REPORTS ON MATHEMATICAL LOGIC
40 (2006), 3–13

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UPPER PART OF THE LATTICE OF EXTENSIONS OF THE POSITIVE RELEVANT LOGIC $\mathbf{R}^+$

A b s t r a c t. In this paper it is proved that the interval $[\mathbf{R}^+, L(\mathbf{2}^+)]$ of the lattice of extensions of the positive (i.e. negationless) relevant logic $\mathbf{R}^+$ has exactly two co-atoms ($L(\mathbf{2}^+)$ denotes here the only Post-complete extension of $\mathbf{R}^+$). One of these two co-atoms is the only maximal extension of $\mathbf{R}^+$ which satisfies the *relevance property*: if $A \to B$ is a theorem then A and B have a common variable. A result of this kind for the relevant logic $\mathbf{R}$ was presented in Swirydowicz [1999].

1. Preliminaries. R^+-algebras

1. Let a set of propositional variables $p, q, r, \ldots$ be given and let F be the set of propositional formulae built up from propositional variables by means of the connectives: $\to$ (implication), $\wedge$ (conjunction), $\vee$ and (disjunction).

Received 31 May 2004

The positive (i.e. negationless) Anderson and Belnap logic $\mathbf{R}^+$ with relevant implication (cf. A.R. Anderson, N.D. Belnap [1975]) is defined as the subset of propositional formulae of F which are provable from the set of axiom schemes indicated below, by application of the rule of Modus Ponens (MP; $A, A \to B/B$) and the Rule of Adjunction ($A, B/A \wedge B$):

$A1.$ $A \to A$
$A2.$ $(A \to B) \to ((B \to C) \to (A \to C))$
$A3.$ $A \to ((A \to B) \to B)$
$A4.$ $(A \to (A \to B)) \to (A \to B)$
$A5.$ $A \wedge B \to A$
$A6.$ $A \wedge B \to B$
$A7.$ $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
$A8.$ $A \to A \vee B$
$A9.$ $B \to A \vee B$
$A10.$ $(A \to B) \wedge (C \to B) \to (A \vee C \to B)$
$A11.$ $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

Lemma 1.1. *The formulae listed below are theorems of* $\mathbf{R}^+$*:*

(t1) $(p \to q) \wedge (r \to s) \to (p \wedge r \to q \wedge s)$,
(t2) $(p \to q) \wedge (r \to s) \to (p \vee r \to q \vee s)$,
(t3) $(p \vee q \to r) \to (p \to r)$,
(t4) $(p \to q \wedge r) \to (p \to r)$,
(t5) $(p \to (q \to r)) \to (q \to (p \to r))$.
(t6) $((p \wedge q) \vee r) \to (p \wedge (q \vee r))$
(t7) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

2. To present an algebraic semantics for the logic $\mathbf{R}^+$ we will exercise some ideas presented in the paper of W. Dziobiak (cf. W. Dziobiak [1983]) and a paper of J. Font and G. Rodriguez (cf. J. Font and G. Rodriguez [1990]).

Definition 1.2. Let $\mathbf{A} = \langle A, \wedge, \vee, \to \rangle$ be an algebra similar to the algebra F of formulae. Then $\mathbf{A}$ is an R^+-algebra if and only if the reduct $\langle A, \wedge, \vee \rangle$ is a distributive lattice and moreover the following equalities and inequalities hold:

$D1.$ $(x \to y) \leq ((y \to z) \to (x \to z))$
$D2.$ $(x \to (x \to y)) \leq (x \to y)$
$D3.$ $(x \to (y \wedge z)) = (x \to y) \wedge (x \to z)$
$D4.$ $((x \vee y) \to z) = ((x \to z) \wedge (y \to z))$
$D5.$ $(x \to (y \to z)) \leq (y \to (x \to z))$
$D6.$ $x \leq ((x \to y) \wedge z) \to y$
$D7.$ $(x \to x) \wedge (y \to y) \to z \leq z$
where $\leq$ denotes the partial order in $\mathbf{A}$.

Let us note that by the Definition the class of R^+-algebras is a variety, because is equationally defined. Let us denote this variety by $\mathcal{R}^+$.

Let $\mathbf{A}$ be a R^+-algebra and let $\nabla_{\mathbf{A}} = [\{a \to a : a \in A\})$ i.e. let $\nabla_{\mathbf{A}}$ be a filter generated by all elements of the form $a \to a$. Then the pair $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ will be called a R^+-*matrix*. It is easy to prove that the relation $\preceq$ defined on A as follows: $x \preceq y$ iff $(x \to y) \in \nabla_{\mathbf{A}}$ is a partial order on A.

Lemma 1.3 (Font and Rodriguez). *1. Let* $\mathbf{A}$ *be an* R^+*-algebra. Then the relations* $\leq$ *and* $\preceq$ *on* A *coincide, i.e.* $x \to y \in \nabla_{\mathbf{A}}$ *iff* $x \leq y$.
2. Let the relation $\sim_{R^+}$ *be defined on the set* F *of formulae as follows:* $A \sim_{R^+} B$ *iff* $A \to B$ *as well as* $B \to A$ *are theorems of* $\mathbf{R}^+$*. Then the algebra* $\mathbf{F}/\sim_{R^+}$ *is a free* R^+*-algebra (*$\mathbf{F}$ *denotes here the algebra of formulae).*

The logic $\mathbf{R}^+$ is algebraizable in the sense of W.J. Blok and D. Pigozzi (cf. Blok and Pigozzi [1989]) and the variety $\mathcal{R}^+$ is the equivalent algebraic semantics for $\mathbf{R}^+$; the proofs presented by P. Font and G. Rodriguez in [1990] for the logic $\mathbf{R}$ work for $\mathbf{R}^+$ as well.

Let $\langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be a R^+-matrix. A filter ∇ on $\mathbf{A}$ is said to be *normal* iff $\nabla_{\mathbf{A}} \subseteq \nabla$. It is known that each normal filter on $\mathbf{A}$ determine a congruence on $\mathbf{A}$ (cf. Dziobiak [1983]). Note that it follows from the results concerning algebraizability of $\mathbf{R}^+$ that in each R^+-algebra the lattice of normal filters and the lattice of congruences are isomorphic.

At last let us note that the following useful Proposition holds.

Proposition 1.4. *Each finitely generated* R^+*-algebra has a least and a greatest element.*

A proof of this Proposition can be obtained by a slight modification of the proof of similar Proposition for R-algebras (cf. Swirydowicz [1999], Proposition 5).

2. Co-atoms in the interval $[\mathbf{R}^+, L(\mathbf{2}^+)]$

Let us begin with the Post-complete extensions of the logic $\mathbf{R}^+$. It is known that the only Post-complete axiomatic extension of $\mathbf{R}^+$ is the logic generated by the two-element algebra $\langle\{0,1\},\wedge,\vee,\rightarrow\rangle$, where the set $\{0,1\}$ is the two-element lattice with $0<1$ and the two-argument operation $\rightarrow$ is defined in the well-known way: $1\rightarrow 0=0$ and $1\rightarrow 1=0\rightarrow 1=0\rightarrow 0=1$. Let us denote here this algebra by $\mathbf{2}^+$. However, I do not know any proof of this fact, so I decided to present a simple algebraic proof of this fact here.

Let $L(\mathbf{2}^+)$ be the logic generated by the algebra $\mathbf{2}^+$.

Lemma 2.1. *The logic $L(\mathbf{2}^+)$ is the only Post-complete extension of the logic $\mathbf{R}^+$.*

Proof. Let $\mathbf{R}^+\subseteq L$ and let L be a nontrivial logic. Let V_L be a variety which determines the logic L and let $\mathbf{A}\in V_L$ be a nontrivial algebra. At last, let $\mathbf{B}$ be a nontrivial finitely generated subalgebra of $\mathbf{A}$. Let us denote by $\nabla_{\mathbf{B}}$ the filter of designated elements of the algebra $\mathbf{B}$. By Proposition 1.4 $\mathbf{B}$, has the least and the greatest element; let us denote them by 0 and 1, respectively. We prove now that the set $\{0,1\}$ is closed under the operation $\rightarrow$.

Since $1\in\nabla_{\mathbf{B}}$, $1\rightarrow 0\leq 0$ (because if $x\in\nabla_{\mathbf{B}}$ then $x\rightarrow y\leq y$ for any $y\in B$). Thus we have $1\rightarrow 0=0$.

To show that $0\rightarrow 0=1$ we argue as follows. It is clear that $(0\rightarrow 0)\in\nabla_{\mathbf{B}}$ and that $0\rightarrow 0\leq 1$. We will show that it is impossible that $0\rightarrow 0<1$. So, let us assume that $0\rightarrow 0<1$. Since $1\in\nabla_{\mathbf{B}}$, $1\rightarrow(0\rightarrow 0)\leq(0\rightarrow 0)$. Moreover, since $x\leq y$ if and only if $(x\rightarrow y)\in\nabla_{\mathbf{B}}$, $1\rightarrow(0\rightarrow 0)\notin\nabla_{\mathbf{B}}$, because we have assumed that $(0\rightarrow 0)<1$. In consequence $1\rightarrow(0\rightarrow 0)\neq(0\rightarrow 0)$, because if $(0\rightarrow 0)\leq 1\rightarrow(0\rightarrow 0)$ then (since $(0\rightarrow 0)\in\nabla_{\mathbf{B}}$) $1\rightarrow(0\rightarrow 0)\in\nabla_{\mathbf{B}}$, and it is a contradiction. Thus $1\rightarrow(0\rightarrow 0)<(0\rightarrow 0)$. On the other hand we have (by commutation and the equality $1\rightarrow 0=0$): $1\rightarrow(0\rightarrow 0)=0\rightarrow(1\rightarrow 0)=0\rightarrow 0$, and it is a contradiction which follows from the assumption that $0\rightarrow 0<1$. Thus $0\rightarrow 0=1$.

The equality $1\rightarrow 1=1$ we prove as follows. It is clear that $1\rightarrow 1\leq 1$. For the converse, let us note that by $(x\rightarrow y)\leq(y\rightarrow z)\rightarrow(x\rightarrow z)$ we have $(0\rightarrow 0)\leq(0\rightarrow 0)\rightarrow(0\rightarrow 0)$, i.e. (by $(0\rightarrow 0)=1$) $1\leq(1\rightarrow 1)$.

And the last equality: $0\rightarrow 1=1$. It is clear that $0\rightarrow 1\leq 1$. For the converse, let us observe that $0\leq 1$, thus $(0\rightarrow 1)\in\nabla_{\mathbf{B}}$. But we have

$(0 \to 1) = (0 \to (1 \to 1)) = 1 \to (0 \to 1)$, thus $(1 \to (0 \to 1)) \in \nabla_{\mathbf{B}}$, thus $1 \leq (0 \to 1)$ and it finishes the proof. □

We describe now two algebras which will play a crucial role in further considerations.

Let $\mathbf{3}_1 = \langle \{0, a, 1\}, \wedge, \vee, \to \rangle$ be an algebra such that $\langle \{0, a, 1\}, \wedge, \vee, \rangle$ is the three-element lattice $(0 < a < 1)$ and let the two-argument operation $\to$ will be defined by the following table:

$\to$	0	a	1
0	1	1	1
a	0	a	1
1	0	0	1

$\mathbf{3}_1$ is the positive reduct of the 3-element Sugihara algebra.

Let $\mathbf{3}_2 = \langle \{0, b, 1\}, \wedge, \vee, \to \rangle$ be an algebra such that $\langle \{0, b, 1\}, \wedge, \vee, \rangle$ is the three-element lattice $(0 < b < 1)$ and let the two-argument operation $\to$ will be defined by the following table:

$\to$	0	b	1
0	1	1	1
b	0	1	1
1	0	b	1

$\mathbf{3}_2$ is a $\perp$-free reduct of 3-element Heyting algebra.

Lemma 2.2. *The algebra $\mathbf{3}_1$ as well as the algebra $\mathbf{3}_2$ are R^+-algebras.*

Theorem 2.3. *If V is a non-trivial R^+-variety and $V \neq V(\mathbf{2}^+)$ then either $\mathbf{3}_1 \in V$ or $\mathbf{3}_2 \in V$.*

Proof. Let V be a R^+-variety and $V \neq V(\mathbf{2}^+)$. Then there exists a nontrivial algebra $\mathbf{A}$ in V, which does not belong to $V(\mathbf{2}^+)$ (it is e.g. a V-free algebra) and there exists a finitely generated nontrivial subalgebra $\mathbf{B}$ of the algebra $\mathbf{A}$ such that $\mathbf{B} \notin V(\mathbf{2}^+)$. By Proposition 1.4 $\mathbf{B}$ contains the least and the greatest element; let us denote them by 0 and 1, respectively. Moreover, by Dziobiak [1983], Lemma 1.2 the filter of designated element of $\mathbf{B}$ is a principal filter.

We will consider two cases: either the filter of designated elements of $\mathbf{B}$ consists of 1 only, or consists of greater number of elements.

Case 1. $\nabla_{\mathbf{B}} \neq [1)$; let $\nabla_{\mathbf{B}} = [a)$, $a \neq 1$.

We will prove that the algebra $\mathbf{3}_1$ is a subalgebra of $\mathbf{B}$.

Note first that $1 \to a \leq a$. It is true because since $a \leq 1$, $1 \to a \leq a \to a$ and $a \to a = a$ (cf. point b). below).

However, $1 \to a \neq a$. It is known that $x \leq y$ iff $(x \to y) \in \nabla_{\mathbf{B}}$, $a \in \nabla_{\mathbf{B}}$. So if $1 \to a = a$, $(1 \to a) \in \nabla_{\mathbf{B}}$ and in consequence $1 \leq a$, but it is impossible, because we assumed that $a \neq 1$.

Thus we have a chain $0 < (1 \to a) < a < 1$. Let us take the subalgebra generated by the set $\{1 \to a, a, 1\}$. It is a three-elements chain, so is closed under lattice operations; we prove that this set is closed under $\to$ as well, and the ,,$\to$-table" is just the table for $\mathbf{3}_1$ (modulo symbols for elements):

$\to$	$1 \to a$	a	1
$1 \to a$	1	1	1
a	$1 \to a$	a	1
1	$1 \to a$	$1 \to a$	1

a) $1 \to 1 = 1$ (cf. Lemma 2.1).

b) $a \to a = a$.

Proof: Since $a \in \nabla_{\mathbf{B}}$, $a \to a \leq a$. Conversely, since a generates the filter of designated elements, $a \leq a \to a$.

c) $a \to 1 = 1$.

Proof: Since a generates $\nabla_{\mathbf{B}}$, $a \leq 1 \to 1$ and by commutation $1 \leq a \to 1$.

d) $(1 \to a) \to a = 1$.

Proof:

It is clear that $(1 \to a) \to a \leq 1$. For the converse, since $1 \to a \leq 1 \to a$, so by commutation $1 \leq (1 \to a) \to a)$.

e) $(1 \to a) \to (1 \to a) = 1$.

Proof:

By a) and d) we have: $1 = 1 \to 1 = 1 \to ((1 \to a) \to a) = (1 \to a) \to (1 \to a)$.

f) $1 \to (1 \to a) = 1 \to a$.

Proof:

$(p \to (p \to q) \to (p \to q) \in \mathbf{R}^+$, thus $1 \to (1 \to a) \leq (1 \to a)$. For the converse, by e) it is known that $1 \leq (1 \to a) \to (1 \to a)$, thus $(1 \to a) \leq 1 \to (1 \to a)$.

g) $(1 \to a) \to 1 = 1$.

Proof:
Since $1 \to a \leq 1$, by a) $1 \to a \leq 1 \to 1$, thus $1 \leq (1 \to a) \to 1$.
h) $a \to (1 \to a) = 1 \to a$.
Proof:
By b), $a \to a = a$, thus by commutation $a \to (1 \to a) = 1 \to (a \to a) = 1 \to a$,
and it finishes the proof for the first case.

Case 2. Let $\nabla_{\mathbf{B}} = \{1\}$, but $\mathbf{B} \notin V(\mathbf{2}^+)$ (such algebras exist; $\mathbf{3}_2$ is an example of such algebra).

Since $\nabla_{\mathbf{B}} = \{1\}$, $\mathbf{B}$ is a Heyting algebra, and generally, algebras with $\{1\}$ as filter of designated elements are Heyting algebras. However, the algebra $\mathbf{3}_2$ belongs to every variety of Heyting algebras that properly contains $V(\mathbf{2}^+)$. That finishes the proof. □

3. Maximal strictly relevant extension of the logic $\mathbf{R}^+$ and its syntactical characterization

1. Let us begin with definitions. An extension L of the logic $\mathbf{R}^+$ is said to be *strictly relevant* iff L preserves the *relevance property*: if $A \to B$ is a theorem of L then A and B have a common variable. It is known that $\mathbf{R}^+$ preserves the relevance property. We will now look for maximal strictly relevant extensions of $\mathbf{R}^+$, i.e. these extensions which preserve relevance property.

It is easy to note that the algebra $\mathbf{3}_1$, i.e. the R^+ algebra $\langle\{0, a, 1\}, \vee, \wedge, \to\rangle$, where $0 < a < 1$ and the operation $\to$ is defined as follows

$\to$	0	a	1
0	1	1	1
a	0	a	1
1	0	0	1

contains two trivial (i.e. one-element) subalgebras: $\langle\{a\}, \vee, \wedge, \to\rangle$ and $\langle\{1\}, \vee, \wedge, \to\rangle$. Moreover, all non-relevant implications (in the language without negation) are falsified in this algebra.

For, let us take an implication $A \to B$ such that A and B not have a common variable. Let us define a valuation v as follows: $v(p_i) = 1$ for all

p_i of A, and $v(q_i) = a$ for all q_i of B and extend v to a homomorphism. We have now: $v(A \to B) = v(A) \to v(B) = 1 \to a = 0$. Since 0 is not a designated element of $\mathbf{3}_1$, $A \to B$ cannot be an $\mathbf{R}^+$-tautology.

Theorem 3.1. *Among extensions of* $\mathbf{R}^+$ *satisfying the relevance principle the logic* $L(\mathbf{3}_1)$ *is the only maximal one.*

Proof. Note that by previous theorem all non-trivial varieties of $\mathbf{R}^+$-algebras different from $V(\mathbf{2}_+)$ contain either $\mathbf{3}_1$ ot $\mathbf{3}_2$. Since $\mathbf{3}_2$ does not falsify all non-relevant implications (e.g. does not falsify $(p \to p) \to (q \to q)$), and $\mathbf{3}_1$ falsifies all of them, the logic $L(\mathbf{3}_1)$ must be the only maximal logic which preserve the relevance principle. It finishes the proof. □

2. Similarly as in the case of maximal strictly relevant extensions of the relevant logic $\mathbf{R}$ (cf. K. Swirydowicz [1999]) we present a syntactical characterization of the maximal strictly relevant extension of the logic $\mathbf{R}^+$.

Lemma 3.2. *Let* $\mathbf{A}$ *be an* R^+*-algebra. If* $\mathbf{A}$ *contains a trivial subalgebra* $\langle \{a\}, \wedge, \vee, \to \rangle$*, where* $a \neq 1$ *(if* $\mathbf{A}$ *contains an unit, of course), algebra* $\mathbf{3}_1$ *is a subalgebra of* $\mathbf{A}$*.*

Proof. Let $\langle \{a\}, \wedge, \vee, \to \rangle$ be a trivial subalgebra od $\mathbf{A}$ and let $a \neq 1$, where $a \neq 1$. Of course, $a \to a = a$, thus a belons to the filter of designated elements of $\mathbf{A}$.

A. Assume that $\mathbf{A}$ has an unit. In such a case the set $\{1 \to a, a, 1\}$ is closed under basic operations of $\mathbf{A}$, i.e. this set is a subalgebra of $\mathbf{B}$ isomorphic to $\mathbf{3}_1$.

B. Let $\mathbf{A}$ does not have a unit. Thus there exists a b in $\mathbf{A}$ such that $a < b$. Let us take a subalgebra $\mathbf{B}$ generated by a and b. By Theorem 1 this subalgebra has a unit; denote it by $1_{\mathbf{B}}$. This case can be reduced to the previous one: we will consider now the set $\{a, 1_{\mathbf{B}}, 1_{\mathbf{B}} \to a\}$. This finishes the proof. □

Now, let
$D(\mathbf{3_1}) = (p \to (p \to p)) \wedge ((p \to p) \to p) \wedge (p \to q) \wedge (p \to p)$,
and let
$\chi(\mathbf{3_1}) = D(\mathbf{3_1}) \wedge (q \to q) \to (q \to p)$.

Let $\mathcal{A} = \langle \mathbf{A}, \nabla_{\mathbf{A}} \rangle$ be an R^+-matrix and let $E(\mathcal{A})$ be the set of all $\mathcal{A}$-tautologies, i.e. the set of all formulae whose value belongs to ∇_A under any valuation. By $E(\mathbf{3_1})$ denote the set of $\mathbf{3_1}$-tautologies, i.e. all the the formulae A which satisfy the condition: $h(A) = 1$ or $h(A) = a$ where a, 1 belong to $\mathbf{3}_1$.

Now we prove a Jankow-style lemma.

Lemma 3.3. *Let* $\mathbf{A}$ *be an* $\mathbf{R}^+$*-algebra and let* $\mathcal{A} = \langle \mathbf{A}, \nabla_\mathbf{A} \rangle$ *be a matrix determined by this algebra. Then the following conditions are equivalent:*

(*i*) $\chi(\mathbf{3}_1) \notin E(\mathcal{A})$

(*ii*) $\mathbf{3_1} \in HS(\mathbf{A})$,

(*iii*) $E(\mathcal{A}) \subseteq E(\mathbf{3}_1)$.

Proof. (*i*) $\Rightarrow$ (*ii*). Let $\mathbf{A}$ be an $\mathbf{R}^+$-algebra and let $\mathcal{A} = \langle \mathbf{A}, \nabla_\mathbf{A} \rangle$ be the matrix determined by this algebra. Let $\chi(\mathbf{3_1}) \notin E(\mathcal{A})$. Then there is a valuation h such that $h(\chi(\mathbf{3_1})) \notin \nabla_\mathbf{A}$.

Note that even if the algebra $\mathbf{A}$ has a unit and a zero (denote them by $1_\mathbf{A}, 0_\mathbf{A}$, respectively), $h(D(\mathbf{3_1}) \wedge (q \to q)) \neq 0_\mathbf{A}$, because if not, then $h(\chi(\mathbf{3_1})) = 1_\mathbf{A}$.

Moreover, $h(p) \neq h(q)$. To prove it, assume that $h(p) = h(q)$ and let $h(p) = b$. Now, since $h(\chi(\mathbf{3_1})) \notin \nabla_\mathbf{A}$, the inequality $(b \to (b \to b)) \wedge ((b \to b) \to b) \wedge (b \to b) \leq (b \to b)$ cannot hold for this b. However, this inequality is simply an instance of a well-known lattice inequality $x \wedge y \leq y$. Thus $h(p) \neq h(q)$.

Finally, note that $h(q \to p)$ does not belong to $\nabla_\mathbf{A}$. For if it does, then $h(q) \leq h(p)$, and since $x \leq y$ entails here $z \to x \leq z \to y$, $h(q \to q) \leq h(q \to p)$. Note that in each lattice the following implication holds: if $x \leq y$, then $z \wedge x \leq y$, thus $h(\chi(\mathbf{3}_1)) \in \nabla_\mathbf{A}$, but it is impossible.

Consider now a subalgebra of $\mathbf{A}$, generated by $h(p), h(q)$; denote this subalgebra by $\mathbf{B}$. Note that the filter $\nabla_\mathbf{B} = [h(p \to p) \wedge h(q \to q))_\mathbf{B}$ is a filter of designated elements of the matrix $\mathcal{B}$, determined by the algebra $\mathbf{B}$. Thus the filter $\nabla = [h(D(\mathbf{3}_1) \wedge (q \to q))_\mathbf{B}$ is a nontrivial normal filter on $\mathbf{B}$ ($0_\mathbf{A} \notin \nabla$, $\nabla_\mathbf{B} \subseteq \nabla$). Note now that $h(q \to p) \notin \nabla$. For if it does, then $h(D(\mathbf{3}_1) \wedge (q \to q)) \leq h(q \to p)$, i.e. $h(\chi(\mathbf{3}_1)) \in \nabla_\mathbf{A}$, but it is impossible. It follows from it that in fact the normal filter ∇ is nontrivial. Thus ∇ determines a (nontrivial) congruence relation $\Theta(\nabla)$ in the algebra $\mathbf{B}$.

We prove now that the algebra $\mathbf{3}_1$ is a subalgebra of the quotient algebra $\mathbf{B}/\Theta(\nabla)$.

Let us denote by ∇^* the filter of designated elements of $\mathbf{B}/\Theta(\nabla)$ and let us introduce the following abbreviations: $a = h(p)/\Theta(\nabla)$, $b = h(q)/\Theta(\nabla)$.

a) Since $h(D(\mathbf{3}_1))/\Theta(\nabla) \in \nabla^*$, the equality $a \to a = a$ as well as the inequality $a \leq b$ hold. Of course, the set $\{a\}$ is closed under lattice operations, thus it is a one-element subalgebra of the algebra $\mathbf{B}/\Theta(\nabla)$.

b) $b \to a$ does not belong to ∇^*. Assume contrary. Then $h(q)/\Theta(\nabla) \leq h(p)/\Theta(\nabla)$, and since $h(p) \to h(q) \in \nabla$, $h(p) \equiv h(q)(\Theta(\nabla))$, thus $h(q) \to h(p) \in \nabla$, but it is impossible (cf. above).

It follows from it that although in our algebra the inequality $a \leq b$ holds, nevertheless the equality $a = b$ does not hold, thus $a < b$. Note that a cannot be an unit of the quotient algebra.

In this way we proved that the assumptions of the previous Lemma are satisfied. In consequence $\mathbf{3}_1 \in SHS(\mathbf{A})$, thus $\mathbf{3}_1 \in HS(\mathbf{A})$.

$(ii) \Rightarrow (iii)$: Obvious.

$(iii) \Rightarrow (i)$: Let h be a valuation of the algebra of formulae $\mathbf{F}$ in the algebra $\mathbf{3}_1$ which satisfies the following conditions: $h(p) = a$, $h(q) = 1$. Thus $h(D(\mathbf{3}_1 \wedge (q \to q)) = a$ and $h(q \to p) = 0$; since in $\mathbf{3}_1$ the equality $a \to 0 = 0$ holds, $\chi(\mathbf{3}_1) \notin E(\mathcal{A})$, and it finishes the proof of this Lemma.□

The last Theorem is a simple consequence of this Lemma.

Theorem 3.4. *Let L be an extension of the relevant logic* $\mathbf{R}^+$*. Then for L the relevance principle holds if and only if the formula* $\chi(\mathbf{3}_1)$*, i.e. the formula* $(p \to (p \to p)) \wedge ((p \to p) \to p) \wedge (p \to q) \wedge (p \to p) \wedge (q \to q) \to (q \to p)$ *is not a theorem of L.*

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