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COMPACTNESS IN EQUATIONAL LOGIC

A b s t r a c t. Three versions of the compactness theorem for finitary equational logics are generalized and formulated as properties of infinitary equational logics and it is shown that these properties are coextensive. The central construction of the paper is derived from a model theoretic proof of the consequence version of the compactness theorem: if an equation is a consequence of a set of equations S, then it is a consequence of some finite subset of S. Modifying Birkhoff's 1935 construction, the terms t and t' are related provided the equation $t = t'$ is a consequence of some finite subset of S. This relation is a congruence on the algebra of terms and the quotient algebra induced by this congruence is a model of exactly those equations which are a consequence of some finite subset of S. This construction is extended to infinitary equational logics to show that if κ is a regular cardinal strictly greater than the degree of each functional constant in the non-logical vocabulary of the language and no greater than the cardinality of the language, then any equation that is a consequence of S is a consequence of a subset of S of cardinality strictly less than κ.

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The Compactness Theorem is generally regarded as one of the most fundamental results in first order model theory. Among the several equivalent formulations of this theorem are: (1) the consequence formulation (if the first order sentence ϕ is a logical consequence of a set of first order sentences, then ϕ is a logical consequence of some finite subset of that set); and (2) the satisfiability formulation (if all finite subsets of a set of first order sentences have models, then the entire set has a model).

These formulations have analogues in equational logic. However, since every set of equations has a model (viz. the trivial one), the "natural" equational analogue of the satisfiability formulation is vacuous. There is a more interesting equational analogue of the satisfiability formulation: if all finite subsets of a set of equations have non-trivial models, then the entire set has a non-trivial model. Since a set of equations has only trivial models provided the equation $x = y$ is a logical consequence of the set, this equational analogue of the satisfiability formulation follows from the equational analogue of the consequence formulation. Further, in contrast to the situation in first order model theory, the equational analogue of the consequence formulation is established directly without first proving the satisfiability formulation.

There are several proofs of the equational analogue of the consequence formulation. Since every equation is a first order sentence, the equational analogue of the consequence formulation is a corollary of the consequence formulation itself. It also follows from the Strong Completeness and Strong Soundness theorems for equational logic together with the observation that proofs are finite. In the following, a third proof is presented. This proof employs only model theoretic notions but avoids the detour through first order model theory. One consequence of this approach is that it can be applied to infinitary equational logics.

This proof owes much to Birkhoff's historic 1935 proof of the Strong Completeness Theorem for equational logic. In essence, Birkhoff used the deductive apparatus of equational logic to construct for each set of equations an algebra that makes true exactly those equations that are provable for that set. Here, however, Birkhoff's proof theoretic techniques are replaced by model theoretic ones. For each set of equations, an algebra is constructed which makes true exactly those equations that are logical consequences of some finite subset of the set. Since an equation is a logical consequence of any set of which it is a member, and each equation is contained in a finite set of equations, this algebra is a model of the given set of equations. Hence, every consequence of the given set is a consequence of some finite subset of that set.

Given S, a set of equations, the terms t, t' are related provided the equation $t = t'$ is a logical consequence of a finite subset of S. This relation is a congruence on the formal polynomial or free term algebra (cf. Henkin, Monk and Tarski [1971] p. 143) and the quotient algebra induced by this congruence is a model of exactly those equations that are consequences of some finite subset of S. Further, when all finite subsets of S have non-trivial models, this quotient algebra is the free algebra over the models of S with α generators (cf. Henkin, Monk and Tarski [1971] p. 131) where α is the cardinality of the set of individual variables.

It is natural to view equational languages as "sublanguages" of first order languages. This "first order" view is found in Tarski [1968], Monk [1976] and Henkin [1977]. On this view, the vocabularies of equational languages contain a countably infinite set of individual variables. Thus, the above construction yields free algebras over the models of S with ω generators. A second view of equational languages is also found in the literature (e.g. Burris and Sankappanovar [19480], Cohen [1981] and Henkin, Monk and Tarski [1971]). On this view (called the "algebraic" to distinguish it from the above), equational languages include those whose vocabularies contain an uncountably infinite set of individual variables. This algebraic view is adopted below. Thus, the above construction also yields free algebras over the models of S with uncountably many generators.

The construction outlined above is easily extended to include infinitary equational languages (i.e. those whose non-logical vocabularies contain functional constants of infinite arity). While the equational analogues of the two formulations of the campactness theorem fail for equational logics with infinitary languages, the above construction can be modified to obtain various compactness like results. Let κ be an infinite cardinal, an equational logic is called *κ-compact* provided for S, a set of equations, and ϕ, an equation, if ϕ is a logical consequence of S, then ϕ is a logical consequence of a subset of S of cardinality strictly less than κ; and the logic is called *κ-satisfiable* provided every set of equations has a non-trivial model when all subsets of that set of cardinality strictly less than κ have a non-trivial model. It is easily verified that every κ-compact equational logic is κ-satisfiable. Further, when κ is the least infinite regular cardinal strictly greater than the degree of all the functional constants in the vocabulary of a given equational logic, the above construction can be modified to show that the logic is κ-compact. In this case, t and t' are identified provided the equation $t = t'$ is a logical consequence of a subset of S of cardinality strictly smaller than κ. As above, if all subsets of S of cardinality strictly smaller than κ have non-trivial models, the resulting quotient algebra is the free algebra over the models of S with α generators.

While the significance of κ-compactness and κ-satisfiability results in equational logic is not comparable to that of the Compactness Theorem in first order logic, at least some of the applications of the Compactness Theorem (e.g. Upward Löwenheim-Skolem Theorems) have analogues in equational logic which can be obtained by the construction outlined above.

1. K is a non-empty set of non-logical constants, called a *type*. Members of K are either functional constants or individual constants and no member of K is both. For each g, a functional constant in K, $d(g)$ is a non-zero ordinal called the *arity* or *degree* of g. The *characteristic* of K is the least infinite cardinal strictly greater than the arity of each functional constant in K. $c(K)$ is the characteristic of K. When $d(g) \geq \aleph_0$, g is called an *infinitary functional constant* and when $d(g) < \aleph_0$, g is called a *finitary functional constant*. Among the infinitary functional constants we distinguish those of constant rank from those of variable rank.

For each infinite cardinal λ, V_λ is a set, disjoint from K, whose members are called *individual variables*. The cardinality of V_λ (cardV_λ) is λ. Further, if $\lambda < \lambda'$, $V_\lambda \subseteq V_{\lambda'}$. $T_K(\lambda)$ denotes the set of *terms* in $K \cup V_\lambda$. $T_K(\lambda)$ is the smallest set containing all individual constants in K, all variables in V_λ and for all g, a functional constant in K, if g is finitary or infinitary and of constant rank, then for all $\{t_\theta : \theta < d(g)\} \subseteq T_K(\lambda), g(t_0, \ldots, t_\theta \ldots)_{\theta<d(g)} \in T_K(\lambda)$ and if g is an infinitary functional constant of variable rank, then for all ordinals § such that $0 < \S \leq d(g)$ and all $\{t_\theta : \theta < \S\} \subseteq T_K(\lambda)$, $g(t_0, \ldots, t_\theta \ldots)_{\theta<\S} \in T_K(\lambda)$. Thus, infinitary functional constants of variable rank form terms from any family of terms indexed by a non-zero ordinal less than or equal to their arity. Infinitary functional constants of constant rank as well as the finitary functional constants form terms from families of terms indexed by their arity. $E_K(\lambda)$ is the set of all equations of the form $t = t'$ where t and t' are in $T_K(\lambda)$. Members of $E_K(\lambda)$ are called *equations* in $K \cup V_\lambda$. When $\lambda < \lambda'$, $T_K(\lambda) \subseteq T_K(\lambda')$ and $E_K(\lambda) \subseteq E_K(\lambda')$. When t is a term in $K \cup V_\lambda$, $V(t)$ is the set of variables occurring in t. When $V(t)$ is empty, t is called a *constant term*.

Υ_K is the collection of all *algebras of type* K. Members of Υ_K are order pairs $\mathfrak{A} = (A, f_\mathfrak{A})$ where (1) A is a non-empty set (the *domain* of $\mathfrak{A}$); and (2) $f_\mathfrak{A}$ is a function defined on K such that (a) for all k, an individual constant in K, $f_\mathfrak{A}(k)$ (the *denotation* of k in $\mathfrak{A}$) is a member of A; and (b) for all g, a functional constant in K, if g is finitary or infinitary but of constant rank, $f_\mathfrak{A}(g) : {}^{d(g)}A \to A$; and if g is an infinitary and of variable rank, $f_\mathfrak{A}(g) : \cup\{{}^\theta A : 0 < \theta \leq d(g)\} \to A$. $f_\mathfrak{A}(g)$ is called the *extension* of g in $\mathfrak{A}$. As usual, the cardinality of $\mathfrak{A}$ (card $\mathfrak{A}$) is the cardinality of the domain of $\mathfrak{A}$ (card A).

When $\mathfrak{A}$ is an algebra of type K and g is a functional constant in K, $f_\mathfrak{A}(g)$ is a finitary (infinitary) operation in $\mathfrak{A}$ provided g is finitary (infinitary). When $f_\mathfrak{A}(g)$ is infinitary, $f_\mathfrak{A}(g)$ is said to be of variable (constant) rank provided g is of variable (constant) rank. The characterization of abstract algebras in Birkhoff [1935] includes algebras with infinitary operations of both variable and constant rank. The cardinal algebras of Tarski [1949], the m-complete Boolean algebras of Scott [1955] and Tarski [1955] and the ordinal algebras of Tarski [1956] all have infinitary operations of variable rank. Infinitary operations of variable rank are called infinitary operation symbols in Karp [1964]. While isolated results about algebras with infinitary operations appeared earlier (e.g. Chang [1954]), Slominski [1959] contains the first systematic presentation of the foundations of the study of algebras with infinitary operations.

An *assignment* for $\mathfrak{A}$ in $E_K(\lambda)$ is a function $\mathbf{a}$ from V_λ to A. For all $\alpha \in V_\lambda$, $\mathbf{a}(\alpha)$ is the value of α on $\mathbf{a}$ (in $\mathfrak{A}$). For each term t in $T_K(\lambda)$, $f_{\mathfrak{A}}\mathbf{a}(t)$ is defined as follows: (1) if t is a variable in V_λ, $f_{\mathfrak{A}}\mathbf{a}(t) = \mathbf{a}(t)$; (2) if t is an individual constant, $f_{\mathfrak{A}}\mathbf{a}(t) = f_{\mathfrak{A}}(t)$; and (3) if t is $g(t_0 \ldots t_\theta \ldots)_{\theta<\delta}$, then $f_{\mathfrak{A}}\mathbf{a}(t) = f_{\mathfrak{A}}(g)(h)$ where $h \in {}^\delta A$ and for each $\theta < \delta$, $h(\theta) = f_{\mathfrak{A}}\mathbf{a}(t_\theta)$. It is easily verified that for all t, a term in $K \cup V_\lambda$, $f_{\mathfrak{A}}\mathbf{a}(t) \in A$. $f_{\mathfrak{A}}\mathbf{a}(t)$ is called the *value of* t on $\mathbf{a}$ (in $\mathfrak{A}$). When no confusion results, $\mathbf{a}(t)$ denotes $f_{\mathfrak{A}}\mathbf{a}(t)$. As $(\mathfrak{A})$ is the set of assignments for $\mathfrak{A}$.

$\mathbf{a}$ *satisfies* $t = t'$ in $\mathfrak{A}$ provided $\mathbf{a}(t) = \mathbf{a}(t')$. $\mathfrak{A} \models t = t'[\mathbf{a}]$ indicates that $\mathbf{a}$ staisfies $t = t'$ in $\mathfrak{A}$; $\mathfrak{A} \not\models t = t'[\mathbf{a}]$, that $\mathbf{a}$ does not satisfy $t = t'$ in $\mathfrak{A}$. An equation is *true* on $\mathfrak{A}$ provided it is satisfied by every assignment for $\mathfrak{A}$; and an equation is *false* on $\mathfrak{A}$ provided it is not satisfied by some assignment for $\mathfrak{A}$. $\mathfrak{A} \models t = t'$ indicates that $t = t'$ is true on $\mathfrak{A}$, and $\mathfrak{A} \not\models t = t'$, that is false on $\mathfrak{A}$. $\mathfrak{A}$ *is a model* of an equation when the equation is true on $\mathfrak{A}$, and $\mathfrak{A}$ is a model of a set of equations when $\mathfrak{A}$ is a model of every member of the set. $\mathfrak{A} \models S$ indicates that $\mathfrak{A}$ is a model of the set of equations S. $S \models \phi$ indicates that ϕ is a logical consequence of S, and $\models \phi$, that ϕ is a logical consequence of the null set.

Let δ be a non-zero ordinal, $\overrightarrow{\alpha}$ be $\alpha_0 \ldots \alpha_\theta \ldots \theta < \delta$ (α_θ and α'_θ are different when θ and θ' are different) and $\overrightarrow{t}$ be $t_0 \ldots t_\theta \ldots \theta < \delta$. $(\phi)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \overrightarrow{t}\end{smallmatrix}]$ is the result of simultaneously replacing every occurrence of α_θ in ϕ by t_θ for all $\theta < \delta$. $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \overrightarrow{t}\end{smallmatrix}]$ is the result of simultaneously replacing every occurrence of α_θ in t by t_θ for all $\theta < \delta$. Notice that $(t = t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \overrightarrow{t}\end{smallmatrix}]$ is $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \overrightarrow{t}\end{smallmatrix}] = (t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \overrightarrow{t}\end{smallmatrix}]$.

$(\phi)[\begin{smallmatrix}t\\ t'\end{smallmatrix}]$ is the result of replacing every occurrence of t in ϕ by t'. $(t'')[\begin{smallmatrix}t\\ t'\end{smallmatrix}]$ is the result of explaining every occurrence of t in t'' by t'. Notice that $(t'' = t''')[\begin{smallmatrix}t\\ t'\end{smallmatrix}]$ is $(t'')[\begin{smallmatrix}t\\ t'\end{smallmatrix}] = (t''')[\begin{smallmatrix}t\\ t'\end{smallmatrix}]$.

$\mathbf{E}_K(\lambda)$ is the equational logic over $K \cup V_\lambda$. $\mathbf{E}_K(\lambda) = (E_K(\lambda), \Upsilon_K)$. When $c(K) = \aleph_0$, $\mathbf{E}_K(\lambda)$ is called a *finitary equational logic*; and when

$c(K) > \aleph_0$, $\mathbf{E}_K(\lambda)$ is called an *infinitary equational logic.*

$\mathfrak{T}_{\mathfrak{K}}(\lambda)$ is the *formal polynomial algebra* over $K \cup V_\lambda$.

$$\mathfrak{T}_{\mathfrak{K}}(\lambda) = (T_K(\lambda)), f_{\mathfrak{T}_{\mathfrak{K}}(\lambda)})$$

where (1) for each k, an individual constant in K, $f_{\mathfrak{T}_{\mathfrak{K}}(\lambda)}(k) = k$; (2) for each g, a finitary or an infinitary functional constant of constant rank, and each h, a member of ${}^{d(g)}T_K(\lambda)$, $f_{\mathfrak{T}_{\mathfrak{K}}(\lambda)}(g)(h) = g(h(0)\dots h(\theta),\dots)_{\theta<d(g)}$; and (3) for each g, an infinitary functional constant of variable rank, each δ, $0 < \delta \le d(g)$, and each h, a member of ${}^{\delta}T_K(\lambda)$,

$$f_{\mathfrak{T}_{\mathfrak{K}}(\lambda)}(g)(h) = g(h(0)\dots h(\theta)\dots)_{\theta<\delta}.$$

Notice that for each λ, $\mathfrak{T}_{\mathfrak{K}}(\lambda) \in \Upsilon_K$.

For κ any infinite cardinal and any set A, $\wp_\kappa(A)$ is the collection of all subsets of A of cardinality strictly less than κ. Given infinite cardinals κ and λ, $S \subseteq E_K(\lambda)$ and t, t', terms in $K \cup V_\lambda, t \sim_{S,\kappa} t'$ iff there is $S' \in \wp_\kappa(S)$ such that $S' \models t = t'$. We show below that when κ is a regular cardinal, $\sim_{S,\kappa}$ is a congruence on the formal polynomial algebra over $K \cup V_\lambda$. It is easily verified that $\sim_{S,\kappa}$ is an equivalence relation on $T_K(\lambda)$ for all infinite κ.

The following Lemma is used below to show that $\sim_{S,\kappa}$ is a congruence on $T_K(\lambda)$ when κ is a regular cardinal.

Lemma 1. *For all δ, a non-zero ordinal, all $\{t_\theta : \theta < \delta\}$, $\{t'_\theta : \theta < \delta\}$ families of terms in $K \cup V_\lambda$, and all g, a functional constant in K, if $g(t_0 \dots t_\theta \dots)_{\theta<\delta}$ is a term in $K \cup V_\lambda$ and for all $\theta < \delta$,*

$$S \models t_\theta = t'_\theta, \text{ then } S \models g(t_0 \dots t_\theta \dots)_{\theta<\delta} = g(t'_0 \dots t'_\theta \dots)_{\theta<\delta}.$$

Proof. Suppose that $g(t_0 \dots t_\theta \dots)_{\theta<\delta} \in T_K(\lambda)$ and for all $\theta < \delta$, $S \models t_\theta = t'_\theta$. Note that $g(t'_0 \dots t'_\theta \dots)_{\theta<\delta} \in T_K(\lambda)$. Let $\mathfrak{A}$ be a model of S and $\mathbf{a}$ an assignment for $\mathfrak{A}$. By supposition, for all $\theta < \delta$, $\mathbf{a}(t_\theta) = \mathbf{a}(t'_\theta)$. By definition $\mathbf{a}(g(t_0 \dots t'_\theta \dots))_{\theta<\delta} = f_{\mathfrak{A}}(g)(h)$ where $h(\theta) = \mathbf{a}(t_\theta)$; and $\mathbf{a}(g(t'_0 \dots t'_\theta \dots))_{\theta<\delta} = f_{\mathfrak{A}}(g)(h')$ where $h'(\theta) = \mathbf{a}(t'_\theta)$. By supposition, $h = h'$. Hence, $g(t_o \dots t_\theta \dots)_{\theta<\delta} = g(t'_o \dots t'_\theta \dots)_{\theta<\delta}$ is satisfied by $\mathbf{a}$ in $\mathfrak{A}$. Hence, $S \models g(t_0 \dots t_\theta \dots)_{\theta<\delta} = g(t'_0 \dots t'_\theta \dots)_{\theta<\delta}$. □

Given S and κ, $[t]_{S,\kappa}$ is the equivalence class of t in $T_K(\lambda)$ and $T_K(\lambda)/S, \kappa$ is the partition on $T_K(\lambda)$ induced by $\sim_{S,\kappa}$.

Lemma 2. *If κ is a regular cardinal and $c(K) \le \kappa$, then $\sim_{S,\kappa}$ is a congruence on $\mathfrak{T}_{\mathfrak{K}}(\lambda)$.*

Proof. Suppose that κ is a regular cardinal greater than or equal to $c(K)$. Let g be a functional constant in K, δ be a non-zero ordinal and $\{t_\theta : \theta < \delta\}$, $\{t'_\theta : \theta < \delta\}$ be families of terms in $K \cup V_\lambda$ such that $g(t_0 \ldots t_\theta \ldots)_{\theta<\delta}$ is a term in $K \cup V_\lambda$.

Suppose that for all $\theta < \delta$, $t_\theta \sim_{S,\kappa} t'_\theta$. We show that

$$g(t_0 \ldots t_\theta \ldots)_{\theta<\delta} \sim_{S,\kappa} g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta}.$$

By supposition, for each $\theta < \delta$ there is $S_\theta \in \wp_\kappa(S)$ such that $S_\theta \models t_\theta = t'_\theta$. Since $\delta < \kappa$ and κ is regular, $S' = \cup\{S_\theta : \theta < \delta\} \in \wp_\kappa(S)$. Thus, for all $\theta < \delta$, $S' \models t_\theta = t'_\theta$. Hence, by Lemma 1, $S' \models g(t_0 \ldots t_\theta \ldots)_{\theta<\delta} = g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta}$ and $\sim_{S,\kappa}$ is a congruence on $\mathfrak{T}_{\mathfrak{K}}(\lambda)$.

Thus, when κ is a regular cardinal, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa$ is the quotient algebra on $\mathfrak{T}_{\mathfrak{K}}(\lambda)$ induced by the congruence $\sim_{S,\kappa}$. $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa = (T_K(\lambda)/S,\kappa, f_{S,\kappa})$ where (1) given k, an individual constant in K, $f_{S,\kappa}(k) = [k]_{S,\kappa;}$ and (2) given g, a functional constant in K, δ, a non-zero ordinal, $\{t_\theta : \theta < \delta\}$, a family of terms in $K \cup V_\lambda$ such that $g(t_0 \ldots t_\theta \ldots)_{\theta<\delta}$ is a term in $T_K(\lambda)$, and $h \in {}^\delta T_K(\lambda)/S,\kappa$ is such that $h(\theta) = [t_\theta]_{S,\kappa}$, $f_S,\ \kappa(g)(h) = [g(t_0 \ldots t_\theta \ldots)_{\theta<\delta}]_{S,\kappa}$ By the axiom of choice, $\operatorname{card} T_K(\lambda)/S,\kappa \leq \operatorname{card} T_K(\lambda)$. Further, if for all $S' \in \wp_\kappa(S)$, S' has a non-trivial model, then $\lambda \leq \operatorname{card} T_K(\lambda)/S,\kappa$. To understand this last point it suffices to notice that if α and α' are different members of V_λ, then $[\alpha]_{S,\kappa} \neq [\alpha']_{S,\kappa}$. Otherwise, there is $S' \in \wp_\kappa(S)$ such that $S' \models \alpha = \alpha'$. But, $\alpha = \alpha'$ is true only on trivial algebras and therefore S' has only trivial models. Finally notice that if some member of $\wp_\kappa(S)$ has only trivial models, $\operatorname{card} T_K(\lambda)/S,\kappa = 1$.

We now show that when κ is regular, for all $\phi \in E_K(\lambda)$, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa \models \phi$ iff there is $S' \in \wp_\kappa(S)$ s.t. $S' \models \phi$. Let $\mathbf{a}_{S,\kappa}$ be that assignment for $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa$ such that for all $\alpha \in V_\lambda$, $\mathbf{a}_{S,\kappa}(\alpha) = [\alpha]_{S,\kappa}$ It is easily verified, by induction on $T_K(\lambda)$ that for all t, a term in $K \cup V_\lambda$, $\mathbf{a}_{S,\kappa}(t) = [t]_{S,\kappa}$. □

The following is then immediate.

Lemma 3. *Given K, λ and $S \subseteq E_K(\lambda)$ if κ is regular, then for all $\phi \in E_K(\lambda)$, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa \models \phi[\mathbf{a}_{S,\kappa}]$ iff there is $S' \in \wp_\kappa(S)$ such that $S' \models \phi$.*

Proof. Let $\phi \in E_K(\lambda)$. Thus, there are t, $t' \in T_K(\lambda)$ such that ϕ is $t = t'$. Now, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa \models t = t'[\mathbf{a}_{S,\kappa}]$ iff $\mathbf{a}_{S,\kappa}(t) = \mathbf{a}_{S,\kappa}(t')$ iff $[t]_{S,\kappa} = [t']_{S,\kappa}$ iff $t \sim_{S,\kappa} t'$ iff there is $S' \in \wp_\kappa(S)$ such that $S' \models t = t'$.

It follows from Lemma 3 that if κ is a regular cardinal greater than or equal to $c(K)$ and $S \subseteq E_K(\lambda)$, then for all ϕ, a sentence in $E_K(\lambda)$, if $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa \models \phi$, then there is $S' \in \wp_\kappa(S)$ such that $S' \models \phi$. □

The following lemmas yield the converse of the above.

Lemma 4. *If κ is a regular cardinal greater than or equal to $c(K)$, $S \subseteq E_K(\lambda)$, δ is a non-zero ordinal $\leq \lambda$, $\{\alpha_\theta : \theta < \delta\}$ is a family of distinct members of V_λ and $\{t_\theta : \theta < \delta\}$ is a family of terms in $K \cup V_\lambda$, then for all* **a**, *an assignment for $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa$, and all t, a term in $K \cup V_\lambda$, if $V(t) \subseteq \{\alpha_\theta : \theta < \delta\}$ and $\mathbf{a}(\alpha_\theta) = [t_\theta]_{S,\kappa}$ for all $\theta < \delta$, then $\mathbf{a}(t) = \mathbf{a}_{S,\kappa}((t)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,])$ where $\overrightarrow{\alpha}$ is $\alpha_0 \ldots \alpha_\theta \ldots (\theta < \delta)$ and $\overrightarrow{t}$ is $t_0 \ldots t_\theta \ldots (\theta < \delta)$.*

Proof. Suppose that κ is a regular cardinal greater than or equal to $c(K)$ and that $S \subseteq E_K(\lambda)$. Let δ be a non-zero ordinal $\leq \lambda$, $V' = \{\alpha_\theta : \theta < \delta\}$ be a family of distinct members of V_λ and $\{t_\theta : \theta < \delta\}$ be a family of terms in $K \cup V_\lambda$. $\overrightarrow{\alpha}$ is $\alpha_0 \ldots \alpha_\theta \ldots (\theta < \delta)$ and $\overrightarrow{t}$ is $t_0 \ldots t_\theta \ldots (\theta < \delta)$. Let **a** be an assignment for $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa$ (in $\mathbf{E}_K(\lambda)$). And suppose that $\mathbf{a}(\alpha_\theta) = [t_\theta]_{S,\kappa}$ for each $\theta < \delta$.

Let $T = \{t \in T_K(\lambda) : \mathit{if} \quad V(t) \subseteq V', \quad \mathit{then} \quad \mathbf{a}(t) = \mathbf{a}_{S,\kappa}((t)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,])\}$. We proceed by showing that $T = T_K(\lambda)$.

Suppose that $t \in V_\lambda$. Suppose that $V(t) \subseteq V'$. Since $t \in V_\lambda$, $V(t) = \{t\}$ and there is $\theta < \delta$ such that t is α_θ. By supposition, $\mathbf{a}(t) = a(\alpha_\theta) = [t_\theta]_{S,\kappa}$. $(t)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,]$ is t_θ and $\mathbf{a}_{S,\kappa}(t_\theta) = [t_\theta]_{S,\kappa}$. Thus $t \in T$.

Suppose that t is an individual constant in K. $V(t) = \Lambda \subseteq V'$. $\mathbf{a}(t) = f_{S,\kappa}(k) = [k]_{S,\kappa} = \mathbf{a}_{S,\kappa}(k)$, where t is the individual constant k. Since $(k)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,]$ is k , $t \in T$.

Let t be $g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta}$ where g is a functional constant in K and $\{t'_\theta : \theta < \delta\} \subseteq T$. Suppose that $V(t) \subseteq V'$. Thus, for all $\theta < \delta$, $V(t'_\theta) \subseteq V'$; and by supposition $\mathbf{a}(t'_\theta) = \mathbf{a}_{S,\kappa}((t'_\theta)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,])$. Notice that $(g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta})[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,]$ is $g((t'_0)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,] \ldots (t'_\theta)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,] \ldots)_{\theta<\delta}$. By definition, $\mathbf{a}(g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta}) = f_{S,\kappa}(g)(h)$ where $h(\theta) = \mathbf{a}(t'_\theta)$. Further,

$$\begin{aligned} \mathbf{a}_{S,\kappa}(g(t'_0 \ldots t'_\theta \ldots)_{\theta<\delta}) &= \mathbf{a}_{S,\kappa}(g((t'_0)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,] \ldots (t'_\theta)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,] \ldots)_{\theta<\delta}) \\ &= f_{S,\kappa}(g)(h') \end{aligned}$$

where $h'(\theta) = \mathbf{a}_{S,\kappa}((t'_\theta)[\,{}^{\overrightarrow{\alpha}}_{\overrightarrow{t}}\,])$. Since $\{t'_\theta : \theta < \delta\} \subseteq T$, $h = h'$. Therefore, $t \in T$. □

Lemma 5. [The Substitution Lemma] *For all $\mathfrak{A} \in \Upsilon_K$, all non-zero ordinals $\delta \leq \lambda$, all $\mathbf{aa}'$, assignments for $\mathfrak{A}$ in $\mathbf{E}_K(\lambda)$ all $\{\alpha_\theta : \theta < \delta\}$, a family* *of distinct variables in* V_λ*, and all* $\{t_\theta : \theta < \delta\}$*, a family of terms in* $K \cup V_\lambda$ *if (1) for all* $\theta < \delta$*,* $\mathbf{a}(\alpha_\theta) = \mathbf{a}'(t_\theta)$ *and (2) for all* $\alpha \in V_\lambda$*, if* $\alpha \notin \{\alpha_\theta : \theta < \delta\}$*, then* $\mathbf{a}(\alpha) = \mathbf{a}'(\alpha)$*, then (1) for all* t*, a term in* $K \cup V_\lambda$*,* $\mathbf{a}(t) = \mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$*; and (2) for all* ϕ*, an equation in* $K \cup V_\lambda$*,* $\mathfrak{A} \models \phi[\mathbf{a}]$ *iff* $\mathfrak{A} \models (\phi)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}][\mathbf{a}']$ *where* $\overrightarrow{\alpha}$ *is* $\alpha_0 \dots \alpha_\theta \dots (\theta < \delta)$ *and* $\overrightarrow{t}$ *is* $t_0 \dots t_\theta \dots (\theta < \delta)$*.*

Proof. Given $\mathfrak{A}$, $\{\alpha_\theta : \theta < \delta\}$, $\{t_\theta : \theta < \delta\}$, $\mathbf{a}$ and $\mathbf{a}'$. Let $\overrightarrow{\alpha}$ be $\alpha_0 \dots \alpha_\theta \dots (\theta < \delta)$ and $\overrightarrow{t}$ be $t_0 \dots t_\theta \dots (\theta < \delta)$. Suppose that $\mathbf{a}(\alpha_\theta) = \mathbf{a}'(t_\theta)$, all $\theta < \delta$ and that $\mathbf{a}(\alpha) = \mathbf{a}'(\alpha)$ for$\alpha \in V_\lambda$ - $\{\alpha_\theta : \theta < \delta\}$. Let $T = \{t : t \in T_K(\lambda) \quad \textit{and} \quad \mathbf{a}(t) = \mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])\}$. We proceed by showing that $T = T_K(\lambda)$.

Suppose $t \in V_\lambda$. Consider $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$. Either there is unique $\theta < \delta$ such that t is α_θ and $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$ is t_θ or $t \notin \{\alpha_\theta : \theta < \delta\}$ and $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$ is t. Suppose t is α_θ. $\mathbf{a}(\alpha_\theta) = \mathbf{a}'(t_\theta) = \mathbf{a}'((\alpha_\theta)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$. Suppose $t \notin \{\alpha_\theta : \theta < \delta\}$. Thus, $\mathbf{a}(t) = \mathbf{a}'(t) = \mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$.

Suppose that t is an individual constant. $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$ is t and $\mathbf{a}(t) = f_{\mathfrak{A}}\mathbf{a}'(t) = \mathbf{a}'(t)$. Suppose that t is $g(t'_0 \dots t'_\theta \dots)_{\theta<\delta}$ where $\{t'_\theta : \theta < \delta\} \subseteq T$. By definition, $\mathbf{a}(t) = f_{\mathfrak{A}}(g)(h)$ where $h(\theta) = \mathbf{a}(t'_\theta)$, all $\theta < \delta$. Since $(g(t'_0 \dots t'_\theta \dots)_{\theta<\delta})[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$ is $g((t'_0)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}] \dots (t'_\theta)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}] \dots)_{\theta<\delta})$, $\mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]) = f_{\mathfrak{A}}(g)(h')$ where $h'(\theta) = \mathbf{a}'((t'_\theta)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$, all $\theta < \delta$. By the induction hypothesis, $h = h'$ and $\mathbf{a}(t) = \mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$. This completes the proof of (1).

Let ϕ be an equation in $K \cup V_\lambda$. Thus, there are t, t' terms in $K \cup V_\lambda$ such that ϕ is $t = t'$. Notice that $(t = t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$ is $(t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}] = (t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]$. By definition, $\mathfrak{A} \models \phi[\mathbf{a}]$ iff $\mathbf{a}(t) = \mathbf{a}(t')$; and $\mathfrak{A} \models (\phi)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}][\mathbf{a}']$ iff $\mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}]) = \mathbf{a}'((t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$. By (1), $\mathbf{a}(t) = \mathbf{a}'((t)[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$ and $\mathbf{a}(t') = \mathbf{a}'((t')[\begin{smallmatrix}\overrightarrow{\alpha}\\ \rightarrow\\ t\end{smallmatrix}])$. Thus,

$\mathfrak{A} \models \phi[\mathbf{a}]$ iff $\mathfrak{A} \models \phi[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}][\mathbf{a}']$.

It follows from Lemma 5 that if $S \models \phi$, then $S \models (\phi)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}]$. To understand this, consider the following. Suppose $S \models \phi$. Let $\mathfrak{A} \models S$. Thus, $\mathfrak{A} \models \phi$. Let $\mathbf{a}' : V_\lambda \to A$. To show that $\mathfrak{A} \models (\phi)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}][\mathbf{a}']$. Let $\mathbf{a} : V_\lambda \to A$ be such that $\mathbf{a}(\alpha_\theta) = \mathbf{a}'(t_\theta)$, for $\theta < \delta$ and $\mathbf{a}(\alpha) = \mathbf{a}'(\alpha)$ for $\alpha \notin \{\alpha_\theta : \theta < \delta\}$. By Lemma 5, $\mathfrak{A} \models \phi[\mathbf{a}]$ iff $\mathfrak{A} \models (\phi)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}][\mathbf{a}']$. Since $\mathfrak{A} \models \phi$, $\mathfrak{A} \models \phi[\mathbf{a}]$ and $\mathfrak{A} \models (\phi)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}][\mathbf{a}']$. □

Theorem 1. *For all $S \subseteq E_K(\lambda)$ and all κ, if $c(K) \leq \kappa$ and κ is regular, then for all ϕ, an equation in $K \cup V_\lambda$, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa \models \phi$ iff there is $S' \in \wp_\kappa(S)$ and $S' \models \phi$.*

Proof. Suppose that ϕ is an equation in $K \cup V_\lambda$. Thus, there are t, t', terms in $K \cup V_\lambda$, such that ϕ is $t = t'$.

Suppose that there is $S' \in \wp_\kappa(S)$ such that S$' \models t = t'$. By Lemma 3, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa \models t = t'[\mathbf{a}_{S,\kappa}]$. Let $V(t) \cup V(t') = \{\alpha_\theta : \alpha < \delta\}$. $\delta \leq \lambda$. Let $\mathbf{a} : V_\lambda \to T_K(\lambda)/S, \kappa$. Thus, for all $\theta < \delta$, there is t_θ, a term in $K \cup V_\lambda$ such that $\mathbf{a}(\alpha_\theta) = [t_\theta]_{S,\kappa}$. Let $\overrightarrow{\alpha}$ be $\alpha_0 \ldots \alpha_\theta \ldots (\theta < \delta)$ and $\overrightarrow{t}$ be $t_0 \ldots t_\theta \ldots (\theta < \delta)$. By Lemma 4, $\mathbf{a}(t) = \mathbf{a}_{S,\kappa}((t)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}])$ and $\mathbf{a}(t') = \mathbf{a}_{S,\kappa}((t')[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}])$. By Lemma 5, $S' \models (\phi)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}]$. Thus, $\mathbf{a}_{S,\kappa}((t)[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}]) = \mathbf{a}_{S,\kappa}((t')[\genfrac{}{}{0pt}{}{\overrightarrow{\alpha}}{\overrightarrow{t}}])$ and $\mathbf{a}(t) = \mathbf{a}(t')$. By definition, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa \models \phi[\mathbf{a}]$. As $\mathbf{a}$ was arbitrarily chosen, $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa \models \phi$. Theorem 1 then follows from Lemma 3. □

When κ and κ' are regular cardinals and $\kappa < \kappa'$, it follows from Theorem 1 that $\sim_{S,\kappa} = \sim'_{S,\kappa}$; hence that $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa = \mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa'$.

We now look at other consequences of Theorem 1.

Corollary 1. *If $S \subseteq E_K(\lambda)$ and κ is a regular cardinal $\geq c(K)$, then*

(1) $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S, \kappa \models S$;

(2) *for all ϕ, an equation in $E_K(\lambda)$, if $S \models \phi$, then there is $S' \in \wp_\kappa(S)$ such that $S' \models \phi$; and*

(3) *if every member of* $\wp_\kappa(S)$ *has a non-trivial model, then* S *has a non-trivial model.*

Proof. Suppose that $S \subseteq E_K(\lambda)$ and κ is a regular cardinal $\geq c(K)$.

(1) Let $\phi \in S$. Since κ is infinite, $\{\phi\} \in \wp_\kappa(S)$. Thus, by Theorem 1, $\mathfrak{T}_\mathfrak{K}(\lambda)/S, \kappa \models \phi$

(2) Let ϕ be an equation in $E_K(\lambda)$. Suppose, $S \models \phi$. By (1) and the definition of logical consequence, $\mathfrak{T}_\mathfrak{K}(\lambda)/S, \kappa \models \phi$ and, by Theorem 1, there is $S' \in \wp_\kappa(S)$ such that $S' \models \phi$.

(3) Suppose that every member of $\wp_\kappa(S)$ has a non-trivial model. Suppose that S has only trivial models. Thus, $S \models x = y$ and by (2) there is $S' \in \wp_\kappa(S)$ such that $S' \models x = y$. Thus, S' has only trivial models. □

Corollary 1 yields compactness results for finitary equational logics (i.e. logics $\mathbf{E}_K(\lambda)$ where $c(K) = \aleph_0$). Corollary 1 (2) yields the (finitary) equational analogue of the consequence formulation of the Compactness Theorem and Corollary 1 (3) yields the (finitary) equational analogue of the satisfiability formulation.

Theorem 1 also yields equational analogues of the Löwenheim-Skolem theorems. For all $\mathfrak{A}$, $\mathfrak{B}$, algebras of type K, $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent* (or *indistinguishable*) in $E_K(\lambda)$ provided for all ϕ, an equation in $E_K(\lambda)$, $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.

Corollary 2. *For all* S*, a set of equations in,* $K \cup V_\lambda$*, and all* $\mathfrak{A}$*, an algebra of type* K*,*

(1) *if* S *has a non-trivial model, then* S *has an infinite model (in particular, there is* $\mathfrak{B}$*, a model of* S *such that* $\lambda \leq \mathit{card}\, \mathfrak{B} \leq \mathit{card}\, T_K(\lambda)$*; and*

(2) *if* $\mathfrak{A}$ *is non-trivial, then there is* $\mathfrak{B}$*, an algebra of type* K*, such that* $\lambda \leq \mathit{card}\, \mathfrak{B} \leq \mathit{card}\, T_K(\lambda)$ *and* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are indistinguishable in* $E_K(\lambda)$*.*

Proof. Suppose that $S \subseteq E_K(\lambda)$ and $\mathfrak{A} \in \Upsilon_K$. Let κ be a regular cardinal such that $c(K) \leq \kappa$.

Suppose S has a non-trivial model. Thus, all members of $\wp_\kappa(S)$ have non-trivial models; and $\mathfrak{T}_\mathfrak{K}(\lambda)/S, \kappa$ is the desired model of S. Suppose that $\mathfrak{A}$ is non-trivial. Let $S = \{\phi : \phi \in E_K(\lambda)$ and $A \models \phi\}$. Notice that S is closed under $\models$. Hence, $\mathfrak{A}$ and $\mathfrak{T}_\mathfrak{K}(\lambda)/S, \kappa$ are indistinguishable in $E_K(\lambda)$.

Notice that if $S \subseteq E_K(\lambda)$ and λ' is a cardinal $> \lambda$, $S \subseteq E_K(\lambda')$. Thus, again by Theorem 1, $\mathfrak{T}_\mathfrak{K}(\lambda')/S, \kappa$ is an algebra of type K and $\lambda' \leq$

$\text{card}\,\mathfrak{T}_{\mathfrak{K}}(\lambda')/S,\kappa \leq \text{card}\,T_K(\lambda')$. Now, for all ϕ an equation in $K \cup V_\lambda$, $\mathfrak{T}_{\mathfrak{K}}(\lambda')/S,\kappa \models \phi$ iff there is $S' \in \wp_\kappa(S)$ and $S' \models \phi$. Hence, $\mathfrak{T}_{\mathfrak{K}}(\lambda')/S,\kappa$ and $\mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa$ are indistinguishable in $E_K(\lambda)$. □

These observations suffice to establish the following.

Corollary 3. *For all S, a set of equations in $K \cup V_\lambda$, all $\mathfrak{A}$, an algebra of type K, and all cardinals $\lambda' \geq \lambda$,*

(1) *if S has a non-trivial model, then there is $\mathfrak{B}$, an algebra of type K, such that $\mathfrak{B}$ is a model of S and $\lambda' \leq \text{card}\,\mathfrak{B} \leq \text{card}\,T_K(\lambda')$; and*

(2) *if $\mathfrak{A}$ is non-trivial, then there is $\mathfrak{B}$, an algebra of type K, such that $\mathfrak{A}$ and $\mathfrak{B}$ are indistinguishable in $E_K(\lambda)$ and $\lambda' \leq \text{card}\,\mathfrak{B} \leq \text{card}\,T_K(\lambda')$.*

When $\text{card}\,T_K(\lambda') = \lambda'$, Corollary 3 provides algebras of cardinality λ'. Such results are easily established for finitary equational logics. Suppose that $\text{card}\,K \leq \aleph_0$ and $c(K) = \aleph_0$. Thus, for all cardinals $\lambda \geq \aleph_0$, $\text{card}\,T_K(\lambda) = \lambda$. If $\text{card}\ K > \aleph_0$ and $c(K) = \aleph_0$, $\text{card}\,T_K(\lambda) = max\{\text{card}\,K, \lambda\}$. Thus, when $\lambda \geq \text{card}\,K$, $\text{card}\,T_K(\lambda) = \lambda$. These observations suffice to establish the following.

Corollary 4. *If $c(K) = \aleph_0$, S is a set of equations in $K \cup V_\lambda$ and $\mathfrak{A}$ is an algebra of type K, then*

(1) *if S has a non-trivial model, then S has a model of cardinality λ' for each $\lambda' \geq \lambda$ such that $\text{card}\,K \leq \lambda'$; and*

(2) *if $\mathfrak{A}$ is non-trivial, then for all cardinals $\lambda' \geq \lambda$ such that $\text{card}\,K \leq \lambda'$, there is $\mathfrak{B}$, an algebra of type K, such that $\mathfrak{A}$ and $\mathfrak{B}$ are indistinguishable in $E_K(\lambda)$ and $\text{card}\,\mathfrak{B} = \lambda'$.*

The question of what happens in infinitary equational logics (i.e. when $c(K) > \aleph_0$) is discussed in a sequel.

The natural model theoretic version of Birkhoff's 1935 construction identifies terms t and t' provided the equation $t = t'$ is a logical consequence of S. When S has a non-trivial model, the algebra constructed in this way is exactly the one constructed above. Define $\equiv_S$ on $T_K(\lambda)$ such that $t \equiv_S t'$ iff $S \models t = t'$. Let κ be the least regular cardinal $\geq c(K)$. If follows from Corollary 1 that $\equiv_S = \sim_{S,\kappa}$. Hence, $\equiv_S$ is a congruence on $\mathfrak{T}_{\mathfrak{K}}(\lambda)$ and $\mathfrak{T}_{\mathfrak{K}}(\lambda)/\equiv_S = \mathfrak{T}_{\mathfrak{K}}(\lambda)/S,\kappa$.

2. While Theorem 1 yields compactness results for finitary equational logics, these results do not extend to the infinitary case. Let K be a type which contains infinitely-many individual constants and g, a functional constant of degree $\aleph_0$. Let $\{c_n : n \geq 0\}$ be the set of individual constants in K. Let S be the union of the following sets of equations: $\{g(xc_1c_3 \ldots c_{2n+1} \ldots) = x\}$, $\{g(xc_2c_4 \ldots c_{2n+2} \ldots) = c_0\}$, $\{c_{2n+1} = c_{2n+2} : n \geq 0\}$. The equation $c_0 = x$ is a consequence of S. Hence S has only trivial models. However, every finite subset of S has a non-trivial model. Therefore, both the equational analogue of the satisifability formulation and the equational analogue of the consequence formulation fail in the infinitary logic $\mathbf{E}_K(\lambda)$.

There are generalizations of the consequence and satisfiability formulations that hold in infinitary equational logics. Let κ be an infinite cardinal. $\mathbf{E}_K(\lambda)$ is *κ-compact* provided (a) $\kappa \leq \operatorname{card} E_K(\lambda)$ and (b) for all $S \subseteq E_K(\lambda)$, all $\phi \in E_K(\lambda)$, if $S \models \phi$ then there is is $S' \in \wp_\kappa(S)$ such that $S' \models \phi$. $\mathbf{E}_K(\lambda)$ is *κ-satisfiable* provided (a) $\kappa \leq \operatorname{card} E_K(\lambda)$ and (b) for all $S \subseteq E_K(\lambda)$, if all members of $\wp_\kappa(S)$ have non-trivial models, then S has a non-trivial model. $\mathbf{E}_K(\lambda)$ is *compact* provided it is $\aleph_0$-compact, and $\mathbf{E}_K(\lambda)$ is *satisfiable* provided it is $\aleph_0$ -satisfiable. Any equational logic that is κ-compact is κ-satisfiable since a set of equations has only trivial models iff the equation $x = y$ is a consequence of S. Further, if $\mathbf{E}_K(\lambda)$ is κ-compact (or κ-satisfiable) and κ' is a cardinal such that $\kappa \leq \kappa' \leq \operatorname{card} E_K(\lambda)$, then $\mathbf{E}_K(\lambda)$ is also κ'-compact (κ'-satisfiable).

It follows from Corollary 1 that $\mathbf{E}_K(\lambda)$ is κ-compact and κ-satisfiable when κ is a regular cardinal between $c(K)$ and $\operatorname{card} E_K(\lambda)$. This together with earlier observations suffices to establish the following.

Corollary 5. *For all cardinal κ,*

(1) *if $c(K)$ is a regular cardinal and $c(K) \leq \kappa \leq \operatorname{card} E_K(\lambda)$, then $\mathbf{E}_K(\lambda)$ is κ-compact and κ-satisfiable; and*

(2) *if $c(K)$ is a singular and $c(K) < \kappa \leq \operatorname{card} E_K(\lambda)$, then $\mathbf{E}_K(\lambda)$ is κ-compact and κ-satisfiable.*

In this section we show that Corollary 5 is the best result obtainable. In particular, it is shown that (1) if $c(K)$ is a regular cardinal and $\aleph_0 \leq \kappa < c(K)$, then $\mathbf{E}_K(\lambda)$ is neither κ-compact nor κ-satisfiable; and (2) if $c(K)$ is a singular cardinal and $\aleph_0 \leq \kappa \leq c(K)$, then $\mathbf{E}_K(\lambda)$ is neither κ-compact nor κ-satisfiable.

Lemma 6. *If κ is an infinite cardinal and there is $g \in K$ such that $\kappa \leq d(g)$, then $\mathbf{E}_K(\lambda)$ is neither κ-compact nor κ-satisfiable.*

Proof. Suppose that κ is an infinite cardinal and that g is a functional constant in K such that $\kappa \leq d(g)$. For each $\lambda < \kappa$, t_λ is defined as follows:

(1) t_0 is $g(t_0^0 \dots t_0^\theta \dots)_{\theta<d(g)}$ where for all θ, t_0^θ is x_0; and

(2) when $\lambda > 0$, t_λ is $g(t_\lambda^0 \dots t_\lambda^\theta \dots)_{\theta<d(g)}$ where for all $\theta < \lambda$, t_λ^θ is x_1 and for all $\theta \geq \lambda$, t_λ^θ is x_0.

Let S be $\{x_0 = t_\lambda : \lambda < \kappa\}$. ϕ is the equation $x_0 = g(t'_0 \dots t'_\theta \dots)_{\theta<d(g)}$ where for all $\theta < \kappa$, t'_θ is t_θ and for all $\theta \geq \kappa$, t'_θ is x_0. ϕ' is the equation $x_1 = g(t'_0 \dots t'_\theta \dots)_{\theta<d(g)}$. We claim that $S \models \phi$. Suppose that $\mathfrak{A} \models S$ and that $\mathbf{a} \in A\mathbf{s}(\mathfrak{A})$. $\mathbf{a}(g(t'_0 \dots t'_\theta \dots)_{\theta<d(g)}) = f_{\mathfrak{A}}(g)(h)$ where $h \in {}^{d(g)}A$ and $h(\theta) = \mathbf{a}(t'_\theta)$, for all $\theta < d(g)$. If $\theta < \kappa$, $\mathfrak{A} \models x_0 = t_\theta$. Hence, $\mathbf{a}(t'_\theta) = \mathbf{a}(t_\theta) = \mathbf{a}(x_0)$. If $\theta > \kappa$, t'_θ is x_0. Thus, $\mathbf{a}(t'_\theta) = \mathbf{a}(x_0)$. Therefore, for all $\theta < d(g)$, $\mathbf{a}(t'_\theta) = \mathbf{a}(x_0)$. Since $\mathfrak{A} \models x_0 = t_0$, $f_{\mathfrak{A}}(g)(h) = \mathbf{a}(x_0)$; and $\mathfrak{A} \models \phi$.

We now proceed to show that for all $S' \in \wp_\kappa(S)$, $S' \not\models \phi$. Let $S' \in \wp_\kappa(S)$. Since $\operatorname{card} S = \kappa$, S' is a proper subset of S. Thus, there is $\lambda^* > 0$ such that the equation $x_0 = t_{\lambda^*} \notin S'$. It suffices to show that $S - \{x_0 = t_{\lambda^*}\} \not\models \phi$. Let $\mathfrak{A} = (A, f_{\mathfrak{A}})$ where $A = \kappa$. Let $h \in {}^{d(g)}A$ and $\theta < d(g)$, θ is *the critical point* of h provided there are $\delta, \delta' \in \kappa$ where $\delta \neq \delta'$, for all $\lambda \leq \theta$, $h(\lambda) = \delta$; for all $\lambda > \theta$, $h(\lambda) = \delta'$ and $\theta + 1 < d(g)$. *h has a critical point* provided there is $\theta < d(g)$ such that θ is the critical point of h. Finally, h is a *constant function* provided there is $\delta \in \kappa$ such that $h(\theta) = \delta$ for all $\theta < d(g)$. $f_{\mathfrak{A}}(g)$ is defined as follows:

(1) $f_{\mathfrak{A}}(g)(h) = h(0)$, if h is a constant function;

(2) $f_{\mathfrak{A}}(g)(h) = h(\theta + 1)$, if θ is the critical point of h and $\theta \neq \lambda^*$;

(3) $f_{\mathfrak{A}}(g)(h) = h(\lambda^*)$, if λ^* is the critical point of h; and

(4) $f_{\mathfrak{A}}(g)(h) = h(\lambda^*)$, if h is not a constant function and h does not have a critical point.

We claim that $\mathfrak{A} \models S - \{x_0 = t_{\lambda^*}\}$. Suppose that $\mathbf{a} \in A\mathbf{s}(\mathfrak{A})$. $\mathbf{a}(t_0) = \mathbf{a}(g(t_0^0 \dots t_0^\theta \dots)_{\theta<d(g)}) = f_{\mathfrak{A}}(g)(h)$ where $h(\theta) = \mathbf{a}(t_0^\theta)$ for all $\theta < d(g)$. By construction, for all $\theta < d(g)$, t_0^θ is x_0. Thus, h is a constant function and $f_{\mathfrak{A}}(g)(h) = h(0) = \mathbf{a}(x_0)$. Therefore, $\mathfrak{A} \models x_0 = t_0$. Let $\lambda < \kappa$. Suppose that $\lambda \neq \lambda^*$. $\mathbf{a}(t_\lambda) = \mathbf{a}(g(t_\lambda^0 \dots t_\lambda^\theta \dots)_{\theta<d(g)}) = f_{\mathfrak{A}}(g)(h)$ where $h(\theta) = \mathbf{a}(t_\lambda^\theta)$. By construction for all $\theta < \lambda$, t_λ^θ is x_1 and t_λ^θ is x_0 for $\theta \geq \lambda$. Since $\lambda < \kappa$ and κ is a cardinal $\lambda + 1 < \kappa \leq d(g)$. Thus, λ is the critical point of h. Hence $f_{\mathfrak{A}}(g)(h) = h(\lambda + 1) = \mathbf{a}(t_\lambda^{\lambda+1}) = \mathbf{a}(x_0)$; and $\mathfrak{A} \models x_0 = t_\lambda$. $\mathfrak{A} \models S - \{x_0 = t_{\lambda^*}\}$.

Finally, we show that $\mathfrak{A} \models \phi'$. Let $\mathbf{a} \in A\mathbf{s}(\mathfrak{A})$. It suffices to show that $\mathbf{a}(g(t'_0 \ldots t'_\theta \ldots)_{\theta<d(g)}) = \mathbf{a}(x_1)$. By construction, for all $\theta < \kappa$, t'_θ is t_θ; and for $\theta \geq \kappa$, t'_θ is x_0. Since $\mathfrak{A} \models S - \{x_0 = t_{\lambda^*}\}$, $\mathbf{a}(t'_\theta) = \mathbf{a}(t_\theta) = \mathbf{a}(x_0)$ when $\theta < \kappa$ and $\theta \neq \lambda^*$. Further, for $\theta \geq \kappa$, $\mathbf{a}(t'_\theta) = \mathbf{a}(x_0)$. $\mathbf{a}(t'_{\lambda^*}) = \mathbf{a}(t_{\lambda^*}) = \mathbf{a}(g(t^0_{\lambda^*} \ldots t^\theta_{\lambda^*} \ldots)_{\theta<d(g)})$. Let $h^* \in {}^{d(g)}\kappa$ be such that $h^*(\theta) = \mathbf{a}(t^\theta_{\lambda^*})$. By construction, $t^\theta_{\lambda^*}$ is $x_{1,}$ for $\theta \leq \lambda^*$, and $t^\theta_{\lambda^*}$ is x_0 ,for $\theta > \lambda^*$. Since $\lambda^* < \kappa \leq d(g)$ and κ is a cardinal, $\lambda^* + 1 < \kappa \leq d(g)$. Thus, λ^* is the critical point of h^*. $\mathbf{a}(t'_{\lambda^*}) = \mathbf{a}(g(t^0_{\lambda^*} \ldots t^\theta_{\lambda^*} \ldots)_{\theta<d(g)}) = f_{\mathfrak{A}}(g)(h^*) = h^*(\lambda^*) = \mathbf{a}(x_1)$. Either $\mathbf{a}(x_1) = \mathbf{a}(x_0)$ or $\mathbf{a}(x_1) \neq \mathbf{a}(x_0)$. Suppose that $\mathbf{a}(x_1) \neq \mathbf{a}(x_0)$. Let $h \in {}^{d(g)}\kappa$ be such that $h(\theta) = \mathbf{a}(t'_\theta)$. h is not a constant function; further, since $\lambda^* > 0$, h has no critical point. Thus, $\mathbf{a}(g(t'_0 \ldots t'_\theta \ldots)_{\theta<d(g)}) = f_{\mathfrak{A}}(g)(h) = h(\lambda^*) = \mathbf{a}(t'_{\lambda^*}) = \mathbf{a}(x_1)$. Suppose that $\mathbf{a}(x_1) = \mathbf{a}(x_0)$. Then, h is a constant function and $f_{\mathfrak{A}}(g)(h) = h(0) = \mathbf{a}(x_0) = \mathbf{a}(x_1)$. Hence, $\mathbf{a}(g(t'_0 \ldots t'_\theta \ldots)_{\theta<d(g)}) = \mathbf{a}(x_0) = \mathbf{a}(x_1)$. Therefore, in either case $\mathfrak{A} \models \phi'[\mathbf{a}]$. Since $\mathfrak{A}$ is non-trivial, there is $\mathbf{a} \in A\mathbf{s}(\mathfrak{A})$ such that $\mathbf{a}(x_0) \neq \mathbf{a}(x_1)$. Hence, $S - \{x_0 = t_{\lambda^*}\} \not\models \phi$ and $\mathbf{E}_K(\lambda)$ is not κ-compact.

Further, all members of $\wp_\kappa(S \cup \{\phi'\})$ have non-trivial models. But, since $S \models \phi$; $S \cup \{\phi'\} \models x_0 = x_1$ and $S \cup \{\phi'\}$ has only trivial models. Hence, $\mathbf{E}_K(\lambda)$ is not κ-satisfiable.

Let κ be an infinite cardinal $< c(K)$. There is g, a functional constant in K, such that $\kappa \leq d(g)$. Hence, by Lemma 6, $\mathbf{E}_K(\lambda)$ is neither κ-compact nor κ-satisfiable. Therefore, if $c(K)$ is a regular cardinal and $\aleph_0 \leq \kappa < c(K)$, then $\mathbf{E}_K(\lambda)$ is neither κ-compact nor κ-satisfiable. Analogous results hold when $c(K)$ is a singular cardinal. It remains to show that if $c(K)$ is a singular cardinal, then $\mathbf{E}_K(\lambda)$ is neither $c(K)$ -compact nor $c(K)$ -satisfiable.

Lemma 7. *If $c(K)$ is a singular cardinal, then $\mathbf{E}_K(\lambda)$ is neither $c(K)$ -compact nor $c(K)$ -satisfiable.*

Proof. Suppose that $c(K)$ is an infinite singular cardinal. Let β be the cofinality of $c(K)$. β is an infinite regular cardinal $< c(K)$. There is $\{\theta_\gamma : \gamma < \beta\}$, a family of strictly increasing infinite cardinals each of which is strictly smaller than $c(K)$, and $\cup\{\theta_\gamma : \gamma < \beta\} = c(K)$. Further, there is $\{g_\gamma : \gamma < \beta\} \cup \{g\}$, a family of different functional constants in K, such that (1) $\beta < d(g)$; and (2) for all $\gamma < \beta$, $\theta_\gamma < d(g_\gamma)$.

For each $\gamma < \beta$ and $\delta < \theta_\gamma$, $t_{\gamma,\delta}$ is defined as follows:

(1) $t_{\gamma,0}$ is $g_\gamma(t^o_{[\gamma,0]} \ldots t^\theta_{[\gamma,0]} \ldots)_{\theta<d(g_\gamma)}$, where for all $\theta < d(g_\gamma)$, $t^\theta_{[\gamma,0]}$ is x_0; and

(2) when $\delta > 0$, $t_{\gamma,\delta}$ is $g_\gamma(t^0_{[\gamma,\delta]} \ldots t^\theta_{[\gamma,\delta]} \ldots)_{\theta<d(g_\gamma)}$, where for all $\theta < \delta$, $t^\theta_{[\gamma,\delta]}$ is x_1 and for all $\theta \geq \delta$, $t^\theta_{[\gamma,\delta]}$ is x_0.

For $\gamma < \beta$, $S_\gamma = \{x_0 = t_{\gamma,\delta} : \delta < \theta_\gamma\}$. $S = \cup\{S_\gamma : \gamma < \beta\}$. card $S_\gamma = \theta_\gamma$ and card $S = c(K)$. ϕ_γ is $x_0 = g_\gamma(t'_{\gamma,0} \dots t'_{\gamma,\theta} \dots)_{\theta<d(g\gamma)}$ where if $\theta < \theta_\gamma$, $t'_{\gamma,\theta}$ is $t_{\gamma,\theta}$ and if $\theta \geq \theta_\gamma$, $t'_{\gamma,\theta}$ is x_0. $S_\gamma \models \phi_\gamma$.

Let ϕ^* be $x_0 = g(t^*_0 \dots t^*_\theta \dots)_{\theta<d(g)}$, where for all $\theta < d(g)$, t^*_θ is x_0.

Let ϕ^{**} be $x_0 = g(t^{*'}_0 \dots t^{*'}_\theta \dots)_{\theta<d(g)}$, where for all $\theta' < \beta$, $t^{*'}_{\theta'}$ is $g_{\theta'}(t'_{\theta',0} \dots t'_{\theta',\theta} \dots)_{\theta<d(g_{\theta'})}$ and for $\theta' \geq \beta$, $t^{*'}_{\theta'}$ is x_0.

Let $\phi^{**'}$ be $x_1 = g(t^{*'}_0 \dots t^{*'}_{\theta'} \dots)_{\theta'<d(g)}$. It is easily verified that $S \cup \{\phi^*\} \models \phi^{**}$ and that $S \cup \{\phi^*, \phi^{**'}\} \models x_1 = x_0$.

It is shown below that if $S' \in \wp_{c(K)}(S)$, then $S' \cup \{\phi^*\} \not\models \phi^{**}$ and that $S' \cup \{\phi^*, \phi^{**'}\}$ has a non-trivial model. Given $S' \in \wp_{c(K)}(S)$, there are $\gamma < \beta$ and δ^* such that $0 < \delta^* < \theta_{\gamma*}$ and $x_0 = t_{\gamma^*,\delta^*}$ is not in S'. Let $\mathfrak{A} = (A, f_{\mathfrak{A}})$ where $A = c(K)$. For $\gamma < \beta$, $\theta < d(g_\gamma)$ and $h \in {}^{d(g_\gamma)}A$, h is a constant function, θ is the critical point of h and h has a critical point are defined as in the proof of Lemma 6. If $h \in {}^{d(g)}A$, $f_{\mathfrak{A}}(g)(h) = h(\gamma)$.

When $\gamma \neq \gamma^*$, $f_{\mathfrak{A}}(g_\gamma)$ is defined as follows:

(1) $f_{\mathfrak{A}}(g_\gamma)(h) = h(0)$, if h is a constant function;

(2) $f_{\mathfrak{A}}(g_\gamma)(h) = h(\theta + 1)$, if θ is the critical point of h; and

(3) $f_{\mathfrak{A}}(g_\gamma)(h) = h(0)$, if h is not a constant function and h has no critical point.

$f_A(g_{\gamma^*})$ is defined as follows:

(1) $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(0)$, if h is a constant function;

(2) $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(\theta + 1)$, if θ is the critical point of h and $\theta \neq \delta^*$;

(3) $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(\delta^*)$, if δ^* is the critical point of h; and

(4) $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(\delta^*)$, if h is not a constant function and h has no critical point.

It is easily verified that $\mathfrak{A} \models (S \cup \{\phi^*\}) - \{x_0 = t_{\gamma^*,\delta^*}\}$. Let $\mathbf{a} \in As(\mathfrak{A})$. To show that $\mathfrak{A} \models \phi^{**'}[\mathbf{a}]$, $\mathbf{a}(g(t^{*'}_0 \dots t^{*'}_\theta \dots)_{\theta<d(g)}) = \mathbf{a}(t^{*'}_{\gamma^*})$. Since $\gamma^* < \beta$, $t^{*'}_{\gamma^*}$ is $g_{\gamma^*}(t'_{\gamma^*,0} \dots t'_{\gamma^*,\theta} \dots)_{\theta<d(g_{\gamma^*})}$. Thus, $\mathbf{a}(t^{*'}_{\gamma^*}) = \mathbf{a}(g_{\gamma^*}(t'_{\gamma^*,0} \dots t'_{\gamma^*,\theta} \dots)_{\theta<d(g_{\gamma^*})})$. By construction, if $\theta < \theta_{\gamma^*}$, $t'_{\gamma^*,\theta}$ is $t_{\gamma^*,\theta}$ and if $\theta \geq \theta_{\gamma^*}$, $t'_{\gamma^*,\theta}$ is x_0.

Let $h \in {}^{d(g_{\gamma^*})}A$ be such that $h(\theta) = \mathbf{a}(t'_{\gamma^*,\theta})$ for all $\theta < d(g_{\gamma^*})$. $\mathbf{a}(t^{*'}_{\gamma^*}) = f_{\mathfrak{A}}(g_{\gamma^*})(h)$. If $\theta \neq \delta^*$, then $\mathbf{a}(t'_{\gamma^*,\theta}) = \mathbf{a}(x_0)$; and

$$\mathbf{a}(t'_{\gamma^*,\delta^*}) = \mathbf{a}(g_{\gamma^*}(t^0_{[\gamma^*,\delta^*]} \dots t^\theta_{[\gamma^*,\delta^*]} \dots)_{\theta<d(g_{\gamma^*})}).$$

For all $\theta \leq \delta^*$, $t^\theta_{[\gamma^*,\delta^*]}$ is x_1 and for all $\theta > \delta^*$, $t^\theta_{[\gamma^*,\delta^*]}$ is x_0. Let $h' \in {}^{d(g_{\gamma^*})}A$ be such that $h'(\theta) = \mathbf{a}(t^\theta_{[\gamma^*,\delta^*]})$. Since $\delta^* < \theta_{\gamma^*} < d(g_{\gamma^*})$ and θ_{γ^*} is a cardinal,

$\delta^* + 1 < d(g_{\gamma^*})$ and δ^* is the critical point of h'. Thus, $f_{\mathfrak{A}}(g_{\gamma^*})(h') = h'(\delta^*) = \mathbf{a}(x_1) = \mathbf{a}(t'_{\gamma^*,\delta^*})$. Either $\mathbf{a}(x_1) = \mathbf{a}(x_0)$ or $\mathbf{a}(x_1) \neq \mathbf{a}(x_0)$. Suppose that $\mathbf{a}(x_1) = \mathbf{a}(x_0)$. Then, h is a constant function and $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(0) = \mathbf{a}(x_0) = \mathbf{a}(x_1)$. Hence, $\mathfrak{A} \models \phi^{**'}[\mathbf{a}]$. Suppose that $\mathbf{a}(x_1) \neq \mathbf{a}(x_0)$. h is not a constant function. And, as $\delta^* > 0$, h does not have a critical point. Therefore, $f_{\mathfrak{A}}(g_{\gamma^*})(h) = h(\delta^*) = \mathbf{a}(x_1) = \mathbf{a}(t'_{\gamma^*,\delta^*})$ and $\mathfrak{A} \models \phi^{**'}[\mathbf{a}]$. Hence, in either case $\mathfrak{A} \models \phi^{**'}[\mathbf{a}]$ and $\mathfrak{A} \models \phi^{**'}$.

Since $\mathfrak{A}$ is non-trivial, there is $\mathbf{a} \in A\mathbf{s}(\mathfrak{A})$ such that $\mathbf{a}(x_0) \neq \mathbf{a}(x_1)$. Thus, $\mathfrak{A} \not\models \phi^{**}$, $S' \cup \{\phi^*\} \not\models \phi^{**}$ and $\mathbf{E}_K(\lambda)$ is not $c(K)$ -compact. Further, for all $S' \in \wp_{c(K)}(S)$, $S' \cup \{\phi^*, \phi^{**'}\}$ has a non-trivial model and $\mathbf{E}_K(\lambda)$ is not $c(K)$ -satisfiable. □

Corollary 1 together with Lemma 6 and Lemma 7 yield the following.

Theorem 2. *If κ is an infinite cardinal $\leq \operatorname{card} E_K(\lambda)$, then $\mathbf{E}_K(\lambda)$ is κ-compact iff $\mathbf{E}_K(\lambda)$ is κ-satisfiable.*

3. There are other formulations of the compactness theorem in first order logic. Among these is the *elementary class* formulation: if Υ, a class of interpretations, is an elementary class (i. e. the models of some finite set of first order sentences), then for every S, a set of first order sentences, if Υ is the class of models of S, then there is S', a finite subset of S, such that Υ is the class of models of S'. The elementary class formulation is implied by the consequence formulation. Suppose that Υ is an elementary class. There is $n \geq 1$ such that $S'' = \{\phi_i : i < n\}$ and Υ is the class of models for S''. Let S be any set of sentences. Suppose that Υ is the class of models for S. For all $i < n$, $S \models \phi_i$. Hence, by the consequence formulation, for each $i < n$ there is S_i, a finite subset of S, such that $S_i \models \phi_i$. Let $S' = \cup\{S_i : i < n\}$. S' is a finite subset of S and Υ is the class of models of S'.

The elementary class formulation yields the satisfiability formulation. To see this note that the empty set is an elementary class. Hence, if a set of first order sentences has no models, by the elementary class formulation, some finite subset of this set has no models.

The elementary class formulation has an analogue in finitary equational logic. Let K be a type whose characteristic is $< \aleph_0$ and let Υ be a class of algebras of type K. Υ is a *finitary_equational class* provided Υ is the class of models of a finite set of equations of type K. The following is the equational analogue of the elementary class formulation: if Υ is a finitary equational class, then for all sets of equations of type K, if Υ is the class of models of this set of equations, then Υ is the class of some finite subset of the given set of equations. Some authors (e.g. Burris and Sankappanavar [1980] p. 98) call the equational analogue of the elementary class formulation the compactness theorem for equational logic. By an argument analogous to the above it is easily shown that the equational analogue of the elementary class formulation is implied by the equational analogue of the consequence formulation. Further, the equational analogue of the satisfiability formulation is implied by the equational analogue of the elementary class formulation. To see this one need only observe that the class of trivial models of type K is a finitary equational class.

The above can be generalized to infinitary equational logics. Let K be a type, Υ be a class of algebras of type K and κ be an infinite cardinal, Υ is a *κ-equational class* (in $\mathbf{E}_K(\lambda)$ provided there is S, a set of equations of type K, such that (1) Υ is the collection of models of S and (2) the cardinality of $S < \kappa$. The finitary equational classes are exactly the $\aleph_0$-equational classes. $\mathbf{E}_K(\lambda)$ is κ - *ec* provided (1) $\kappa \leq \text{card}\, E_K(\lambda)$: and (2) if Υ is a κ-equational class in $\mathbf{E}_K(\lambda)$, then for all S, a set of equations in K, if Υ is the class of models of S, there is $S' \in \wp_\kappa(S)$ such that Υ is the class of models of S'. When $c(K)$ is $\aleph_0$, $\mathbf{E}_K(\lambda)$ is $\aleph_0$- e c. Further, $\mathbf{E}_K(\lambda)$ is also κ -e c for any κ between $\aleph_0$ and $\text{card}\, E_K(\lambda)$. These results are easily generalized.

Lemma 8. *If $\mathbf{E}_K(\lambda)$ is κ-compact and κ is a regular cardinal, then*

(1) *$\mathbf{E}_K(\lambda)$ is κ - e c; and*

(2) *if κ' is a cardinal such that $\kappa \leq \kappa' \leq \text{card}\, E_K$, then $\mathbf{E}_K(\lambda)$ is κ' - e c.*

Proof. Suppose that $\mathbf{E}_K(\lambda)$ is κ -compact and that κ is a regular cardinal.

(1) Let Υ be a κ -equational class in $\mathbf{E}_K(\lambda)$. Thus, there is θ, a cardinal $< \kappa$, and $S'' = \{\theta_\gamma : \gamma < \theta\}$, a set of equations of type K, such that Υ is the class of models of S''. Let S be a set of equations of type K and suppose that Υ is the class of models of S. By supposition, for each $\gamma < \theta$ there is $S_\gamma \in \wp_\kappa(S)$ and $S_\gamma \models \theta_\gamma$. Let $S' = \cup\{S_\gamma : \gamma < \theta\}$. Υ is the class of models of S'. Since $\theta < \kappa$ and κ is regular, $\text{card}\, S' < \kappa$.

(2) Let κ' be a cardinal such that $\kappa < \kappa' \leq \text{card}\, E_K(\lambda)$. Suppose that Υ is a κ' -equational class in $\mathbf{E}_K(\lambda)$. Thus, there is θ, a cardinal strictly less than κ', and $S'' = \{\phi_\gamma : \gamma < \theta\}$ such that Υ is the class of models of S''. Let S be a set of equations in $E_K(\lambda)$ and suppose that Υ is the collection of models of S. Each member of S'' is a consequence of S. Since $\mathbf{E}_K(\lambda)$ is κ- compact, for all $\gamma < \theta$ there is $S_\gamma \in \wp_\kappa(S)$ such that $S_\gamma \models \theta_\gamma$. Let $S' = \cup\{S_\gamma : \gamma < \theta\}$. Υ is the class of models of S'. $\text{card}\, S' \leq max\{\kappa, \theta\} < \kappa'$.

It follows from Lemma 8 and Corollary 1 that if $c(K)$ is a regular cardinal, $\mathbf{E}_K(\lambda)$ is κ -e c for each κ between $c(K)$ and $\mathbf{E}_K(\lambda)$; and that if $c(K)$ is a singular cardinal, $\mathbf{E}_K(\lambda)$ is κ -e c for each κ between the cardinal successor of $c(K)$ and $\text{card}\, E_K(\lambda)$. □

The next lemma implies that this is the best result obtainable.

Lemma 9. *If* $\mathbf{E}_K(\lambda)$ *is* κ *- e c, then* $\mathbf{E}_K(\lambda)$ *is* κ *-satisfiable.*

Proof. Suppose that $\mathbf{E}_K(\lambda)$ is κ -e c. Since κ is infinite, the class of trivial algebras of type K is a κ -equational class. Hence, if the set of equations S has only trivial models, then some member of $\wp_\kappa(S)$ has only trivial models and $\mathbf{E}_K(\lambda)$ is κ -satisfiable. □

It follows from Lemma 9 and Lemma 7 that if $c(K)$ is a singular cardinal, then $\mathbf{E}_K(\lambda)$ is not $c(K)$ -e c. Further, Theorem 2 together with the above yield the following.

Theorem 3. *If* κ *is an infinite cardinal* $\leq \operatorname{card} E_K(\lambda)$*,* $\mathbf{E}_K(\lambda)$ *is* κ *-compact iff* $\mathbf{E}_K(\lambda)$ *is* κ *-e c iff* $\mathbf{E}_K(\lambda)$ *is* κ*-satisfiable.*

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