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CORRIGENDUM: RESIDUATION IN COMMUTATIVE ORDERED MONOIDS WITH MINIMAL ZERO

The assertional logic $\mathbf{S}(\mathsf{BCIA})$ of the quasivariety of **BCI**-algebras (in Iseki's sense) is axiomatized, relative to pure implicational logic **BCI**, by the rule

$$x, y, \vdash x \to y \tag{G}$$

(see [1]). Alternatively, the role of (G) can be played by

$$x \vdash x \to (y \to y) \tag{1}$$

(see [2]). The formula

$$(x \to x) \to (y \to y) \tag{2}$$

is a theorem of $\mathbf{S}(\mathsf{BCIA})$.

In [2, Proposition 22] we claimed erroneously that, relative to **BCI**, the axiom (2) is equivalent to (G) (i.e. to (1)), and we concluded that $\mathbf{S}(\mathsf{BCIA})$ is an axiomatic extension of **BCI**. This conclusion is also false. To correct this we verify here

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Proposition. $\mathbf{S}(\mathsf{BCIA})$ *is not an axiomatic extension of* **BCI**.

Proof. Let Z and ω denote the set of all integers and the set of all non-negative integers, and $\mathbf{Z}$ the algebra $\langle Z; \to \rangle$, where $a \to b = b - a$ for all $a, b \in Z$. Observe that both $\langle \mathbf{Z}; \omega \rangle$ and $\langle \mathbf{Z}; \{0\} \rangle$ are **BCI**-matrices, i.e. both ω and $\{0\}$ contain all instances in Z of the axioms (B), (C) and (I), and are closed modus ponens. Also $\langle \mathbf{Z}; \{0\} \rangle$ is a $\mathbf{S}(\mathsf{BCIA})$-matrix, because it is closed under (G) (equivalently, under (1)).

Suppose that we could axiomatize $\mathbf{S}(\mathsf{BCIA})$ by adding only new $(\to)$-*axioms* to (B), (C), (I) and modus ponens. These new axioms are, of course, theorems of $\mathbf{S}(\mathsf{BCIA})$ so their instances in Z must be all 0, because $\langle \mathbf{Z}; \{0\} \rangle$ is a $\mathbf{S}(\mathsf{BCIA})$-matrix. It would then follow that $\langle \mathbf{Z}; \omega \rangle$ is a $\mathbf{S}(\mathsf{BCIA})$-matrix, because $\{0\} \subseteq \omega$. This is a contradiction, since $\langle \mathbf{Z}; \omega \rangle$ is clearly not closed under (1): $1 \in \omega$ but $1 \to (1 \to 1) = -1 \notin \omega$. □

Footnote 27 of [2] should be disregarded for similar reasons. No further results of [2] depended on the erroneous claim.

The name **BCIP** was used in [2] for the logic $\mathbf{S}(\mathsf{BCIA})$. The use of 'P', which connotes 'point' was motivated by the fact that truth is represented by a single element in the algebraic models of this logic and that $\mathbf{S}(\mathsf{BCIA})$ is the least 'pointedly algebraizable' simple extension of **BCI**. If this usage is worth preserving then the labels of [2] should be changed: (P) should mean the rule (1) rather than (2). (The label (G) already stands for Gödel.)

We also correct here an inaccurate remark in [2, pp. 26, 53] (on which nothing in that paper relied) to the effect that **BCI** has no local deduction-detachment theorem (LDDT). It is the appropriate extension of **BCI** by one or both of the connectives $\wedge$, $\vee$ (without the Ackermann constant t) that demonstrably has no LDDT. This is shown, for example, in [3, Example 2]. It is well known that the pure implication logic **BCI** has the following LDDT:

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff for some } n \in \omega,\ \Gamma \vdash \varphi \to^n \psi.$$

Here $\varphi \to^n \psi$ denotes ψ and $\varphi \to^{n+1} \psi$ abbreviates $\varphi \to (\varphi \to^n \psi)$.

References

[1] J.K. Kabziński, *BCI-algebras from the point of view of logic*, Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio. **12** (1983), pp. 126–129.

[2] J.G. Raftery, C.J. van Alten, *Residuation in Commutative Ordered Monoids*, Reports on Mathematical Logic **34** (2000), pp. 23–57

[3] C.J. van Alten, J.G. Raftery, *Embedding theorems and rule separation in logics without weakening*, Studia Logica **76** (2), to appear.

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