $\vdash_{CL} s_v(A_i)$, for $i = 1, \dots, n$. Since s_v converts all formulas into variable-free formulas, we invoke the fact mentioned above, that restricted to such formulas classical and intuitionistic logic agree, to conclude that $\vdash_{IL} s_v(A_i)$ for each i.

from Wojtylak [55] (following a suggestion by A. Wroński). Although these results are not new, then, it is convenient for our purposes to see them together "under one roof" here. Their common prerequisite is what we might call Prucnal's Lemma (= Lemma 4 in [37]):

Lemma 2.5. *For any formula A, let s_A be the unique substitution satisfying $s_A(p_i) = A \to p_i$ for each variable p_i. Then we have, where $\vdash$ is the implicational fragment of $\vdash_{IL}$, for all formulas A, B: $s_A(B) \dashv\vdash A \to B$.*

Proof. For an arbitrary but fixed A, show by induction on the construction of B that $s_A(B) \dashv\vdash A \to B$. □

This lemma of Prucnal's, used in [37] to show the weak structural completeness of the implicational fragment of $\vdash_{IL}$, can be put to the same use for any implicational intermediate logic, just as Wojtylak [55] uses it to show (DT) for all such implicational intermediate logics (a result already established in Wroński [57]):

Theorem 2.6. *Let $\vdash$ be any substitution-invariant extension of the implicational fragment of $\vdash_{IL}$, in the same language. Then (i) $\vdash$ satisfies (DT), and (ii) $\vdash$ is weakly structurally complete.*

Proof. (i) Suppose that for $\vdash$ as described, we have $\Gamma, A \vdash B$. By substitution-invariance, we have $s_A(\Gamma), s_A(A) \vdash s_A(B)$. By Lemma 2.5 and the fact that $\vdash$ extends the implicational fragment of $\vdash_{IL}$, we have that $C \vdash s_A(C)$ for each $C \in \Gamma$, so $\Gamma, s_A(A) \vdash s_A(B)$. By the same considerations we have that $\vdash s_A(A)$ (since $\vdash A \to A$) and $s_A(B) \vdash A \to B$, allowing us to conclude, as desired, that $\Gamma \vdash A \to B$.

(ii) Again with $\vdash$ as described, suppose that $A_1, ..., A_n \nvdash B$, with a view to showing that $A_1, ..., A_n \nvdash^{\mathsf{sc}} B$. Let $s = s_{A_1} \circ \ldots \circ s_{A_n}$, where the substitutions s_{A_i} are as in Lemma 2.4 (and $\circ$ is for composition). Note that by that Lemma, $s(A_k)$ $(k = 1, \ldots, n)$ is intuitionistically equivalent to $A_1 \to (A_2 \to \ldots (A_n \to A_k) \ldots)$. But for each k, this formula is intuitionistically provable outright (i.e., is an $\vdash_{IL}$-consequence of $\varnothing$), so since these are formulas in the implicational fragment and $\vdash$ extends that fragment of $\vdash_{IL}$, we have $\vdash s(A_k)$ in each case. To complete the proof that $A_1, ..., A_n \nvdash^{\mathsf{sc}} B$, it remains only to show that $\nvdash s(B)$. Now, by Lemma 2.5 again, the formula $s(B)$ is $\vdash_{IL}$- and therefore $\vdash$-equivalent to $A_1 \to (A_2 \to \ldots (A_n \to B) \ldots)$; thus $\vdash s(B)$ would imply $A_1, ..., A_n \vdash B$, since $\vdash$ satisfies the 'detachment' half of (DDT), as remarked in Section 1. That would contradict our initial supposition that $A_1, ..., A_n \nvdash B$; so $\nvdash s(B)$, and hence $A_1, ..., A_n \nvdash^{\mathsf{sc}} B$. □

The "weakly" in Theorem 2.6(ii) cannot be replaced by "strongly". See [55] for a proof that the implicational fragment of $\vdash_{IL}$ itself is not strongly structurally complete. On the positive side, we observe (as Prucnal [37] does) that Lemma 2.5 can be extended beyond the $\rightarrow$ fragment to include any connectives which are consequent-distributive according to $\vdash_{IL}$, where an n-ary connective # is *consequent-distributive* according to a consequence relation $\vdash$ just in case for all formulas $A, B_1, \ldots, B_n$ in the language of $\vdash$, we have

$$A \rightarrow \#(B_1, \ldots, B_n) \dashv\vdash \#((A \rightarrow B_1), \ldots, (A \rightarrow B_n)).$$

Thus, Lemma 2.5 applies to the $\{\rightarrow, \wedge\}$ and $\{\rightarrow, \leftrightarrow\}$ fragments of $\vdash_{IL}$, as well as to the $\{\rightarrow, \wedge, \vee\}$ fragment of the Dummett intermediate logic $\vdash_{LC}$. Versions of Theorem 2.6 can accordingly be provided, suitably tailored to these variant fragments – and with the last case, variant choice of a 'base' logic. (Slaney and Meyer [47] offer a further variation of some interest, for showing weak structural completeness of the implication–conjunction fragment of the relevant logic **R** – given a particular understanding of what rule derivability consists in for this application.)

The example provided by $\vdash^{\mathsf{sc}}_{IL}$ of an extension of intuitionistic logic not enjoying the Deduction Theorem is in some ways simple and other ways less so. It has a simple enough description as the structural completion of the usual intuitionistic consequence relation, and a simple matrix semantics: it is the consequence relation determined by the Lindenbaum matrix for intuitionistic logic (thought of as a set of formulas).[23] On the other hand, the present author at any rate has no idea about what a characterization of $\vdash^{\mathsf{sc}}_{IL}$ in terms of Kripke semantics might look like, especially in view of the failure of (DT); for information on the prospects of providing a manageable proof system for this consequence relation, see Wojtylak [55]

[23]See Theorem T3 in Tokarz [51]. This is a countable matrix, so since $\vdash_{IL}$ has no countable characteristic matrix (Wroński [56]), one could conclude independently of familiarity with particular counterexamples to structural completeness for intuitionistic logic that such counterexamples must exist, since these cardinality considerations show $\vdash^{\mathsf{sc}}_{IL} \neq \vdash_{IL}$.

and Iemhoff [23]. In the next section we shall seek an example without such complexities.

Postscript on Superclassical Options. Can the example provided in 2.3 of a logic extending $\vdash_{IL}$ though in the same language and not satisfying (DT) be varied to yield a similar example where the logic to be extended is not $\vdash_{IL}$ but $\vdash_{CL}$? Naturally we assume that $\to$ is a connective of the language of $\vdash_{CL}$ in raising this question, and the Post-completeness of classical propositional logic returns a negative answer, assuming that the only consequence relations deserving to be considered as logics are those which are substitution-invariant. There is nevertheless a simple variation on the $(\cdot)^{\mathsf{sc}}$ theme which yields a (non-substitution-invariant) superclassical consequence relation – an example of some interest in its own right because of its connection with the notion of structural completeness.

Or perhaps that should read: its claimed connection with the notion of structural completeness. Bergman [2], begins a discussion of analogues of the property of structural completeness for quasivarieties (including varieties) of algebras by recapitulating what he takes to be the situation with this property as it applies to consequence relations which (as above) he does not require to be substitution-invariant, though he does require them to be finitary (so the strong/weak distinction does not arise). But he gives a definition of structural completeness (for consequence relations) which itself makes no reference to substitutions, thus ending up with a very different property, and whose name no longer merits the 'structural' part (see note 6). It does (as noted in [2]) have a Makinson-style 'maximality' characterization, like that in Proposition 2.1(ii) above. What we shall accordingly call the *Bergman completion* of a consequence relation $\vdash$, and denote by $\vdash^{\mathsf{bc}}$, is the largest consequence relation over the same language as $\vdash$ which agrees with $\vdash$ on the consequences of the empty set. It follows that

$\Gamma \vdash^{\mathsf{bc}} A$ *if and only if either, for some* $C \in \Gamma \nvdash C$, *or else* $\vdash A$.

The Bergman complete consequence relations would then be those coinciding with their Bergman completions. Thus, the preceding complaint about the 'structural' terminology in this context notwithstanding, a certain generalization of structural completeness would subsume the standard notion as well as admitting this notion as a somewhat degenerate case. Namely: let $\vdash$ be a consequence relation and S be a class of substitutions (defined on the language of $\vdash$). Then call $\vdash$ *S-structurally complete* just in case whenever $\Gamma \nvdash A$ there is some $s \in S$ with $\vdash s(C)$ for all $C \in \Gamma$ while $\nvdash s(A)$. The usual notion of (strong) structural completeness arises as the case of S-structural completeness in which S is the set of all subtitutions, while Bergman completeness is the case in which S contains just the identity substitution. As to whether some intermediate cases might be of independent interest, we shall not speculate.

By contrast with the case of structural completions, the Bergman completion of a substitution-invariant consequence relation will not in general itself be substitution-invariant. For $\vdash$ as either $\vdash_{IL}$ or $\vdash_{CL}$, for example, we have $p \vdash^{\mathsf{bc}} q$ since p is neither intuitionistically nor classically provable. Substitution-invariance would then give $p \to p \vdash^{\mathsf{bc}} q$, which is not the case since now the left-hand side is provable but not the right. The Bergman completion does not lead to what one would normally regard as a logic, then. But for what it's worth, we note here that taking $\vdash$ as $\vdash_{CL}$ gives us, in the shape of $\vdash^{\mathsf{bc}}$, a superclassical consequence relation not satisfying (DT). For while we have $p \vdash^{\mathsf{bc}} q$, we do not have $\vdash^{\mathsf{bc}} p \to q$. (Recall that $\vdash^{\mathsf{bc}}$ always agrees with $\vdash$ on the consequences of $\varnothing$.) That concludes the main business of this postscript, what follows being a matter of incidental interest.

To present the above example we did not actually give Bergman's own definition of (what he called) structural completeness for a consequence relation $\vdash$, which definition ([2], p.61, also [3], p.146), requires $\vdash$ to satisfy (a differently notated version of) the following condition, in which we have used "$\gneqq$" for the relation $\supsetneqq$ between consequence relations over the same language:

For every $\vdash' \gneqq \vdash$, *there is some* A *such that* $\vdash' A$ *and* $\nvdash A$.

While [2] shows that the consequence relations satisfying this condition are exactly those which are Bergman complete, it is not immediately obvious that this would coincide with the usual notion of structural completeness, even if a reference to arbitrary substitutions were reinstated (i.e., if we passed from S-structural completeness with only the identity map in S, to the case of S containing all substitutions). Bergman's and Makinson's versions of these concepts would then appear as follows (labelled (B) and (M)), in which "$\equiv_0$" (following [2]) stands for the relation of agreeing on consequences of the empty set; these are to be read as candidate definitions of structural completeness for a consequence relation $\vdash$:

(B) *For all substitution-invariant* $\vdash'$: $\vdash' \equiv_0 \vdash \Rightarrow$ *not* $\vdash \lneqq \vdash'$.

(M) *For all substitution-invariant* $\vdash'$: $\vdash' \equiv_0 \vdash \Rightarrow \vdash' \leq \vdash$.

While it is obvious that (M), as a condition on $\vdash$, implies (B), the converse is not so evident. In response to a query on this matter, Makinson (in early 1997) kindly showed the present author a proof that (M) and (B) are indeed equivalent.

3. Positive Implication with an Additional Constant

Since the formulation of (DT) involves $\rightarrow$, we shall need at least this connective to be present. Could we perhaps manage with no further connectives, to find an extension of $\vdash_{IL}$ – or rather its implicational fragment – not satisfying (DT)? No: Theorem 2.6(i) rules this out, since we certainly don't want to sacrifice substitution-invariance. So we shall have to have at least one further connective alongside $\rightarrow$. We shall be working with the sequent-to-sequent rule (††) below, corresponding to (DT), which would be called $\rightarrow$-Introduction (or Conditional Proof) in a natural deduction system, and $\rightarrow$-Right in a sequent calculus:

(††) $$\frac{\Gamma, A \succ B}{\Gamma \succ A \rightarrow B}$$

Let us say that a sequent *holds* in a model $(U, \leq, V)$[24] when any point in the model verifying all the formulas on the left of the "$\succ$", verifies that on the right. What makes the rule (††) preserve, for any given model, the property of holding in that model, in view of the usual clause for $\rightarrow$ in the definition of truth (at a point), is the fact that all formulas are *persistent* (or "hereditary"), in the sense of being guaranteed to be true at a point v in a model whenever true at u in that model, where $u \leq v$. In particular, for an application of the above rule to preserve the property of holding in a model, what we need is that all the formulas in Γ are in this sense persistent. Accordingly one might think that the way to deal model-theoretically with an extension of intuitionistic logic without (DT) is to allow non-persistent

[24]In calling $(U, \leq, V)$ a *model* we mean that $(U, \leq)$ – the *frame* of the model – is a poset, and that V assigns to each propositional variable an upward closed subset of U. We give more details presently, when considering a slightly different conception of model.

formulas: then (††) will not be guaranteed to preserve holding in a model when such formulas are amongst the side-formulas Γ. Thus we might obtain a simple extension of intuitionistic logic for which (DT) fails by adding a connective that forms non-persistent compounds.

The idea just sketched does not work out, however. If a formula A is not persistent then the sequent $A \succ p \to A$ will fail to hold in every model. But this means that the consequence relation $\vdash$ defined by (**) in Section 1, except with the reference to provability replaced by one to holding in each model in some specified class, will not be a substitution-invariant extension of $\vdash_{IL}$, since we will have $q \vdash p \to q$ but not $A \vdash p \to A$. Thus we must try another approach.[25]

Instead of working with models of the above form, we will expand each model to accommodate a distinguished element, truth of any formula at which is sufficient for that formula to be true throughout the model. We arrange this at the level of frames by saying that the new frames will have the form $(U, \leq, u_0)$ where $(U, \leq)$ is a partially ordered set with least element u_0 (i.e., $u_0 \in U$ and for all $x \in U$, $u_0 \leq x$). Models on such frames then take the form $(U, \leq, u_0, V)$, where V assigns to each propositional variable an upward closed subset of U (securing persistence for the p_i). Truth of a formula A at an element $x \in U$ in a model $\mathcal{M} = (U, \leq, u_0, V)$, written as "$\mathcal{M} \models_x A$", is defined in the usual way for propositional variables and implicational formulas, and we shall add to the language a new nullary connective (sentential constant) Ω to exploit this additional structure:

- $\mathcal{M} \models_x p_i$ iff $x \in V(p_i)$.
- $\mathcal{M} \models_x A \to B$ iff for all $y \in U$ with $x \leq y$ and $\mathcal{M} \models_y A$, we have $\mathcal{M} \models_y B$.
- $\mathcal{M} \models_x \Omega$ iff $x \neq u_0$.

Note that the condition given here for Ω makes this formula (like the propositional variables and implicational formulas) *persistent* in the sense defined above – i.e., its truth at x in a model guarantees its truth at any $y \geq x$.[26] (If not, we have x, y, with $x \leq y$, Ω true at x but not at y, so

[25] A concrete example of the kind of substitution failure involved here may be found in [19], where a logic for combined classical and intuitionistic negation is considered, the former forming non-persistent compounds, forcing restrictions on rules governing (*inter alia*) the latter, giving rise to provable sequents with unprovable susbtitution-instances.

[26] On the other hand, if we had put "=" for "≠" in this clause, the formula Ω would only have been persistent in one-element models.

$x \neq u_0$ while $y = u_0$. Since u_0 is a least element, this gives $u_0 \leq x \leq u_0$, so by antisymmetry $u_0 = x$: a contradiction.) Recall that we wanted this persistence property, and will be tracing the failure of (DT), mentioned below, to a different source (failure of a certain 'generation theorem', in fact).

Now that we have a distinguished point in our models, we take a sequent to *hold* in a model (as currently conceived) just in case we do not have all the formulas on the left true and the formula on the right false *at that point*. For the usual language of intuitionistic logic, a sequent's holding in every model of the earlier type (i.e., holding throughout each model) and a sequent's holding in every model of the current type (i.e. holding at the distinguished point) are equivalent, and in fact the semantics as presented in Kripke [25] were closer to the the latter than the former.[27] The rule (††) no longer preserves the property of holding in a model, for models as currently conceived, but it manages to preserve the property of holding in all models, by an argument that appeals to a well known result[28] on generated submodels, for present purposes defined as follows. Given a model $\mathcal{M} = (U, \leq, u_0, V)$ with $x \in U$, the submodel of $\mathcal{M}$ generated by x, is the model $\mathcal{M}_x = (U_x, \leq_x, x, V_x)$ where $U_x = \{y \in U \,|\, x \leq y\}$, and $\leq_x$ and V_x are the restrictions of $\leq$ and V to U_x. (Note that the generating point has been promoted to the status of distinguished element here.) Then for the usual language of $\vdash_{IL}$, we have, as one shows by induction on the construction of A:

Theorem 3.1. *Let $\mathcal{M}_x = (U_x, \leq_x, x, V_x)$ be the submodel of $\mathcal{M} = (U, \leq, u_0, V)$ generated by $x \in U$. Then for any formula A of the language of $\vdash_{IL}$, and any $y \in U_x$, we have $\mathcal{M}_x \models_y A$ if and only if $\mathcal{M} \models_y A$.*

Corollary 3.2. *When restricted to sequents over the language of $\vdash_{IL}$, the rule (††) preserves the property of holding in every model (with distinguished element).*

[27] In fact, in the initial presentation of the models in [25] there is a distinguished point but no requirement that every point be accessible to it, and a later observation that without change to the logic, attention can be restricted to models in which accessibility is the ancestral of the 'immediately dominates' relation in a (rooted) tree, whose root is taken as the distinguished element.

[28] See, e.g., Lemma 2.7 in [44], where the formulation is in terms of models without distinguished elements.

Proof. Suppose the conclusion sequent $\Gamma \succ A \to B$ of an application of (††) does not hold in some model, $\mathcal{M}$, say, where $\mathcal{M} = (U, \leq, u_0, V)$, so $\mathcal{M} \models_{u_0} C$ for each $C \in \Gamma$, while $\mathcal{M} \not\models_{u_0} A \to B$. Thus for some $x \in U$ with $u_0 \leq x$, $\mathcal{M} \models_x A$ but $\mathcal{M} \not\models_x B$. By persistence, for all $C \in \Gamma$, $\mathcal{M} \models C$. (That is where the argument would stop if we weren't working with models with distinguished elements.) That does not entitle us to conclude that the premiss sequent for this application of (††) doesn't hold in all models on the grounds that it doesn't hold in $\mathcal{M}$, because to show that it doesn't hold in this model we need all of $\Gamma \cup \{A\}$ true at the distinguished point u_0 of the model, with the formula B false there. But we simply trade $\mathcal{M}$ in for the submodel thereof generated by x, and appeal to Theorem 3.1 to conclude that $\Gamma \cup \{A\}$ are true at the distinguished point of this new model, while B is not, so the premiss sequent does not hold in every model. □

The above is all perfectly familiar and we review it here to note the failure of Theorem 3.1 to extend to the case of the language with ($\to$ and) Ω, which deprives us of Corollary 3.2 for the language of $\vdash_\Omega$, where this is the consequence relation defined in (†††):

(†††) $\Gamma \vdash_\Omega C$ iff for some $\Gamma_0 \subseteq \Gamma$, the sequent $\Gamma_0 \succ C$ holds in every model.

Taking $x \neq u_0$ in a model $\mathcal{M} = (U, \leq, u_0, V)$, at which Ω is accordingly true, we pass to the generated submodel $\mathcal{M}_x$ and relative to this model Ω is now true at x. Such a failure of Theorem 3.1 makes trouble, as promised, for Corollary 3.2 and thus for $\vdash_\Omega$'s satisfying (DT):

Example 3.3. The sequent $\Omega \succ p$ holds in all models, while the sequent $\succ \Omega \to p$ does not. The first sequent has the property of holding in all models because its left-hand formula is never true at the distinguished element, while second lacks the property—which is accordingly not preserved by the rule (††) because the clause for $\to$ in the truth-definition directs us (*inter alia*) to other ('later') points, at all of which Ω is true. Thus we can easily invalidate the sequent by making $V(p)$ empty – for example. We can reformulate the bearing of this on (DT) as follows: the consequence relation $\vdash_\Omega$ does not satisfy the condition (DT). (Note that Prucnal's Lemma – 2.5 above – does not go through with Ω present: $s_A(\Omega)$ is just Ω, and not equivalent to $A \to \Omega$.[29])

[29]The same would apply in the case of $\bot$ of course. As to whether a version of

The choice – more extreme than necessary to make the point – of $V(p)$ as $\varnothing$, in the exposition of this Example, draws our attention to the fact that if we extended our language to accommodate the constant $\bot$ with the usual semantics ($\bot$ to be false everywhere in any model) then we should have $\Omega \dashv\vdash \bot$, where $\vdash$ is defined over this richer language in the same way that (†††) defined $\vdash_\Omega$. The fact that Ω and $\bot$ are thus equivalent does not mean that they are interreplaceable preserving equivalence in longer formulas: the consequence relation just described is not congruential (in the sense of Segerberg [46][30]). For example, this equivalence notwithstanding, we do not have $A \to \Omega \dashv\vdash A \to \bot$ (specifically, because the "$\vdash$" direction fails, e.g. with A and p). The right-hand side of this last failed equivalence is often used in a definition of $\neg A$, and despite its failure, we might think of the left-hand side as similarly definitive of another kind of negation, symbolized by "$\neg_\Omega A$", say (so that the usual negation would more explicitly be written as "$\neg_\bot$"). Like the usual negation, $\neg_\Omega$ enjoys an *Ex Falso Quodlibet* style property – $A, \neg_\Omega A \vdash B$ for all A, B – and it also enjoys, an 'excluded middle' property, formulable if we add disjunction (with the usual semantic treatment) to our language (and we continue to write $\vdash$ for the extended consequence relation): $\vdash A \vee \neg_\Omega A$. For suppose that we have a model $\mathcal{M} = (U, \leq, u_0, V)$ with $\mathcal{M} \not\models_{u_0} A \vee \neg_\Omega A$. Then $\mathcal{M} \not\models_{u_0} A$ and $\mathcal{M} \not\models_{u_0} \neg_\Omega A$, i.e., $\mathcal{M} \not\models_{u_0} A \to \Omega$. Thus for some $x \geq u_0$, $\mathcal{M} \models_x A$ but $\mathcal{M} \not\models_x \Omega$. This last means that $x = u_0$, since Ω is false at only the distinguished element in our models – but now we have a contradiction, since $\mathcal{M} \not\models_{u_0} A$ while $\mathcal{M} \models_x A$.[31] We can give voice to this 'law of excluded middle' behaviour without disjunction actually being present, by means of a sequent-to-sequent rule saying that whatever follows from other assumptions (to use the idiom of natural deduction) and A and also follows from those other assumptions and $\neg_\Omega A$, follows from the other assumptions alone. This rule we accordingly call $(\text{LEM})_\Omega$ below, where we have stated it using only the primitive notation of $\to$ and Ω. The above *Ex Falso Quodlibet* behaviour, can also be encapsulated more succintly as a general

Theorem 2.6(i) can be proved in some other way for the $\{\to, \bot\}$ or $\{\to, \neg\}$ fragments of all intermediate logics, the author has no information.

[30]Wójcicki [54] uses the term "self(-)extensional" for this property.

[31]The upshot of defining $\neg_\Omega A$ as $A \to \Omega$ is that $\neg_\Omega A$ is true at a point x (in a model whose distinguished element is u_0) iff either $x \neq u_0$ or else $x = u_0$ and A is not true at u_0.

form of the sequent mentioned at the start of Example 3.3; we label this (zero-premiss) rule $(\mathrm{EFQ})_\Omega$ below. From this discussion, then, we extract the following two rules, the first justified—to summarise our reasoning—by the fact that (in any model) if $x = u_0$, Ω is false at x, and the second by the fact that if $x \neq u_0$, Ω is true at x:

$$(\mathrm{EFQ})_\Omega \quad \Omega \succ A \qquad\qquad (\mathrm{LEM})_\Omega \quad \frac{\Gamma, A \succ C \qquad \Gamma, A \to \Omega \succ C}{\Gamma \succ C}$$

What we wanted from our second example of the failure of (DT) in an extension of (here, the implicational fragment of) intuitionistic logic, as opposed to that reviewed in Section 2, was a simple Kripke semantics and an equally simple proof-theoretic description. Since $\vdash_\Omega$ was defined in terms of the model theory in the first place, we certainly have that for the current example of failure of (DT). And the above rules give us the desired syntactic description, as we shall record in Theorem 3.5. There will be a few preliminary definitions and observations for that, but before those let us pause to observe that our two rules uniquely characterize the constant they govern in the Hiż–Belnap sense of [1]. Refining that discussion a little (as in [22], §§3, 4) to take into account the possibility of non-congruential logics, let us say that rules governing a connective # uniquely characterize that connective *to within equivalence* when adding reduplicated rules for a new connective $\#'$ (of the same arity as #), and assuming available the usual structural rules (mentioned before the formulation of (**) in Section 1), gives a proof system according to whose associated consequence relation (*à la* (**)) #-compounds and $\#'$ compounds with the same components (in the same order) are equivalent (in the "$\dashv\vdash$" sense). And we say that the rules uniquely characterize # *to within synonymy* when according to this same consequence relation any similarly related #-compounds and $\#'$ compounds are synonymous in the sense of Smiley [48]: meaning that either can be replaced by the other (not necessarily uniformly) inside arbitrary formulas without affecting the correctness of $\vdash$-claims.[32]

[32]We will sometimes just call formulas synonymous in the proof system in question when this holds for the associated consequence relation – in other words, when they are freely interreplaceable within arbitrary formulas in all sequents, *salva provabilitate* – or *salva validitate* in the case of a consquence relation semantically characterized. Note that when represented in schematic form the rules of the combined or reduplicated system are understood as having the schematic letters involved range over formulas of the combined

Now, clearly the rule $(\mathrm{EFQ})_\Omega$ already characterizes Ω uniquely to within equivalence, since this rule and its reduplicated version give $\Omega \succ \Omega'$ and the converse sequent respectively. But we have already seen, with the discussion of Ω and $\bot$ following Example 2.2, that this does not suffice for free interreplaceability within arbitrary embedded positions. We observe now, however, that taken together, the rules $(\mathrm{EFQ})_\Omega$ and $(\mathrm{LEM})_\Omega$ do uniquely characterize Ω to within synonymy, when taken in conjunction with the treatment of $\to$ in the proof system based on these rules which forms the subject of Theorem 3.5 below. It is not hard to verify that formulas A and B are synonymous in that proof system just when $\succ A \leftrightarrow B$ is provable (or $\succ A \to B$ and $\succ B \to A$ are provable). By symmetry, it suffices, then, to show that in the reduplicated system with supplementary rules for Ω', the sequent $\succ \Omega \to \Omega'$ is provable.

$$\frac{\Omega \succ \Omega \to \Omega' \qquad \Omega \to \Omega' \succ \Omega \to \Omega'}{\succ \Omega \to \Omega'}$$

The left premiss here is a case of $(\mathrm{EFQ})_\Omega$, with A taken as $\Omega \to \Omega'$ and the right premiss is a case of (the structural rule) identity. The whole figure depicts an application of $(\mathrm{LEM})_{\Omega'}$, with A as Ω and C as $\Omega \to \Omega'$. So the conclusion is provable.

We now return to the preliminaries needed for showing the completeness of a proof system for $\vdash_\Omega$. A set of formulas Γ will be said to be *deductively closed* (relative to $\vdash$) just in case for all A, $\Gamma \vdash A$ implies $A \in \Gamma$, and to be *ponentially closed* just in case for all A, B, if $A \in \Gamma$ and $A \to B \in \Gamma$ then $B \in \Gamma$; the *deductive closure* (*ponential closure*) of a set is its smallest deductively closed (ponentially closed) superset. By "$\vdash_{\mathsf{mp}}$" we mean the least consequence relation (over a language with $\to$, and for our current application, also Ω) satisfying the condition that B is a consequence of $\{A, A \to B\}$ for all formulas A, B. Thus Γ is ponentially closed when $\Gamma \vdash_{\mathsf{mp}} A$ implies $A \in \Gamma$. And by $IL_\to$ we mean the set of all consequences of $\varnothing$ according to the implicational fragment of $\vdash_{IL}$. For the proof of part (ii) of the following, the standard proof (as found for example in [16], pp. 23–24 or [41] pp. 50–51) of (DT) for a *Modus Ponens* based axiomatization of intuitionistic logic will do – i.e., induction on the length of deductions of B from $\Gamma \cup \{A\}$, using the availability of all formulas of the forms $C \to (D \to C)$ and $(C \to (D \to E)) \to ((C \to D) \to (C \to E))$.

language (with # and #′, that is).

Lemma 3.4. (*i*) *If* $\vdash$ *extends the implicational fragment of* $\vdash_{IL}$ *then any set of formulas which is deductively closed relative to* $\vdash$ *is ponentially closed.*
(*ii*) *If* $IL_{\rightarrow} \subseteq \Gamma$ *and* $\Gamma, A \vdash_{\mathsf{mp}} B$ *then* $\Gamma \vdash_{\mathsf{mp}} A \rightarrow B$.

We are now in a position to prove the completeness of the above proof system using a version of the Scott–Makinson method. This is not exactly a canonical model completeness proof, since no single characteristic model is produced; rather we provide a countermodel for any given unprovable sequent. Except for the need to keep track of the deductive/ponential closure distinction, the details are much as in Section 2 of Segerberg [44]. In the interests of brevity, from now on we shall call those sequents holding in all models *valid* sequents.[33]

Theorem 3.5. *A sequent is valid if and only if it has a proof in the proof system with, as initial sequents, all substitution instances (in the language with* $\rightarrow$ *and* Ω*) of sequents in the* $\rightarrow$*-fragment of intuitionistic logic, and, alongside the structural rules, the rules (EFQ)*$_{\Omega}$ *and (LEM)*$_{\Omega}$.

Proof. The "if" direction (soundness) is clear enough, so we confine our attention to the "only if" direction (completeness). Suppose $\Gamma \succ C$ is unprovable. Let u_0 be a deductively closed superset of Γ not containing C and containing, for any formula A, either A itself or else $A \rightarrow \Omega$. (Such a superset exists, by a Lindenbaum argument, appealing to the rule (LEM)$_{\Omega}$.) The consequence relation in terms of which deductive closure is here to be understood is that associated with the present proof system in accordance with the definition (**) in Section 1; for the present proof we will call this consequence relation $\Vdash$. (Thus the proof will establish that $\Vdash = \vdash_{\Omega}$.) The universe, U, of the model $\mathcal{M} = (U, \leq, u_0, V)$ we shall now construct, and in which $\Gamma \succ C$ will be seen not to hold, comprises u_0 together with all ponentially closed supersets of $u_0 \cup \{\Omega\}$. For any $x, y \in U$ we define: $x \leq y \Leftrightarrow x \subseteq y$ and $V(p_i) = \{x \in U \,|\, p_i \in x\}$; as usual, the latter stipulation secures the basis case for an induction on the complexity of (= number of connectives used to construct) the formula D establishing that $D \in x \Leftrightarrow \mathcal{M} \models_x D$, for the inductive parts of which it suffices to show:

[33] A more refined notion, not needed for present purposes, is validity on a given frame $(U, \leq, u_0)$, in the sense of holding in every model based on that frame.

(a) $\Omega \in x \Leftrightarrow x \neq u_0$; (b) $A \to B \in x \Leftrightarrow$ for all $y \geq x, A \in y \Rightarrow B \in y$.

As to (a), if for $x \in U$, $x \neq u_0$, then certainly $\Omega \in x$, since aside from u_0, we allowed into U only sets of formulas containing Ω; on the other hand $\Omega \notin u_0$ since $C \notin u_0$ by construction, $\Omega \Vdash C$ (in view of $(\mathrm{EFQ})_\Omega$) and u_0 is deductively closed.

We turn to (b). If $A \to B \in x$ and $A \in y$ for $y \geq x$, i.e., $y \supseteq x$, then $A \to B \in y$ and so since all elements of U are ponentially closed (including u_0, by Lemma 33.4(i)), $B \in y$. Suppose on the other hand that $A \to B \notin x$. We distinguish two cases (1) $x = u_0$, (2) $x \neq u_0$. For case (1), since $B \Vdash A \to B$, we have $B \notin x$ ($= u_0$). So if $A \in u_0$, then u_0 itself is a suitable (i.e., $\geq u_0$) point containing A but not B. So suppose instead that $A \notin u_0$; in that case, we note for future reference that $A \to \Omega \in u_0$ (from the way u_0 was originally specified). If $u_0, A \vdash_{\mathsf{mp}} B$ then $u_0 \vdash_{\mathsf{mp}} A \to B$ so (by Lemma 3.4(ii), since $IL_{\to} \subseteq u_0$) we should have $A \to B \in u_0$; thus $u_0, A \nvdash_{\mathsf{mp}} B$, and so the ponential closure of $u_0 \cup \{A\}$ will do as a $y \geq u_0$ with $A \in y$, $B \notin y$. In fact, in this case $y \gneqq u_0$, so we must make sure that $y \in U$, which required not only that y be a ponentially closed superset of u_0, but that $\Omega \in y$. But on this issue, recall that $A \to \Omega \in u_0$, so since $A \in y$, we do have $\Omega \in y$ after all.
Case (2): $x \neq u_0$. The argument here is as for the second half of case (1), except that we do not have to worry about checking that Ω belongs to the desired $y \geq x$ with $A \in y$, $B \notin y$, since $\Omega \in x$ already.
We have now shown that $D \in x \Leftrightarrow \mathcal{M} \models_x D$ for all formulas D, so since $\Gamma \subseteq u_0$ and $C \notin u_0$, $\mathcal{M} \models_{u_0} A$ for each $A \in \Gamma$ while $\mathcal{M} \not\models_{u_0} C$, our unprovable sequent $\Gamma \succ C$ does not hold in $\mathcal{M}$, and is therefore not valid.□

In the discussion after Example 3.3 above, we noted that extending $\vdash_\Omega$ (which we now see from Theorem 3.5 coincides with the $\Vdash$ defined in the course of its proof) with the addition of $\perp$ gave a non-congruential consequence relation, and we note here that $\vdash_\Omega$ is itself already non-congruential, especially as making this observation will provide us with another counterexample to (DT) for this consequence relation:

Example 3.6. Note that the sequent $(p \to \Omega) \to \Omega \succ p$ (a form of double negation elimination for negation as $\neg_\Omega$) is valid, as is the converse sequent also (this time with any formula in place of Ω). Thus $(p \to \Omega) \to \Omega$ and

p are equivalent according to $\vdash_\Omega$. But these formulas are not synonymous (in Smiley's sense, as explained above): in particular, while $\vdash_\Omega p \to p$, $\nvdash_\Omega ((p \to \Omega) \to \Omega) \to p$ – thereby providing us, incidentally, with another example (beyond that of Example 3.3) of the failure of (DT) for $\vdash_\Omega$ (alias $\Vdash$).

The final point we make about the above proof system is that, in taking on board all of what would otherwise have been the results of applying the rule (††) "safely" within the confines of the Ω-free fragment of $\vdash_\Omega$, by simply treating all sequents of implicational intuitionistic logic as initial sequents,[34] we have missed out on the elegant systematization of that logic provided by (††) and the upside down form of the same rule (or equivalently, *Modus Ponens* or the sequent calculus ($\to$ Right) rule, depending on one's preferred approach). To meet this objection, one could instead modify the proof system, so that alongside the needed structural rules and one of the last-mentioned rules, (††) itself is used with a proviso: that the premiss sequent $\Gamma, A \succ B$ has itself been proved without any appeal to either of the distinctive Ω rules, $(\mathrm{EFQ})_\Omega$ and $(\mathrm{LEM})_\Omega$. This change would mean that the rule (††) would no longer have the status of a local constraint on derivations, since whether we can apply the rule to premiss sequents depends on the prior derivational history of those sequents: it would be the logical analogue of one of the *global derivational constraints* urged in the context of linguistic theory in Lakoff [26]. Such a move, this time restricting the rule of necessitation, was made in Kamp [24] in the axiomatic presentation of a range of modal logics that are quasi-normal (defined below) though not normal, to use the terminology of Segerberg [45] (adapted from Scoggs [43]), where such logics are themselves treated with the use of models with distinguished elements, serving as something of an inspiration for our definition of $\vdash_\Omega$.[35] Segerberg's example could not be followed closely in the present setting because it relies on a formula A's having the property of

[34] Or rather: all substitution instances of such sequents in the language which also has Ω.

[35] See Section 2 of Chapter III of [45] for details. Segerberg allows a set of distinguished elements rather than a single such element, but this difference is not material. Kamp is interested in modal logics with a temporal interpretation (tense logics, that is), the distinguished element in a model being thought of as the 'present' moment (the moment of utterance of the expression being semantically evaluated). The global derivational constraint idea can be seen in the first and last paragraphs on p.244 of [24].

always being true at the distinguished point without $\Box A$'s also having this property (thus allowing for failures of the rule of necessitation), because the distinguished point can have other points accessible to it where A is false. Because of the requirement of persistence, no such arrangement could be made here, and we relied instead (for failures of (DT)) on the fact that the distinguished point could *falsify* a formula (namely Ω) which points accessible to it did not. We assume known the concept a normal modal logic, and remind the reader that a *quasi-normal* modal logic is a modal logic which extends the smallest normal modal logic **K**. (While normality has an unequivocal interpretation for modal logics as sets of formulas, there is room for disagreement as to what it should amount to for modal logics as consequence relations: see note 40 below.) In these terms, recalling the fact that the implicational fragment of intuitionistic logic is sometimes—as in the title of the present section–called the logic of positive implication, we could define a *positive* consequence to be one satisfying (DDT), and call $\vdash$ *quasi-positive* if $\vdash$ extends the smallest positive consequence relation. The latter is of course nothing but the implicational fragment of $\vdash_{IL}$. There remains a further disanalogy with the modal case, in that to get our extension to fail to satisfy (DT), in view of Theorem 2.6(i), we had to understand "extension" in the liberal way mentioned in Section 1 – allowing for additional logical vocabulary: otherwise, by that result, we should have it that every quasi-positive consequence relation was positive. In the modal case, however, the non-normal extensions of **K** are cast in the very same language as **K** itself. (We are also considering here treatments of modal logic in the 'logics as sets of formulas' framework rather than the 'logics as sets of sequents' – essentially, as consequence relations, that is – which is the habitat for our logic of "$\rightarrow$" and "Ω".[36])

[36]One might initially think of the usual rule of $\forall$ introduction in a natural deduction systematization of predicate logic as involving a global derivational constraint, in sanctioning the transition from $A(t)$ to $\forall v(A(v))$, where $A(t)$ is $A(v)$ with free occurrences of v replaced by the parameter t, on the grounds that there is a precondition that $A(t)$ does not depend, at the point at which this rule is applied, on any assumptions containing t. As will be evident from our earlier discussion, however, we prefer to think of natural deduction rules no less than sequent calculus rules as sequent-to-sequent rules, in which case the constraint is local rather than a global derivational constraint since the set of assumptions concerned is present on the left of the premiss-sequent for an application of $\forall$ introduction. Interested readers can make up their own minds on whether the rule, due to Fitch, that Curry and Feys [9], p.261, describe as "Rule **P** [= *Modus Ponens*] ...

A word is in order on the proviso on (††) in the alternative proof system just canvassed (whose equivalence, in respect of provable sequents, to that described in Theorem 3.5 we leave the interested reader to verify). Our restriction forbade an application to any sequent in whose derivational history either of the rules $(\mathrm{EFQ})_\Omega$, $(\mathrm{LEM})_\Omega$, had figured, whereas our illustration of what could go wrong in violating this restriction, Example 3.3, only showed the problem in the case of the former rule. However, the restriction really does need to encompass both rules, as we now show.

Example 3.7. Being of the form $A \to B$, $A \succ B$, both of the following sequents are initial sequents of the proof system described in Theorem 3.5:

$$(p \to \Omega) \to q, p \to \Omega \succ q \qquad\qquad p \to q,\, p \succ q.$$

From these two, by thinning and the rule $(\mathrm{LEM})_\Omega$, we get $(p \to \Omega) \to q, p \to q \succ q$. From this last sequent (whose validity could alternatively just be checked straight away using the semantics) the rule (††) delivers the invalid sequent $(p \to \Omega) \to q \succ (p \to q) \to q$, even though there is no earlier appeal to $(\mathrm{EFQ})_\Omega$. Thus we do need to block prior application not only of $(\mathrm{EFQ})_\Omega$ but also of $(\mathrm{LEM})_\Omega$ for a safe (i.e., validity-preserving) application of (††).

We have chosen to discuss the Ω example at length because adding one nullary connective seemed the simplest possible setting in which to illustrate failures of (DT) for a consequence relation extending $\vdash_{IL}$ in an expanded language. Adding instead a 1-ary connective would be less 'simple' in purely quantitative terms, but in fact there is a variation on the Ω example which utilises such a connective and has the advantage of considerable familiarity. It is with setting out this example that we close our discussion. We have already had occasion to mention that Kamp in [24] wanted to treat the 'formal properties of "now"', where (an operator representing) "now" attaches to a formula A to make a formula true at an arbitrary point just in case A is true at the distinguished point of the model. The same

subjected to an ad hoc and complex restriction" should count as a global derivational constraint. As the authors ([9], p.261, note 10 , put it: "It is complex in the sense that the criterion for applicability of rule **P** does not depend on the premises alone, but also on their relation to the whole proceding proof." After completing the present paper, the author encountered similar uses of the terminology of global constraints or rules (in logic) in two further sources, namely Bystrov [8] and Meyer Viol [32], p.146.

device has been used in alethic modal logic for the treatment of "actually", the distinguished element now being thought of as the actual world. As is mentioned in the survey article [21], where several alternative approaches, in particular to the definition of validity, are contrasted, this idea goes back somewhat further historically, namely to [31], in which Meredith and Prior introduce a sentential constant $\boldsymbol{n}$ stipulated in the definition of truth to be true at the distinguished element and nowhere else. (See also Section 7 [30]. The idea was first mooted in a note circulated by Meredith in 1956.) But they noted that – what we are calling – an actuality operator, which for present purposes we shall write as "R" (think of the Real world), answering to the above description is definable: define RA as $\Box(\boldsymbol{n} \to A)$, where $\Box$ is interpreted by universal quantification over the points in the model and $\to$ is for material implication. We have already had occasion to observe that in the models we are considering, a formula true at precisely the distinguished element would be non-persistent except in the trivial case of one-element models (as remarked in note 25), but if we take something along the lines of R (now thinking of u_0 as the Root, in the representative case in which the poset $(U, \leq)$ is a tree) as primitive rather than defined, with a semantical clause reading as follows, we obtain persistent R-compounds. Here we take $\mathcal{M}$ to be $(U, \leq, u_0, V)$ as before, and x any element of U:

$$\mathcal{M} \models_x RA \text{ iff } \mathcal{M} \models_{u_0} A, \text{ for all formulas } A.$$

Defining validity as before (holding in all models, with holding in a model consisting in truth-preservation at the distinguished point), we have another simple example, in the present language with $\to$ and R – though we could of course add $\wedge$, $\vee$ and $\neg$ – of a failure of (DT) in an extension of $\vdash_{IL}$:

Example 3.8. In view of the way validity has been defined, the sequents $Rp \succ p$ and $p \succ Rp$ are both valid. The results of applying the rule (††) to these premiss sequents differ in respect of validity, $\succ Rp \to p$ being valid since p's being true at u_0 (as it must be for Rp to be true anywhere) means that p is true everywhere (as u_0 is a least element). But the sequent $\succ p \to Rp$ is not valid: allowing p to 'become' true at $x \gneqq u_0$ will make the antecedent true and the consequent at x, so the implication is false at u_0. Thus the consequence relation defined in terms of the present semantics as $\vdash_\Omega$ was defined above (i.e., using (†††)) fails to satisfy (DT).

It is not difficult to provide a (sound and) complete proof system for this 'intuitionistic actuality logic' but we shall not do so here, contenting ourselves with the observation that both this logic and the logic of the Ω example, when formulated in the full language of intuitionistic logic (and also the novel connective in each case) lack the Disjunction Property. We pointed this out for Ω with the disjunction $p \vee (p \to \Omega)$ (alias $p \vee \neg_\Omega p$), and could supply for the present case the example of $Rp \vee \neg Rp$. This formula is evidently valid (or rather, strictly speaking, the sequent $\succ Rp \vee \neg Rp$ is valid, and note further that the first R can be omitted here) while neither of its disjuncts is. According to Definition 4 on p.131 of Gabbay [16], this makes neither Ω nor R acceptable as a "new intuitionistic connective" (the title of Gabbay's section); using the phrase actually defined by Definition 4, this would be put by saying that neither the logic of Ω nor that of R counts as an "intuitionistic extension". Many proposed extensions (in the sense of Section 1 above) of intuitionistic logic similarly fail to count as "intuitionistic extensions" according to Gabbay's definition, even though they may be conservative over intuitionistic logic in its customary language and defined by proof systems whose rules uniquely characterize (to within equivalence) the novel connectives – two other conditions featuring in Definition 4. This includes some cases discussed in other parts of [16] itself, such as Rauszer's dual intuitionistic negation (Brouwer negation, as she called it in, e.g., [38][37]). Note that we can formulate the Disjunction Property without "$\vee$", using instead the 'deductive disjuntion' (let us say) of formulas A, B, meaning the formula $(A \to p_k) \to ((B \to p_k) \to p_k)$ where p_k is the first variable (in the ordering $p_1, p_2, ...$) not occurring in either A or B. The Disjunction Property is then the property that whenever the deductive disjunction of A, B, is provable (is a consequence of $\varnothing$) then so is either A or B.[38]

[37] Gabbay discusses this topic in [16], though he concentrates on dual intuitionistic implication (see the index entries under "connectives, dual"). For a dedicated discussion of dual intuitionistic negation, see [18]. Some further discussion of this connective and of Gabbay's conditions on new intuitionistic connectives in general may be found in Section 4 of [20].

[38] The antecedent in this formulation is obviously equivalent for any substitution-invariant $\vdash \supseteq \vdash_{IL}$, for a fixed A, B, to (1): for all formulas C in the language of $\vdash$, we have $\vdash (A \to C) \to ((B \to C) \to C)$, and less obviously equivalent – whether or not $\vee$ is present and (more surprisingly) whether or not $\vdash$ satisfies (DT))– in turn to (2) $A \to C, B \to C \vdash C$ for all formulas C. (For the implication (1) $\Rightarrow$ (2), the Detachment

Whichever formulation of the Disjunction Property one prefers, the fact that it fails for dual intuitionistic negation, which we shall notate by $\neg_{\mathsf{d}}$, is not unconnected with its failure for the logic of R. We recall that for a model $\mathcal{M} = (U, \leq, u_0, V)$ and $x \in U$, the semantical clause for $\neg_{\mathsf{d}}$ in the definition of truth at a point reads thus:

$\mathcal{M} \models_x \neg_{\mathsf{d}} A$ iff for some $y \leq x$, we have $\mathcal{M} \not\models_y A$.

Such formulas are persistent in the usual way (i.e., pass truth at u on to truth at $v \geq u$) but engender a failure of the Disjunction Property – here given its explicit $\vee$-based formulation – for the class of valid formulas, in view of the validity of the excluded middle principle $p \vee \neg_{\mathsf{d}} p$. The connection with the R logic above is that for models of the form $(U, \leq, u_0, V)$ with u_0 a least element, we can define $\neg_{\mathsf{d}}$ in terms of R and $\neg$:

Proposition 3.9. *For any $x \in U$, with $\mathcal{M} = (U, \leq, u_0, V)$ and u_0 a least element, we have, for all formulas A, $\mathcal{M} \models_x \neg_{\mathsf{d}} A$ if and only if $\mathcal{M} \models_x \neg RA$.*

Proof. Suppose $\mathcal{M} \models_x \neg_{\mathsf{d}} A$. Then for some $y \in U$ with $y \leq x$, $\mathcal{M} \not\models_y A$. Since $u_0 \leq y$, by persistence, $\mathcal{M} \not\models_{u_0} A$, so for no $v \in U$ do we have $\mathcal{M} \models_v RA$. But this implies that $\mathcal{M} \models_x \neg RA$. Conversely, suppose that $\mathcal{M} \models_x \neg RA$. Then in particular $\mathcal{M} \not\models_x RA$, so $\mathcal{M} \not\models_{u_0} A$; thus we have, in the shape of u_0, some $y \leq x$ with A not true at y, meaning that $\mathcal{M} \models_x \neg_{\mathsf{d}} A$. □

The semantical clause for $\neg_{\mathsf{d}}$ above, though deployed in Proposition 3.9 only for models (on frames) with a least element is not given in Rauszer [38] with any such restriction, and one must bear this in mind to avoid jumping to false conclusions. For example in view of the definability noted in Proposition 3.9 of "$\neg_{\mathsf{d}}$" as "$\neg R$",[39] one might hastily conclude from the intuitionistic law of triple negation that $\neg\neg\neg_{\mathsf{d}} p$ must be equivalent to $\neg_{\mathsf{d}} p$ on

half of (DDT) suffices, inherited by $\vdash$ from $\vdash_{IL}$ as noted in Section 1, while for (2) $\Rightarrow$, choose a new schematic letter D and put $(A \to D) \to ((B \to D) \to D)$ for C, which allows us to conclude to a re-lettered form of (1) with D written in place of C.)

[39]That result showed that these prefixes yielded, when attached to any given formula, formulas with the same truth value at arbitrary elements of a model and not just at u_0, so synonymy rather than mere equivalence is secured, as befits the relation of *definiens* and *definiendum*.

the grounds that the former can be rewritten as $\neg\neg\neg Rp$ (and is therefore equivalent to $\neg Rp$, alias $\neg_{\mathsf{d}} p$).

Postscript on Superclassical Options. There is no difficulty in providing substitution-invariant extensions of $\vdash_{CL}$ not satisfying (DT) analogous to those considered *à propos* of $\vdash_{IL}$ in this section, i.e., with additional logical vocabulary. The fact supervaluational semantics ([15], pp. 94–96) gave rise to such examples was widely commented on in the 1970s: see van Fraassen [15], p.170 two-thirds down, or Thomason [50], p.273 (and the contrast between (6.3) and (6.5) on p.275 thereof). What gives rise to the failure of (DT) is the use of universal quantification (over bivalent valuations) in these cases, so the same arises with the 'model consequence' or 'global consequence' relation in modal logic. (See for example Sections 3 and 5 of [22].) We mean here the relation holding between a set of modal formulas Γ and a single such formula A when in any Kripke model (for modal logic) at every point of which all of Γ are true, A too is true at every point. Since $\Box A$ is a global consequence in this sense of the formula A for any A, while $A \to \Box A$ is not in general (e.g., take A as p) a global consequence of $\varnothing$, we have a failure of (DT). This example may seem more controversial than the supervaluational cases in view of Smiley's comments (see note 15 above), which reflect a general preference against treating necessitation 'horizontally' rather than 'vertically', as it is put in Scott [42], and Scott's own recommendations on score in [42] (see the comment on Table II, p.149)[40]. However, all we wanted for our purposes was a subsitution-invariant consequence relation extending $\vdash_{CL}$ without (DT)—not one respecting the informal idea of 'inferential' consequence (what can be inferred from a set of suppositions, that is)—so the global consequence relation just mentioned meets our current needs (as well as revealing, incidentally, an infelicity in Gabbay's terminology – mentioned in note 15 – of 'provability rules' *vs.* 'consequence rules').

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[40]In view of Scott's contribution to the general theory of normal modal logics, it is particularly disheartening to see the term "normal" being defined (for consequence relations) in [6], [7], in such a way as to require that $A \vdash \Box A$, rather than requiring that $\Gamma \vdash A$ should imply $\Box\Gamma \vdash \Box A$

Mathematical Logic, who suggested some corrections.

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