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ON THE ASYMPTOTIC DENSITY OF TAUTOLOGIES IN LOGIC OF IMPLICATION AND NEGATION

A b s t r a c t. The paper solves the problem of finding the asymptotic probability of the set of tautologies of classical logic with one propositional variable, implication and negation. We investigate the proportion of tautologies of the given length n among the number of all formulas of length n. We are especially interested in asymptotic behavior of this proportion when $n \to \infty$. If the limit exists it represents the real number between 0 and 1 which we may call *density of tautologies* for the logic investigated. In the paper [2] the existence of this limit for classical (and at the same time intuitionistic) logic of implication built with exactly one variable is proved. The present paper answers the question *"how much does the introduction of negation influence the "density of tautologies" in the propositional calculus of one variable?"* While in the case of implicational calculus the limit exists and is about 72.36% (see Theorem 4.6 in the paper [2]), in our case the limit exists as well, but negation lowers the *density of tautologies* to the level of about 42.32%.

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1. Introduction

In this paper we examine the density of tautologies among all propositional formulas built up from one propositional variable by means of implication and negation. Our main result is a precise estimate of the number of tautologies among formulas of length n with one propositional variable. This result was partially motivated by the paper [2] in which it has been proved that among formulas with one variable the fraction of tautologies tends to $1/2 + \sqrt{5}/10$ as the length of formulas approaches infinity.

2. Implicational - negational formulas

In this section we present some properties of numbers characterizing the number of formulas of the given length of our language.

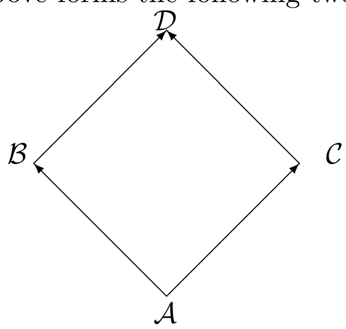
The language of implicational - negational formulas of one propositional variable a consists of formulas $\mathcal{F}$ built from a by means of negation and implication only.

$$\begin{array}{rcl} a & \in & \mathcal{F} \\ \phi \to \psi \in \mathcal{F} & if & \phi \in \mathcal{F} \text{ and } \psi \in \mathcal{F} \\ \neg\phi \in \mathcal{F} & if & \phi \in \mathcal{F} \end{array}$$

We start this section by dividing the set of all formulas into four classes according to their behavior on the two possible valuations. Since we have formulas built with exactly one propositional variable a we can evaluate formulas by two valuations: ν_0 associating 0 to a and ν_1 associating 1 to a. Therefore:

Definition 2.1. For any $i, j \in \{0, 1\}$ by $\mathcal{F}^{i,j}$ we mean the set of formulas ϕ such that $\nu_0(\phi) = i$ and $\nu_1(\phi) = j$. The four classes $\mathcal{F}^{i,j}$ are ordered by assuming that $\mathcal{F}^{i,j} \leq \mathcal{F}^{i',j'}$ if $i \leq i'$ and $j \leq j'$. On our four classes $\mathcal{F}^{i,j}$ we can establish an operation of implication $\Rightarrow$ by $\mathcal{F}^{i,j} \Rightarrow \mathcal{F}^{i',j'} = \mathcal{F}^{i \to i', j \to j'}$ where $\to$ stands for the classical implication defined on the set $\{0, 1\}$ and an operation of negation by $\neg\mathcal{F}^{i,j} = \mathcal{F}^{\neg i, \neg j}$. For simplicity we will denote classes $\mathcal{F}^{i,j}$ respectively by: $\mathcal{A} := \mathcal{F}^{0,0}$, $\mathcal{B} := \mathcal{F}^{0,1}$, $\mathcal{C} := \mathcal{F}^{1,0}$ and $\mathcal{D} := \mathcal{F}^{1,1}$. Note that $\mathcal{D}$ is the class of all tautologies in $\mathcal{F}$ and the atomic formula $a \in \mathcal{B}$.

The order defined above forms the following two dimensional poset:



Classes are defined in such a way that if $\phi \in \mathcal{F}^{i,j}$ and $\psi \in \mathcal{F}^{i',j'}$ then $\phi \to \psi \in \mathcal{F}^{i,j} \Rightarrow \mathcal{F}^{i',j'}$ and $\neg\phi \in \neg\mathcal{F}^{i,j}$. The implication $\Rightarrow$ and negation $\neg$ defined on classes can be displayed by the following table.

$\Rightarrow$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\neg$
$\mathcal{A}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{C}$
$\mathcal{C}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{B}$
$\mathcal{D}$	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{A}$

Table 1

For technical reasons we are going to find the total number of formulas also in classes obtained from the poset by projecting it in two possible ways. Let us define new classes of formulas $\mathcal{K} = \mathcal{A} \cup \mathcal{C}$ and $\mathcal{L} = \mathcal{B} \cup \mathcal{D}$. The second projected poset is defined as $\mathcal{I} = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{J} = \mathcal{C} \cup \mathcal{D}$.

The orders defined on projected classes form the following two one-dimensional posets:

We define in the natural way implication $\Rightarrow$ and negation $\neg$ on the projected classes $\{\mathcal{I}, \mathcal{J}\}$ and $\{\mathcal{K}, \mathcal{L}\}$.

$\Rightarrow$	$\mathcal{I}$	$\mathcal{J}$	$\neg$
$\mathcal{I}$	$\mathcal{J}$	$\mathcal{J}$	$\mathcal{J}$
$\mathcal{J}$	$\mathcal{I}$	$\mathcal{J}$	$\mathcal{I}$

$\Rightarrow$	$\mathcal{K}$	$\mathcal{L}$	$\neg$
$\mathcal{K}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{L}$	$\mathcal{K}$	$\mathcal{L}$	$\mathcal{K}$

3. Counting formulas

First we have to establish the way the length of formulas is measured. By $|\phi|$ we mean the length of the formula ϕ, which we define as the total number of characters in the formula, including implication and negation signs. Parentheses which are sometimes necessary are not included in the length of formula.

$$|a| = 1 \tag{3.1}$$
$$|\phi \to \psi| = |\phi| + |\psi| + 1 \tag{3.2}$$
$$|\neg\phi| = |\phi| + 1 \tag{3.3}$$

Definition 3.1. By $\mathcal{F}_n$ we mean the set of formulas of length $n-1$. Subclasses of formulas of length n are defined accordingly by:

$$\mathcal{A}_n = \mathcal{F}_n \cap \mathcal{A} \tag{3.4}$$
$$\mathcal{B}_n = \mathcal{F}_n \cap \mathcal{B} \tag{3.5}$$
$$\mathcal{C}_n = \mathcal{F}_n \cap \mathcal{C} \tag{3.6}$$
$$\mathcal{D}_n = \mathcal{F}_n \cap \mathcal{D} \tag{3.7}$$
$$\mathcal{I}_n = \mathcal{F}_n \cap \mathcal{I} \tag{3.8}$$
$$\mathcal{J}_n = \mathcal{F}_n \cap \mathcal{J} \tag{3.9}$$
$$\mathcal{K}_n = \mathcal{F}_n \cap \mathcal{K} \tag{3.10}$$
$$\mathcal{L}_n = \mathcal{F}_n \cap \mathcal{L} \tag{3.11}$$

Definition 3.2. The number F_n is given by the recurrence:

$$F_0 = 0, \quad F_1 = 0, \quad F_2 = 1 \tag{3.12}$$
$$F_n = F_{n-1} + \sum_{i=1}^{n-1} F_i F_{n-i} \tag{3.13}$$

Lemma 3.3. *The number of formulas of length $n-1$ is F_n. So $F_n = \#\mathcal{F}_n$.*

Proof. Any formula of length $n-1$ for $n > 2$ is either a negation of some formula of length $n-2$, for which the fragment F_{n-1} is responsible, or is the implication between pair of formulas of lengths $i-1$ and $n-i-1$, respectively. The length of any of such implicational formula must be $(i-1)+(n-i-1)+1$ which is exactly $n-1$. Therefore the total number of such formulas is $\sum_{i=1}^{n-1} F_i F_{n-i}$. □

4. Generating functions

The main tool we use for dealing with asymptotics of sequences of numbers are *generating functions.* A nice exposition of the method can be found in [6] and [1]. Also see [2] for the presentation of this method in logics.

Many questions concerning the asymptotic behavior of a sequence A can be efficiently resolved by analyzing the behavior of generating function f_A at the complex circle $|z| = R$. The key tool will be the following result due to Szegö [5] [Thm. 8.4], see as well [6] [Thm. 5.3.2] which relates the generating functions of numerical sequences with limit of fractions. For the technique of proof described below please consult also [2]. We need the following much simpler versions of the Szegö Lemma with one and two singularities

Lemma 4.1. *(simplified Szegö lemma) Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ the only singularity at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has an expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p (1-z)^{\frac{p}{2}}, \tag{4.1}$$

where $p > 0$, and the branch chosen above for the expansion equals to $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}). \tag{4.2}$$

Lemma 4.2. *Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ and $z = -1$ the only singularities at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has the expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p^{(1)} (1-z)^{p/2}, \tag{4.3}$$

where $p > 0$, and the branch chosen above for the expansion equals $v(0)$ for $z = 0$, and again $v(z)$ in the vicinity of $z = -1$ has the expansion of the form

$$v(z) = \sum_{p \geq 0} v_p^{(-1)} (1+z)^{p/2}, \tag{4.4}$$

where $p > 0$, and the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = \left(v_1^{(1)} + v_1^{(-1)}\right) \begin{pmatrix} 1/2 \\ n \end{pmatrix} ((-1)^n + 1) + O(n^{-2}). \tag{4.5}$$

From the formula above we can immediately see that for odd n we have $[z^n]\{v(z)\} = O(n^{-2})$.

The symbol $[z^n]\{v(z)\}$ stands for the coefficient of z^n in the exponential series expansion of $v(z)$. For technical reasons we will need to know the rate of growth of the function $\begin{pmatrix} 1/2 \\ n \end{pmatrix} (-1)^n$ which appears at the formula (4.2) and (4.5).

Lemma 4.3. *For $n \in \mathbb{N}$ we have*

$$\begin{pmatrix} 1/2 \\ n \end{pmatrix} (-1)^{n+1} \quad = \quad O(n^{-3/2}) \tag{4.6}$$

Proof. It can be obtained by the Stirling approximation formula

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \quad < n! < \quad \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \tag{4.7}$$

(see [3] for details, consult also Lemma 7.5 page 589 of [2]). □

5. Calculating generating functions

We start by calculating the generating functions we will need.

Lemma 5.1. *The generating function f_F for the numbers F_n is*

$$f_F(x) = \frac{1-x}{2} - \frac{\sqrt{(x+1)(1-3x)}}{2}, \tag{5.1}$$

Proof. The recurrence $F_n = F_{n-1} + \sum_{i=1}^{n-2} F_i F_{n-i}$ becomes the equality

$$f_F(x) = x f_F(x) + f_F^2(x) + x^2, \tag{5.2}$$

since the recurrence fragment $\sum_{i=1}^{n-2} F_i F_{n-i}$ exactly corresponds to the multiplication of power series. The term F_{n-1} corresponds to the function $x f_F(x)$. The quadratic term x^2 corresponds to the first non-zero coefficient in the power series of f_F. Solving the equation we get two possible solutions: $f_F(x) = (1-x)/2 - \sqrt{-3x^2 - 2x + 1}/2$ or $f_F(x) = (1-x)/2 + \sqrt{-3x^2 - 2x + 1}/2$. We have to choose the first solution, since it corresponds to the assumption $f_F(0) = F_0 = 0$ (see equality (3.12)). □

Definition 5.2. The numbers I_n and J_n are given by the recurrence:

$$I_0 = 0, \quad I_1 = 0, \quad I_2 = 1 \tag{5.3}$$

$$J_0 = 0, \quad J_1 = 0, \quad J_2 = 0 \tag{5.4}$$

$$I_n = \sum_{i=1}^{n-1} I_i J_{n-i} + J_{n-1} \tag{5.5}$$

$$J_n = F_n - I_n \tag{5.6}$$

Lemma 5.3. *The numbers of formulas from classes $\mathcal{I}_n, \mathcal{J}_n$ are respectively $I_n = \#\mathcal{I}_n$ and $J_n = \#\mathcal{J}_n$.*

Proof. A formula from $\mathcal{I}_n$ is either a negation of some formula from $\mathcal{J}_{n-1}$ (see the fragment J_{n-1}) or it is an implication between a pair of formulas of lengths i and $n-i$ respectively. The total number of such formulas is $\sum_{i=1}^{n-1} I_i J_{n-i}$. □

Lemma 5.4. *The generating functions f_I, f_J for numbers I_n, J_n are*

$$f_I(x) = \frac{(f_F(x) - 1 - x)}{2} + \frac{\sqrt{4x^2 + x f_F(x) - f_F(x) + 2x + 1}}{2}, \tag{5.7}$$

$$f_J(x) = \frac{(f_F(x) + 1 + x)}{2} - \frac{\sqrt{4x^2 + x f_F(x) - f_F(x) + 2x + 1}}{2}. \tag{5.8}$$

Proof. The recurrence $I_n = \sum_{i=1}^{n-1} I_i J_{n-i} + J_{n-1}$ (see 5.5) becomes the equality

$$f_I(x) = f_I(x) f_J(x) + z f_J(x) + x^2 \tag{5.9}$$

since the recurrence fragment $\sum_{i=1}^{n-1} I_i J_{n-i}$ exactly corresponds to the multiplication of power series and J_{n-1} corresponds to the function $x f_J(x)$. The quadratic term x^2 corresponds to the first non-zero coefficient in the power series of f_I. Together with the recurrence $J_n = F_n - I_n$ it forms the system of equations:

$$f_I(x) = f_I(x) f_J(x) + x f_J(x) + x^2, \tag{5.10}$$

$$f_J(x) = f_F(x) - f_I(x). \tag{5.11}$$

Solving the system we get

$$f_I(x) = f_I(x) f_F(x) - f_I^2(x) + x f_F(x) - x f_I(x) + x^2 \tag{5.12}$$

with two possible solutions:

$$f_I(x) = (f_F(x) - 1 - x)/2 - \sqrt{4x^2 + x f_F(x) - f_F(x) + 2x + 1}/2,$$

and

$$f_I(x) = (f_F(x) - 1 - x)/2 + \sqrt{4x^2 + x f_F(x) - f_F(x) + 2x + 1}/2.$$

We have to choose the second solution, since it corresponds to the assumption $f_I(0) = I_0 = 0$ (see equation (5.3)).

A solution for f_J is similarly:

$$f_J(x) = (f_F(x) + 1 + x)/2 - \sqrt{(4x^2 + x f_F(x) - f_F(x) + 2x + 1)}/2. \quad \square$$

Definition 5.5. The numbers K_n and L_n are given by the recursion:

$$K_0 = 0, \; K_1 = 0, \; K_2 = 0 \tag{5.13}$$

$$L_0 = 0, \; L_1 = 0, \; L_2 = 1 \tag{5.14}$$

$$K_n = \sum_{i=1}^{n-1} L_i K_{n-i} + K_{n-1} \tag{5.15}$$

$$L_n = F_n - K_n \tag{5.16}$$

Lemma 5.6. *The numbers of formulas from classes $\mathcal{K}_n, \mathcal{L}_n$ are $K_n = \#\mathcal{K}_n$ and $L_n = \#\mathcal{L}_n$ respectively.*

Proof. The proof is similar to the proof of Lemma 5.3 □

Lemma 5.7. *The generating functions f_K, f_L for numbers K_n, L_n are*

$$f_K(x) = \frac{f_F(x) - 1 - x}{2} + \frac{\sqrt{xf_F(x) - f_F(x) + 2x + 1}}{2}, \tag{5.17}$$

$$f_L(x) = \frac{f_F(x) + 1 + x}{2} - \frac{\sqrt{xf_F(x) - f_F(x) + 2x + 1}}{2}. \tag{5.18}$$

Proof. Calculations are similar as in the proof of Lemma 5.4. The system of equations to be solved is:

$$f_K(x) = f_K(x)f_L(x) + xf_L(x), \tag{5.19}$$
$$f_L(x) = f_F(x) - f_K(x). \tag{5.20}$$

Notice the lack of the x^2 term in the first equation. □

In order to make use of already solved classes, we design the recurrence, in such a way that each class is defined in terms of the class $\mathcal{C}$.

Definition 5.8. The numbers A_n, B_n, C_n and D_n are given by the recursions:

$$\begin{array}{lcll} A_0 = A_1 = 0 & , & A_n = & K_n - C_n \\ B_0 = B_1 = 0 & , & B_n = & I_n - A_n = I_n - K_n + C_n \\ C_0 = C_1 = 0 & , & C_n = & \sum_{i=1}^{n-1} C_i D_{n-i} + \sum_{i=1}^{n-1} A_i B_{n-i} \\ & & & + \sum_{i=1}^{n-1} B_i C_{n-i} + B_{n-1} \\ D_0 = D_1 = 0 & , & D_n = & I_n - C_n \end{array}$$

Lemma 5.9. *The numbers of formulas from classes $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n$ are respectively: $A_n = \#\mathcal{A}_n, B_n = \#\mathcal{B}_n, C_n = \#\mathcal{C}_n, D_n = \#\mathcal{D}_n$.*

Proof. It follows from the table 1 in Definition 2.1. Any formula from $\mathcal{C}_n$ is either a negation of some formula from $\mathcal{B}_{n-1}$ or it is an implication between a pair of formulas of lengths i and $n - i$ respectively. According to Table 1 there are three possible situations. The implication can be only between formulas which are taken from the classes $\mathcal{D}$ and $\mathcal{C}$ or $\mathcal{B}$ and $\mathcal{A}$ or $\mathcal{B}$ and $\mathcal{C}$ respectively. The total number of such formulas must be therefore $\sum_{i=1}^{n-1} C_i D_{n-i} + \sum_{i=1}^{n-1} A_i B_{n-i} + \sum_{i=1}^{n-1} B_i C_{n-i}$. □

Lemma 5.10. *The generating functions f_A, f_B, f_C, f_D for the numbers A_n, B_n, C_n, D_n are*

$$f_A = f_K - f_C, \tag{5.21}$$
$$f_B = f_L - f_D, \tag{5.22}$$
$$f_C = \frac{(-1 + f_J + f_K + x)}{2} + \frac{\sqrt{(1 - f_J - f_K - x)^2 + 4(f_I f_K - f_K^2 + x f_I - x f_K)}}{2}, \tag{5.23}$$
$$f_D = f_J - f_C. \tag{5.24}$$

Proof. The recurrence given in Definition 5.8 and Lemma 5.9 becomes after a closer examination the system of functional equations:

$$f_A(x) = f_K(x) - f_C(x), \tag{5.25}$$
$$f_B(x) = f_I(x) - f_K(x) + f_C(x), \tag{5.26}$$
$$f_C(x) = f_D(x) f_C(x) + f_A(x) f_B(x) + f_B(x) f_C(x) + x f_B(x), \tag{5.27}$$
$$f_D(x) = f_J(x) - f_C(x), \tag{5.28}$$

By substituting into (5.27) and simplification we get:

$$f_C(x) = [f_J(x) - f_C(x)] f_C(x) + [f_I(x) - f_K(x) + f_C(x)](f_K(x) + x) \tag{5.29}$$

which is in fact a quadratic equation with respect to the unknown function f_C. Solving with boundary condition $f_C(0) = C_0 = 0$ we get:

$$f_C = \frac{(-1 + f_J + f_K + x)}{2} + \frac{\sqrt{(1 - f_J - f_K - x)^2 + 4(f_I f_K - f_K^2 + x f_I - x f_K)}}{2} \tag{5.30}$$

From this and equations (5.25), (5.26) and (5.28) we find other functions. In particular the function f_D which describes the numbers of tautologies is the following:

$$f_D(x) = \frac{(1+f_J-f_K-x)}{2} - \frac{\sqrt{(1-f_J-f_K-x)^2-4(-f_If_K+f_K^2-xf_I+xf_K)}}{2}, \tag{5.31}$$

where f_I, f_J, f_K are given in Lemmas 5.4 and 5.7.

After appropriate simplification we get the following formula for $f_D(x)$:

$$f_D(x) = \frac{1}{8}\Big(8-\sqrt{2}\sqrt{1+6x-x^2-Y}-\sqrt{2}\sqrt{1+6x+7x^2-Y} \tag{5.32}$$

$$-2\sqrt{1-10x+3x^2-Y+\sqrt{1+6x-x^2-Y}\sqrt{1+6x+7x^2-Y}}\Big),$$

where $Y = (1-x)\sqrt{(1+x)(1-3x)}$. □

6. From generating functions to asymptotic densities

In this section we have gathered several proofs making substantial use of generating functions. In order to apply Szegö Lemma 4.1 in the simplified version we must calibrate functions f_F, f_D in such a way to obtain the singularity of the smallest modulus at $z = 1$. Since the singularity of the smallest modulus for functions f_F, f_D is at $x = 3$ we substitute $x = 3z$ and obtain new calibrated functions:

Definition 6.1.

$$\widehat{f_D}(x) = f_D(x/3), \tag{6.1}$$
$$\widehat{f_F}(x) = f_F(x/3). \tag{6.2}$$

After simplification of the formula we get:

$$\widehat{f_D}(x)=\frac{1}{24}\Bigg[24-\sqrt{2}\sqrt{9+18x-x^2-Y}-\sqrt{2}\sqrt{9+18x+7x^2-Y} \qquad (6.3)$$
$$-2\sqrt{9-30x+x^2-Y+\sqrt{9+18x-x^2-Y}\sqrt{9+18x+7x^2-Y}}\Bigg]$$

$$\widehat{f_F}(x)=\frac{1}{6}\left(3-x-\sqrt{3}\sqrt{(1-x)(3+x)}\right) \qquad (6.4)$$

where $Y=\sqrt{3}(x-3)\sqrt{(1-x)(3+x)}$.

Note that the relation between the coefficients of the power series $\widehat{f_D}$ and f_D is as follows: $[x^n]\{f_D(x)\}=\left([x^n]\{\widehat{f_D}(x)\}\right)3^n$. The same relation holds for $\widehat{f_F}$ and f_F: $[x^n]\{f_F(x)\}=\left([x^n]\{\widehat{f_F}(x)\}\right)3^n$.

Lemma 6.2. *$x=1$ is the only singularity of $\widehat{f_D}$ and $\widehat{f_F}$ located in $|x|\le 1$.*

Proof. It is obvious that $\widehat{f_F}$ (see 6.4) has only singularities at $x=1$ and $x=-3$. For $\widehat{f_D}$ (see (6.3) apart from $x=1$ and $x=3$ there can only be singularity when the expression under the outer root signs become 0. Therefore another possible singularities of $\widehat{f_D}$ are solutions of the following equations.

$$9+18x-x^2-Y=0 \quad (6.5)$$
$$9+18x+7x^2-Y=0 \quad (6.6)$$
$$9-30x+x^2-Y+\sqrt{9+18x-x^2-Y}\sqrt{9+18x+7x^2-Y}=0 \quad (6.7)$$

where $Y=\sqrt{3}(x-3)\sqrt{(1-x)(3+x)}$. There are 7 complex solutions of those equations. However the minimal modulus among them is approximately 1.2245. So indeed $x=1$ is the only singularity of $\widehat{f_D}(x)$ and $\widehat{f_F}(x)$ in $|x|\le 1$. □

Now we find the expansions of functions $\widehat{f_D}$ and $\widehat{f_F}$ into power series in the vicinity of $x=1$.

Theorem 6.3. *Expansions of $\widehat{f_D}$ and $\widehat{f_F}$ in the vicinity of $x=1$ are:*

$$\begin{aligned}\widehat{f_F}(x) &= f_0+f_1\sqrt{1-x}+\dots\\ \widehat{f_D}(x) &= d_0+d_1\sqrt{1-x}+\dots\end{aligned}$$

where

$$
\begin{aligned}
f_0 &= \frac{1}{3},\\
f_1 &= -\frac{1}{\sqrt{3}}, \qquad (6.8)\\
d_0 &= \frac{\left(12-\sqrt{13}-\sqrt{17}-\sqrt{2(\sqrt{221}-9)}\right)}{12},\\
d_1 &= -\frac{1}{2}\sqrt{\frac{1593+107\sqrt{221}+\sqrt{2781590+187110\sqrt{221}}}{23205}}. \qquad (6.9)
\end{aligned}
$$

Proof. Two first terms of the power series expansions have been found using the *Mathematica*® [1] package. □

Now let us enjoy fruits of our (and *Mathematica*) hard labor.

Theorem 6.4. [Main result - the density of tautologies]

$$
\begin{aligned}
\lim_{n\to\infty}\frac{D_n}{F_n} &= \frac{1}{(4\sqrt{13})}+\frac{1}{(4\sqrt{17})}+ \qquad (6.10)\\
&+\frac{1}{2\sqrt{2\left(\sqrt{221}-9\right)}}+\frac{15}{2\sqrt{442\left(\sqrt{221}-9\right)}}
\end{aligned}
$$

Proof. By the Szegö lemma sequences D_n and F_n can be expressed in terms of d_1, f_1 (see (6.9), (6.8)). We get

$$
\begin{aligned}
D_n &= [z^n]\{f_D(z)\} = \left([z^n]\{\widehat{f_D}(z)\}\right)3^n \qquad (6.11)\\
&= \left(d_1\binom{1/2}{n}(-1)^n+O(n^{-2})\right)3^n \qquad (6.12)
\end{aligned}
$$

and

$$
\begin{aligned}
F_n &= [z^n]\{f_F(z)\} = \left([z^n]\{\widehat{f_F}(z)\}\right)3^n \qquad (6.13)\\
&= \left(f_1\binom{1/2}{n}(-1)^n+O(n^{-2})\right)3^n. \qquad (6.14)
\end{aligned}
$$

[1] *Mathematica* is a registered trademark of Wolfram Research

Having f_1 and d_1 computed in Theorem 6.3 and by Lemma 4.3 we get:

$$\frac{D_n}{F_n} = \frac{\left(d_1 \binom{1/2}{n} (-1)^n + O(n^{-2})\right) 3^n}{\left(f_1 \binom{1/2}{n} (-1)^n + O(n^{-2})\right) 3^n} = \frac{d_1}{f_1}(1 + o(1)) \approx 0.4232... \tag{6.15}$$

After simplification we get:

$$\frac{d_1}{f_1} = \frac{1}{(4\sqrt{13})} + \frac{1}{(4\sqrt{17})} + + \frac{1}{2\sqrt{2\left(\sqrt{221}-9\right)}} + \frac{15}{2\sqrt{442\left(\sqrt{221}-9\right)}} \tag{6.16}$$

This concludes the proof. The numerical value of $\frac{d_1}{f_1}$ is 0.423238538401941...

□

Using the same method we can find the densities of the classes $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

Theorem 6.5. *[Distribution of densities]*

$$\lim_{n\to\infty} \frac{A_n}{F_n} = -\frac{1}{4\sqrt{13}} - \frac{1}{4\sqrt{17}} + \frac{1}{2\sqrt{2\,\left(\sqrt{221}-9\right)}} + \frac{15}{2\sqrt{442\,\left(\sqrt{221}-9\right)}} \approx 0.1632\ldots \tag{6.17}$$

$$\lim_{n\to\infty} \frac{B_n}{F_n} = \frac{1}{2} + \frac{1}{4\sqrt{13}} - \frac{1}{4\sqrt{17}} - \frac{1}{2\sqrt{2\,\left(\sqrt{221}-9\right)}} - \frac{15}{2\sqrt{442\,\left(\sqrt{221}-9\right)}} \approx 0.2154\ldots \tag{6.18}$$

$$\lim_{n\to\infty}\frac{C_n}{F_n} = \frac{1}{2}-\frac{1}{4\sqrt{13}}+\frac{1}{4\sqrt{17}}-\frac{1}{2\sqrt{2\left(\sqrt{221}-9\right)}} - \frac{15}{2\sqrt{442\left(\sqrt{221}-9\right)}} \approx 0.1980\ldots \tag{6.19}$$

$$\lim_{n\to\infty}\frac{D_n}{F_n} = \frac{1}{(4\sqrt{13})}+\frac{1}{(4\sqrt{17})}+\frac{1}{2\sqrt{2\left(\sqrt{221}-9\right)}} + \frac{15}{2\sqrt{442\left(\sqrt{221}-9\right)}} \approx 0.4232\ldots \tag{6.20}$$

Notice the lack of symmetry between classes $\mathcal{B}$ and $\mathcal{C}$.

7. Implicational fragment $\mathcal{F}^{\to}$ of $\mathcal{F}$

In this section we want to find the asymptotic density of the implicational fragment of the propositional calculus of one variable. Let us define $\mathcal{F}^{\to}$ to be the set of formulas with one propositional variable and with implication only. In compliance with the convention of Definition 3.1 we define $\mathcal{F}^{\to}_n = \mathcal{F}^{\to} \cap \mathcal{F}_n$.

Definition 7.1. The number $F^{\to}_n$ is given by the recurrence:

$$F^{\to}_0 = 0,\quad F^{\to}_1 = 0,\quad F^{\to}_2 = 1 \tag{7.1}$$

$$F^{\to}_n = \sum_{i=1}^{n-1} F^{\to}_i F^{\to}_{n-i} \tag{7.2}$$

Lemma 7.2. *The number of formulas from the class $\mathcal{F}^{\to}$ of length $n-1$ is $F^{\to}_n$.*

Proof. The proof is straightforward. □

Lemma 7.3. *The generating function $f_{F^{\to}}$ for the numbers $F^{\to}_n$ is*

$$f_{F^{\to}}(x) = \frac{1}{2} - \frac{\sqrt{1-4x^2}}{2} \tag{7.3}$$

Proof. The recurrence $F_n^{\to} = \sum_{i=1}^{n-2} F_i^{\to} F_{n-i}^{\to}$ becomes the equality

$$f_{F^{\to}}(x) = f_{F^{\to}}^2(x) + x^2 \tag{7.4}$$

The calculation of the solution is straightforward. □

As previously we will calibrate the function $f_{F^{\to}}$ in order to satisfy the second simplified version of the Szegö Lemma 4.2.

Definition 7.4.

$$\widehat{f_{F^{\to}}}(x) \quad := \quad f_{F^{\to}}(x/2). \tag{7.5}$$

After simplification of the formula (7.3) we get:

$$\widehat{f_{F^{\to}}}(x) = \frac{1}{2} - \frac{\sqrt{1-x^2}}{2} \tag{7.6}$$

Note that $[x^n]\{f_{F^{\to}}(x)\} = \left([x^n]\{\widehat{f_{F^{\to}}}(x)\}\right) 2^n$. Now we will find two expansions of $\widehat{f_{F^{\to}}}$ into power series in the vicinity of $x = 1$ and $x = -1$. By the second simplified version of Szegö Lemma 4.2 we are particularly interested in the first terms of those expansions. Expansions are of the form:

$$\begin{aligned} \widehat{f_{F^{\to}}}(x) &= v_0^{(1)} + v_1^{(1)}\sqrt{1-x} + \ldots \\ \widehat{f_{F^{\to}}}(x) &= v_0^{(-1)} + v_1^{(-1)}\sqrt{1+x} + \ldots \end{aligned}$$

Therefore

$$F_n^{\to} = \left(\left(v_1^{(1)} + v_1^{(-1)}\right)\binom{1/2}{n}((-1)^n + 1) + O(n^{-2})\right) 2^n. \tag{7.7}$$

Theorem 7.5. *Implicational fragment $\mathcal{F}^{\to}$ has asymptotic density* 0 *in $\mathcal{F}$.*

Proof. By (7.7) and (6.14) we get:

$$\frac{F_n^{\to}}{F_n} = \frac{\left(\left(v_1^{(1)} + v_1^{(-1)}\right)\binom{1/2}{n}((-1)^n + 1) + O(n^{-2})\right) 2^n}{\left(f_1\binom{1/2}{n}(-1)^n + O(n^{-2})\right) 3^n}, \tag{7.8}$$

which for even n is of the order (see Lemma 4.3 for the order of $\binom{1/2}{n}(-1)^{n+1}$)

$$\left(\frac{v_1^{(1)}+v_1^{(-1)}}{f_1}(1+o(1))\right)\left(\frac{2}{3}\right)^n$$

while for odd n it is $O(n^{-2})$. So altogether $\lim_{n\to\infty}\frac{F_n^{\to}}{F_n}=0$.

We can also present an elementary (without Szegö Lemma 4.2) proof of this theorem. The numbers $F_n^{\to}$ are the terms of the Catalan sequence for even n, while for odd n $F_n^{\to}=0$. Since the rate of growth of Catalan numbers is as $4^n n^{-3/2}$, we get the following:

$$F_n^{\to}=O\left(4^{\frac{n}{2}}\left(\frac{n}{2}\right)^{-3/2}\right)=O\left(2^n n^{-3/2}\right)$$

for even n. And the result is immediate. □

Let $D_n^{\to}$ mean the number of implicational tautologies of size n. In compliance with the convention of Definition 3.1 we define $\mathcal{D}_n^{\to}=\mathcal{F}^{\to}\cap\mathcal{D}_n$.

Theorem 7.6. *The set of implicational tautologies has asymptotic density* 0 *in the set of all tautologies.*

Proof. This is the consequence of Theorem 7.5 and the main Theorem 6.4. The limit $\lim_{n\to\infty}\frac{D_n^{\to}}{F_n}$ exists and is equal to 0 since $D_n^{\to}\le F_n^{\to}$ for every n. Therefore $0\le\frac{D_n^{\to}}{F_n}\le\frac{F_n^{\to}}{F_n}$. We already calculated in Theorem 6.4 the limit $\lim_{n\to\infty}\frac{D_n}{F_n}=0.4232\ldots$. Therefore

$$\lim_{n\to\infty}\frac{D_n^{\to}}{D_n}=\frac{\lim_{n\to\infty}\left(\frac{D_n^{\to}}{F_n}\right)}{\lim_{n\to\infty}\left(\frac{D_n}{F_n}\right)}\le\frac{\lim_{n\to\infty}\left(\frac{F_n^{\to}}{F_n}\right)}{0.4232\ldots}=\frac{0}{0.4232\ldots}=0.$$

□

8. Tautologies in $\mathcal{F}$ are less dense than in $\mathcal{F}^{\to}$

In this section we are going to compare the density of the set of tautologies in the systems $\mathcal{F}$ and $\mathcal{F}^{\to}$. This will answer the question *"to what* *extent the introduction of negation influences the density of tautologies in the propositional calculus of one variable."*

In Theorem 7.5 we proved that the numbers $F_n^{\to}$ are the terms of the Catalan sequence for even n while $F_n^{\to} = 0$ for odd n. So $D_n^{\to} = 0$ for odd n. Therefore we are going to count the number of tautologies in $\mathcal{F}^{\to}$ for even n only. The plan is to prove the existence and the exact value of $\lim_{n\to\infty} \frac{D_{2n}^{\to}}{F_{2n}^{\to}}$.

We start by defining the other length measure for implicational formulas from $\mathcal{F}^{\to}$. Then we are going to prove that both measures are equivalent meaning that the asymptotic results for any class of formulas are identical.

By the new length measure $\|\phi\|$ for formulas from $\mathcal{F}^{\to}$ we mean the number of occurrences of the atomic formula a.

$$\|a\| = 1 \tag{8.1}$$

$$\|\phi \to \psi\| = \|\phi\| + \|\psi\| \tag{8.2}$$

Lemma 8.1. *For every formula ϕ from $\mathcal{F}^{\to}$ $|\phi| = 2\|\phi\| - 1$.*

Proof. By simple induction on the structure of ϕ. □

Definition 8.2. By $\mathcal{F}_{(n)}$ we mean the set of formulas ϕ such that $\|\phi\| = n - 1$.

Definition 8.3. The number $F_{(n)}$ is given by the recurrence:

$$F_{(0)} = 0, \quad F_{(1)} = 0, \quad F_{(2)} = 1, \tag{8.3}$$

$$F_{(n)} = \sum_{i=1}^{n-2} F_{(i+1)} F_{(n-i)}. \tag{8.4}$$

Lemma 8.4. *The number of formulas ϕ such that $\|\phi\| = n - 1$ is $F_{(n)}$. So $F_{(n)} = \#\mathcal{F}_{(n)}$.*

Proof. The proof is straightforward. □

Lemma 8.5. *The generating function $f_{(F)}$ for the numbers $F_{(n)}$ is*

$$f_{(F)}(x) = x\left(\frac{1}{2} - \frac{\sqrt{1-4x}}{2}\right) \tag{8.5}$$

Proof. The proof is by solving equation $f_{(F)}(x) = \frac{(f_{(F)}(x))^2}{x} + x^2$ which describes the recurrences (8.4) and (8.3). □

Definition 8.6. In compliance with the convention of Definition 3.1 we define $\mathcal{A}^{\to} = \mathcal{F}^{\to} \cap \mathcal{A}$, $\mathcal{B}^{\to} = \mathcal{F}^{\to} \cap \mathcal{B}$, $\mathcal{C}^{\to} = \mathcal{F}^{\to} \cap \mathcal{C}$ and $\mathcal{D}^{\to} = \mathcal{F}^{\to} \cap \mathcal{D}$. It is easy to observe that $\mathcal{A}^{\to} = \mathcal{C}^{\to} = \emptyset$. By $B^{\to}_{(n)}$ and $D^{\to}_{(n)}$ we denote the numbers of formulas respectively in classes $\mathcal{B}^{\to}$ and $\mathcal{D}^{\to}$ of the new length $\|.\|$ equal to $n-1$. Since the definition of lengths $|.|$ and $\|.\|$ are different (see the definition in Section 7), remember to distinguish between numbers $B^{\to}_n$, $D^{\to}_n$ and $B^{\to}_{(n)}$, $D^{\to}_{(n)}$.

Now our implication $\Rightarrow$ defined on classes (see 2.1) can be reduced to the following 2×2 truth table.

$\Rightarrow$	$\mathcal{B}^{\to}$	$\mathcal{D}^{\to}$
$\mathcal{B}^{\to}$	$\mathcal{D}^{\to}$	$\mathcal{D}^{\to}$
$\mathcal{D}^{\to}$	$\mathcal{B}^{\to}$	$\mathcal{D}^{\to}$

Theorem 8.7. *The numbers $B^{\to}_{(n)}$ and $D^{\to}_{(n)}$ are given by the mutual recursion:*

$$B^{\to}_{(0)} = 0, \quad B^{\to}_{(1)} = 0, \quad B^{\to}_{(2)} = 1 \tag{8.6}$$

$$D^{\to}_{(0)} = 0, \quad D^{\to}_{(1)} = 0, \quad D^{\to}_{(2)} = 0 \tag{8.7}$$

$$B^{\to}_{(n)} = \sum_{i=1}^{n-1} B^{\to}_{(i)} D^{\to}_{(n-i)} \tag{8.8}$$

$$D^{\to}_{(n)} = F_{(n)} - B^{\to}_{(n)} \tag{8.9}$$

Proof. For $n \geq 3$ a formula from $\mathcal{B}^{\to}_{(n)}$ is an implication between a pair of formulas of the lengths i and $n-i$ respectively. The total number of such formulas is $\sum_{i=1}^{n-1} B^{\to}_{(i)} D^{\to}_{(n-i)}$. □

Lemma 8.8. *For every $n \geq 1$ $F^{\to}_{(n)} = F^{\to}_{2n-2}, D^{\to}_{(n)} = D^{\to}_{2n-2}$ and $B^{\to}_{(n)} = B^{\to}_{2n-2}$*

Proof. By a simple application of Lemma 8.1. □

Lemma 8.9. *The generating functions $f_{(B)}, f_{(D)}$ for the numbers $B^{\to}_{(n)}$ and $D^{\to}_{(n)}$ are*

$$f_{(D)}(x) = x\left(\frac{3}{4} - \frac{1}{4}\sqrt{1-4x} - \frac{1}{4}\sqrt{2+2\sqrt{1-4x}+12x}\right) \quad (8.10)$$

$$f_{(B)}(x) = x\left(-\frac{1}{4} - \frac{1}{4}\sqrt{1-4x} + \frac{1}{4}\sqrt{2+2\sqrt{1-4x}+12x}\right) \quad (8.11)$$

Proof. The standard resolving of the recurrence above. □

Theorem 8.10. *[The density of tautologies for $\mathcal{F}^{\to}$ with the length measure $\|.\|$]*

$$\lim_{n\to\infty} \frac{D^{\to}_{(n)}}{F^{\to}_{(n)}} = \frac{1}{2} + \frac{\sqrt{5}}{10} \quad (8.12)$$

Proof. Theorem 4.6 in [2] states that the limit of fractions of two sequences identical with $D^{\to}_{(n-1)}$ and $F^{\to}_{(n-1)}$ is $\frac{1}{2} + \frac{\sqrt{5}}{10}$. □

Theorem 8.11. *[Statman [4]] Implicational intuitionistic and classical logics of one variable coincide.*

Proof. We present the *numerical* proof of this well known logical fact. The generating function for the number of tautologies in intuitionistic logic (see Lemma 7.4 in [2]) is identical with the generating function for the number of tautologies in classical logics $D^{\to}_{(n-1)}$. Therefore the numbers of tautologies in both logics are identical for all n. Since the set of tautologies of intuitionistic logic is a subset of the set of tautologies in the classical one it proves the statement. □

Theorem 8.12. *The set of tautologies in $\mathcal{F}$ has smaller asymptotic density than the asymptotic density of the set of implicational tautologies in $\mathcal{F}^{\to}$.*

Proof. Trivially follows from Theorem 6.4. For the set $\mathcal{F}^{\to}$ we can apply Lemma 8.8 and Theorem 8.10 to obtain

$$\lim_{n\to\infty} \frac{D^{\to}_{2n}}{F^{\to}_{2n}} = \lim_{n\to\infty} \frac{D^{\to}_{(n+1)}}{F^{\to}_{(n+1)}} = \lim_{n\to\infty} \frac{D^{\to}_{(n)}}{F^{\to}_{(n)}} = \frac{1}{2} + \frac{\sqrt{5}}{10} = 0.7236...$$

□

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