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$(N+1)$-BOUNDED WAJSBERG ALGEBRAS WITH A U-OPERATOR

A b s t r a c t. Wajsberg algebras are just a reformulation of Chang MV-algebras where implication is used instead of disjunction. MV-algebras were introduced by Chang to prove the completeness of the infinite-valued Łukasiewicz propositional calculus. Bounded Wajsberg algebras are equivalent to bounded MV-algebras. The class of $(n+1)$-bounded Wajsberg algebras endowed with a U-operator, which plays the role of the universal quantifier, is studied. The simple algebras and the subalgebras of the finite simple algebras are characterized. It is proved that this variety of algebras is semisimple and locally finite.

1. Preliminaries

Wajsberg algebras (see [6,13,9,18]) are an equivalent reformulation of Chang MV-algebras based on implication instead of disjunction. As proved

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by Chang, idempotent MV-algebras are the same as boolean algebras. MV-algebras were introduced by Chang [3,4] to prove the completeness of the infinite valued Łukasiewicz propositional calculus. R. Grigolia [11] axiomatized $(n+1)$-valued Łukasiewicz propositional calculi, using the $(n+1)$-boundedness axiom, together with other special axioms (for details see [6]).

Y. Komori [13] introduced the CN-algebras as algebraic models of Łukasiewicz infinite-valued propositional calculus formulated in terms of the operations implication and negation. A. J. Rodriguez [18] called Wajsberg algebras what was previously known as CN-algebras (see also [9]). Bounded and $(n+1)$-valued Wajsberg algebras are equivalent to bounded and $(n+1)$-valued MV-algebras, respectively.

In what follows, we list several known results about Wajsberg algebras and $(n+1)$-bounded Wajsberg algebras. The basic papers on MV-algebras are Chang's papers [3,4] (see the book [6] for further references). For a reformulation in the context of Wajsberg algebras (or CN-algebras) see [18,9,13]. Let us remember that an algebra $\langle A, \rightarrow, \sim, 1\rangle$ of type $(2,1,0)$ is a Wajsberg algebra (or W-algebra for short) if the identities $1 \rightarrow x = x$, $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$, $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ and $(\sim y \rightarrow \sim x) \rightarrow (x \rightarrow y) = 1$ are satisfied.

We denote by C_{n+1} the W-algebra with universe $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ and the operations defined by $x \rightarrow y := min\ \{1, 1-x+y\}$ and $\sim x := 1-x$. It is verified that C_{t+1} is a W-subalgebra of C_{n+1} if and only if t divides n.

Let $\langle A, \rightarrow, \sim, 1\rangle$ be a W-algebra. Then $(A, \vee, \wedge, \sim, 0, 1)$ is a Kleene algebra where $0 = \sim 1$, $x \vee y = (x \rightarrow y) \rightarrow y$, $x \wedge y = \sim (\sim x \vee \sim y)$ and $x \leq y$ if and only if $x \rightarrow y = 1$.

A subset $D \subseteq A$ is a *deductive system* of A if $1 \in D$ and $a, a \rightarrow b \in D$ implies $b \in D$. We represent by $\mathcal{D}(A)$ and $\mathcal{M}(A)$ the families of deductive systems and maximal deductive systems of A, respectively. $\mathcal{D}(A)$ is an algebraic lattice with the set-inclusion, and it is isomorphic to the algebraic lattice of all congruence relations on A.

For each $x \in A$, we let $0x = 0$, and for each integer $n \geq 1$, $nx = \sim x \rightarrow ((n-1)x)$.

Let n be a positive integer. A W-algebra A is $(n+1)$-bounded (or W_{n+1}-algebra for short), if $\sim a \vee na = 1$, for any $a \in A$.

(P1) A W_{n+1}-algebra A is simple if and only if A is isomorphic to C_{r+1}, for some integer $1 \leq r \leq n$.

We say that an algebra A can be represented in $\prod_{i\in I} A_i$, if A is subdirect product of the family $\{A_i\}_{i\in I}$.

(P2) Let A be a non trivial W-algebra. Then, A is a W_{n+1}-algebra if and only if A can be represented in $\prod_{i=1}^{n} C_{i+1}^{M_{i+1}}$, where $M_{i+1} = \{D \in \mathcal{M}(A) : A/D \simeq C_{i+1}\}$.

W_{n+1}-algebras are locally finite. We denote by $\mathcal{L}_{W_{n+1}}(k)$ the $(n+1)$-bounded Wajsberg algebra freely generated by a set of cardinal $k > 0$.

(P3) [18] $\mathcal{L}_{W_{n+1}}(k)$ is isomorphic to $\prod_{i=1}^{n} C_{i+1}^{w_k(i+1)}$, where

$$w_k(i+1) = (i+1)^k - \sum_{j/i,j\neq i} w_k(j+1)$$

As proved by Grigolia [11], the classes of $(n+1)$-bounded and $(n+1)$-valued Wajsberg algebras are the subvarieties of the variety of W-algebras generated by chains of length less or equal to $n+1$ and by the chain of length $n+1$, respectively.

If A_1 and A_2 are two algebras with the same universe A, algebra A_1 is said to be a *reduct* of A_2 if every fundamental operation of A_1 is a term in the language of A_2.

(P4) [19,14] If A is an $(n+1)$-bounded W-algebra, then A admits an $(m+1)$-valued Łukasiewicz algebra reduct, where m is the least common multiple of all integers r such that $1 \leq r \leq n$.

(P5) [19] Let A be an $(n+1)$-bounded W-algebra. If there is an element $c \in A$ such that $(n-1)(\sim c) = \sim c$ then A is polinomially equivalent to a Post algebra of order $(n+1)$ whose fundamental chain is $\{e_k\}_{0\leq k\leq n}$, where for each $0 \leq k \leq n$, $e_k = k(\sim c) = \sim((n-k)(\sim c))$.

Generalizing the notion of universal quantifier on a Tarski algebra, A. V. Figallo [7,8] defined U-operators on I-algebras (the implicative reduct $\langle A, \rightarrow, 1\rangle$ of a Wajsberg algebra $\langle A, \rightarrow, \sim, 1\rangle$ is an I-algebra). The U-operators on Wajsberg algebras have been studied in [15,16,14].

Definition 1.1. Let $\langle A, \rightarrow, \sim, 1\rangle$ be a W-algebra. A U-operator on A is a function $\forall : A \longrightarrow A$ that verifies the identities

(U1) $\forall x \to x = 1$ and

(U2) $\forall(\forall x \to y) = \forall x \to \forall y$.

An algebra $\langle A, \to, \sim, \forall, 1\rangle$ is a UW-algebra if the reduct $\langle A, \to, \sim, 1\rangle$ is a Wajsberg algebra and $\forall$ is a U-operator on A.

Let A be a UW-algebra. For each $x, y \in A$ we have

(U3) $x \leq y$ implies $\forall x \leq \forall y$.

Following Chang, an element $a \in A$ is called a boolean element if there is $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of boolean elements of A by $B(A)$; $K(A) = \{x \in A : x = \forall x\}$ and $H(A) = B(A) \cap K(A)$. $B(A)$, $K(A)$ and $H(A)$ are UW-subalgebras of A.

A deductive system D of A is a *U-deductive system* if $x \in D$ implies $\forall x \in D$.

We represent by $\mathcal{D}_U(A)$ and $\mathcal{M}_U(A)$ the families of U-deductive systems and maximal U-deductive systems of A, respectively. $\mathcal{D}_U(A)$ is an algebraic lattice with the set-inclusion, and it is isomorphic to the algebraic lattice of all congruence relations on A.

(P6) [16] If S is a UW-subalgebra of A then $\mathcal{D}_U(S) = \{D \cap S : D \in \mathcal{D}_U(A)\}$, and $\mathcal{M}_U(S) = \{D \cap S : D \in \mathcal{M}_U(A)\}$.

It is known that all $(n+1)$-valued W-algebras have a reduct of $(n+1)$-valued Łukasiewicz algebra [19,12]. Monadic $(n+1)$-valued Łukasiewicz algebras were studied by L. Monteiro [17] for $n = 2$ and by M. Abad [1] for $n \geq 1$, respectively. In [14] conditions are given so that a U-operator commutes with Moisil's modal operators that can be defined on an $(n+1)$-valued Wajsberg algebra (the additional axioms are (U4) $\forall(mx) = m\forall x$ and (U5) $\forall(\sim m \sim x) = \sim m \sim \forall x$, for each $m \geq 1$).

Monadic $(n+1)$-valued MV-algebras (see [20,21,10]) are defined as pairs $\langle A, \exists\rangle$, where A is an $(n+1)$-valued MV-algebra and $\exists$ verifies, among other conditions equivalent to (U1) and (U2), the dual of axioms (U4) and (U5) for $m = 2$. It is proved in [10] that the monadic $(n+1)$-valued MV-algebras are polynomially equivalent to the $(n+1)$-valued Łukasiewicz algebras for $n = 2$ and $n = 3$.

2. $UW_{(n+1)}$-algebras: definition and simple algebras

Definition 2.1. [15] An algebra $\langle A, \rightarrow, \sim, \forall, 1\rangle$ is a UW_{n+1}-algebra if the reduct $\langle A, \rightarrow, \sim, 1\rangle$ is an $(n+1)$-bounded Wajsberg algebra and $\forall$ is a U-operator on A.

Let K be a class of algebras and let S and A be K-algebras. We write $S \triangleleft_{\mathbf{K}} A$ to indicate that S is a subalgebra of A, and $A \simeq_{\mathbf{K}} B$ whenever there is a K-isomorphism between K-algebras A and B. We denote by $[X]_{\mathbf{K}}$ the K-subalgebra of A generated by X, for some $X \subseteq A$. In all the cases, when K is the class of UW_{n+1}-algebras we shall tacitly omit the symbol K.

Remark 2.1. It is easy to see that $\langle C_3, \rightarrow, \sim, \forall, 1\rangle$ is a UW_3-algebra, where $\forall 0 = \forall \frac{1}{2} = 0$ and $\forall 1 = 1$. If we consider the structure of three-valued Łukasiewicz algebra of C_3, we have that $\sigma_2^3\left(\forall \frac{1}{2}\right) = 0$ whereas $\forall \sigma_2^3\left(\frac{1}{2}\right) = 1$. Therefore, if we have $\exists x = \sim \forall \sim x$, it results that the class of UW_{2+1}-algebras is not equivalent to the class of monadic three-valued Łukasiewicz algebras introduced by L. Monteiro in [17].

The underlying lattice of a W_{n+1}-algebra is a dual Heyting algebra [22,23]. Then, for every W_{n+1}-algebra A, there is a bijective correspondence between W-subalgebras S of A which verify the condition: for each $a \in A$, the set $(a] \cap S$ has a biggest element, and U-operators on A. Moreover, the U-operator corresponding to W-subalgebra S is defined, for each $a \in A$, as the biggest element in $(a] \cap S$, and $K(A) = S$ (see [5]).

Example 2.1. Let t be an integer, $1 \leq t \leq n$. From the above discussion we know that there are as many U-operators on C_{t+1} as W-subalgebras; i.e. there are as many U-operators on C_{t+1} as divisors of t. If r divides t (r/t for short), we denote by $\langle C_{t+1}, \forall^r\rangle$ the UW_{n+1}-algebra in which the U-operator is determined by C_{r+1}; i.e. $\forall^r x$ is the biggest element in $(x] \cap C_{r+1}$.

Example 2.2. Let a nonempty set I and integers r, t_i, $1 \leq t_i \leq n$ such that r/t_i for every $i \in I$. For each $i \in I$, let $A_i = \langle C_{t_i+1}, \forall_i^r\rangle$. Then, the nonempty family of UW_{n+1}-algebras $\{A_i\}_{i \in I}$ determines a UW_{n+1}-algebra, which we denote by $\mathcal{P}_{r,\{t_i\}_{i\in I}}$, in the following way. Let us consider the direct product of W_{n+1}-algebras C_{t_i+1}, $P_{\{t_i\}_{i\in I}} = \prod\limits_{i\in I} C_{t_i+1}$, and let $\forall^r$

be the unary operator on $P_{\{t_i\}_{i\in I}}$ defined by the prescription $(\forall^r f)(i) = \bigwedge_{i\in I} \forall_i^r(f(i))$. It is easy to show that $\mathcal{P}_{r,\{t_i\}_{i\in I}} = \langle P_{\{t_i\}_{i\in I}}, \forall^r\rangle$ is a UW_{n+1}-algebra by taking into account that in every W-algebra A, if $\{a_i\}_{i\in I} \subseteq A$ is a nonempty family of elements of A and there is $\bigwedge_{i\in I} a_i$, then for every $a \in A$ there exists $\bigwedge_{i\in I} (a\to a_i)$ and $\bigwedge_{i\in I} (a\to a_i) = a\to \bigwedge_{i\in I} a_i$ holds.

If $A_i = \langle C_{t+1}, \forall^t\rangle$ for every $i \in I$ and for some integer t, $1 \le t \le n$, then the $\mathcal{P}_{r,\{t_i\}_{i\in I}}$ algebra is precisely the functional algebra $P_{t,I} = C_{t+1}^I$, which is a $(t+1)$-valued Wajsberg algebra.

In the variety of $(n+1)$-bounded Wajsberg algebras every deductive system is a *Stone filter* (i.e. a lattice filter generated by a set of boolean elements), therefore every U-deductive system coincides with the lattice filter generated by its boolean elements. This is, for every $D \in \mathcal{D}_U(A)$, $D = F(D\cap B(A)) = \{x \in A :$ there is $a \in D\cap B(A)$ such that $a \le x\}$. Besides, it is easy to show that $\mathcal{D}_U(K(A)) = \mathcal{D}(K(A))$ and $\mathcal{D}_U(H(A)) = \mathcal{D}(H(A))$. Using this result and property (P6) we obtain

Theorem 2.1. *Let A be a UW_{n+1}-algebra. Let us consider the functions*

$$\varphi_1 : \mathcal{D}_U(A) \to \mathcal{D}(K(A)),\ \varphi_1(D) = D\cap K(A);$$

$$\varphi_2 : \mathcal{D}_U(A) \to \mathcal{D}_U(B(A)),\ \ \varphi_2(D) = D\cap B(A);$$

$$\varphi_3 : \mathcal{D}(K(A)) \to \mathcal{D}(H(A)),\ \ \varphi_3(D) = D\cap H(A);$$

$$\varphi_4 : \mathcal{D}_U(B(A)) \to \mathcal{D}(H(A)),\ \varphi_4(D) = D\cap H(A).$$

If the sets $\mathcal{D}_U(A), \mathcal{D}_U(B(A)), \mathcal{D}(K(A))$ and $\mathcal{D}(H(A))$ are ordered by set-inclusion, then the functions $\varphi_1, \varphi_2, \varphi_3$ and φ_4 are order-isomorphisms, and the following square is commutative.

$$\begin{array}{ccc} & \varphi_1 & \\ \mathcal{D}_U(A) & \longrightarrow & \mathcal{D}(K(A)) \\ \varphi_2 \downarrow & & \downarrow \varphi_3 \\ \mathcal{D}_U(B(A)) & \longrightarrow & \mathcal{D}(H(A)) \\ & \varphi_4 & \end{array}$$

Hence, the notions of maximal U-deductive system, irreducible U-deductive system and completely irreducible U-deductive system coincide in the variety of UW_{n+1}-algebras, because they coincide in Boolean algebras and the function $\varphi_3 \circ \varphi_1$ (or $\varphi_4 \circ \varphi_2$) preserves these concepts.

Therefore, each proper U-deductive system is the intersection of maximal U-deductive systems and, the intersection of all maximal U-deductive systems is the set $\{1\}$.

Thus, by known results of universal algebra (see [2]) we have that every non trivial algebra A is isomorphic to a subalgebra of $\prod\limits_{M\in\mathcal{M}_U(A)} A/M$, i.e. the following is verified.

Theorem 2.2. *Every UW_{n+1}-algebra is semisimple.*

If A is a finite UW_{n+1}-algebra, it is easy to prove that the maximal U-deductive systems are the lattice filters generated by elements of $H(A)$. Moreover, if A is a finite non trivial UW_{n+1}-algebra, then $A \simeq \prod\limits_{i=1}^{t} A/M_i$, where $\{M_i\}_{1\leq i\leq t}$ is the family of maximal U-deductive systems of A and t is the number of atoms in $H(A)$.

Theorem 2.3. *Let A be a non trivial UW_{n+1}-algebra. The following conditions are equivalent:*

(i) *A is a simple UW_{n+1}-algebra,*

(ii) *$K(A)$ is a simple W_{n+1}-algebra,*

(iii) *$H(A)$ is a simple boolean algebra.*

Proof. Follows at once from Theorem 2.1. □

Corollary 2.1. *The $\mathcal{P}_{r,\{t_i\}_{i\in I}}$ algebras (defined in Example 2.2) are simple.*

Theorem 2.4. *If $\langle A, \rightarrow, \sim, \forall, 1\rangle$ is a simple UW_{n+1}-algebra, then it is isomorphic to a subalgebra of $\mathcal{P}_{r,\{t_i\}_{i\in I}}$, for some nonempty set I and some integers r, $\{t_i\}_{i\in I}$.*

Proof. Let A be a simple UW_{n+1}-algebra. Then $\mathcal{M}(A) \neq \emptyset$ and from Theorem 2.3 and (P6) it is clear that $D \cap K(A) = \{1\}$ for each $D \in \mathcal{M}(A)$; hence the function $g_D : K(A) \longrightarrow A/D$ defined by $g_D(x) = q_D(x)$ for each $x \in K(A)$ (with q_D we denote the natural epimorphism from A onto A/D), is an injective W-homomorphism. Besides, $C_{r+1} \simeq K(A) \simeq S_D \lhd A/D \simeq C_{t_D+1}$, for some integers r, t_D such that $1 \leq r, t_D \leq n$ and r/t_D for each $D \in \mathcal{M}(A)$. Let $g : K(A) \longrightarrow C_{r+1}$ be the isomorphism; we have that $K(A) \simeq S_D \simeq C_{r+1}$ for every $D \in \mathcal{M}(A)$. Therefore, identifying isomorphic algebras and making abuse of notation we write $g_D(k) = g(k)$, for each $k \in K(A)$. Thus we obtain

(1) $A_D = \langle A/D, \forall^r_D \rangle$ is a UW_{n+1}-algebra for each $D \in \mathcal{M}(A)$, and $\forall^r_D((q_D(x))$ is the biggest element in $(q_D(x)] \cap S_D$ (see Example 2.1).

Let $\mathcal{P}_{r,\{t_D\}_{D\in\mathcal{M}(A)}} = \langle P_{\{t_D\}_{D\in\mathcal{M}(A)}}, \forall^r \rangle$ be the algebra defined as in Example 2.2; i.e.

$$P_{\{t_D\}_{D\in\mathcal{M}(A)}} = \prod_{D\in\mathcal{M}(A)} A/D \simeq \prod_{D\in\mathcal{M}(A)} C_{t_D+1},$$

and

$$(\forall^r f)(D) = \bigwedge_{D\in\mathcal{M}(A)} \forall^r_D(f(D)).$$

Then $\mathcal{P}_{r,\{t_D\}_{D\in\mathcal{M}(A)}} = \langle P_{\{t_D\}_{D\in\mathcal{M}(A)}}, \forall^r \rangle$ is a UW_{n+1}-algebra.

From (P2) we know that there is an injective W-homomorphism

$$\varphi : A \longrightarrow P_{\{t_D\}_{D\in\mathcal{M}(A)}}$$

defined by $\varphi(x) = h_x$ for each $x \in A$, where $h_x : \mathcal{M}(A) \longrightarrow \bigcup_{D\in\mathcal{M}(A)} A/D$ is the application defined by $h_x(D) = q_D(x)$, for each $D \in \mathcal{M}(A)$.

To prove that $\varphi(\forall x) = \forall^r \varphi(x)$ for every $x \in A$, it is enough to verify

$$q_M(\forall x) = \bigwedge_{D\in\mathcal{M}(A)} \forall^r_D(q_D(x)),$$

for each $M \in \mathcal{M}(A)$. We will prove that $\alpha_0 = \alpha_1$, being $\alpha_0, \alpha_1 \in S_M$,

$$\alpha_0 = \bigwedge_{D\in\mathcal{M}(A)} \forall^r_D(q_D(x)),$$

and $\alpha_1 = q_M(\forall x)$. In fact, since $\alpha_1 = q_M(\forall x) \leq q_M(x)$, from (1) we have that $\alpha_1 \leq \forall^r_M(q_M(x))$ for every $M \in \mathcal{M}(A)$, because $\alpha_1 \in S_M$; therefore $\alpha_1 \leq \alpha_0$. On the other hand, as $\alpha_0 \in S_M \simeq K(A)$ there is $k \in K(A)$ such that $\alpha_0 = q_M(k)$. Hence,

$$\alpha_0 = q_M(k) = \bigwedge_{D\in\mathcal{M}(A)} \forall^r_D(q_D(x)) \leq \forall^r_D(q_D(x)) \leq q_D(x)$$

for every $D \in \mathcal{M}(A)$. Then, $k \to x \in \bigcap_{D\in\mathcal{M}(A)} D = \{1\}$. From (U1) and (U5) it results $\alpha_0 = q_M(k) \leq q_M(\forall x) = \alpha_1$.

Therefore the W-homomorphism φ is also a UW-homomorphism. □

3. Subalgebras of the finite simple algebras

Let t and r be positive integers such that $1 \leq t \leq n$ and r/t; let us consider the UW_{n+1}-algebra $\langle C_{t+1}, \forall^r\rangle$, defined as in Example 2.1. If α is a positive divisor of r, the restriction of U-operator $\forall^r$ to the subalgebra $C_{\alpha+1}$ is the identity; i.e., $\langle C_{\alpha+1}, \forall^\alpha\rangle$ is a subalgebra of $\langle C_{t+1}, \forall^r\rangle$. If there is a positive integer s such that s/t and r/s, then $C_{r+1} \lhd_{\mathbf{W}} C_{s+1} \lhd_{\mathbf{W}} C_{t+1}$ and therefore $\langle C_{s+1}, \forall^r\rangle$ is a subalgebra of $\langle C_{t+1}, \forall^r\rangle$.

Lemma 3.1. *If S is a subalgebra of $\langle C_{t+1}, \forall^r\rangle$ then $S \simeq \langle C_{s+1}, \forall^r\rangle$, for some positive integer s which verifies r/s and s/t, or $S \simeq (C_{\alpha+1}, \forall^\alpha)$, for some positive integer α such that α/r, $\alpha \neq r$ and we say that S is a Type 1 or Type 2 subalgebra of $\langle C_{t+1}, \forall^r\rangle$, respectively.*

Proof. Let S be a subalgebra of $\langle C_{t+1}, \forall^r\rangle$; in particular $S \lhd_{\mathbf{W}} C_{t+1}$, then there is a positive integer s divisor of t such that $S \simeq_{\mathbf{W}} C_{s+1}$. It is clear that $K(S)$ is a subalgebra of $K(C_{t+1}) \simeq C_{r+1}$, hence $K(S) \simeq C_{\alpha+1}$ for some α divisor of r. If $\alpha \neq r$ then must be $\alpha = s$ and $S \simeq \langle C_{\alpha+1}, \forall^\alpha\rangle$, because the U-operator on S must be the restriction of $\forall^r$ to S. If $\alpha = r$ then $S \simeq \langle C_{s+1}, \forall^r\rangle$. □

Given a nonempty set I and integers r, t_i, $1 \leq t_i \leq n$ such that r/t_i for every $i \in I$, let $\mathcal{P}_{r,\{t_i\}_{i\in I}}$ be the algebra determined by the family of algebras $\{\langle C_{t_i+1}, \forall^r_i\rangle\}_{i\in I}$ as shown by Example 2.2. In the following, we will suppose that the set I is finite, $I = \{1, 2, \ldots, m\}$, and sometimes we will write $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$ instead of $\mathcal{P}_{r,\{t_i\}_{i\in I}}$. We say that:

(i) $\{S_i\}_{1\leq i\leq m}$ is a *Type* 1 *family of subalgebras* of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ if S_i is a Type 1 subalgebra of $\langle C_{t_i+1}, \forall_i^r\rangle$ for each i, $1 \leq i \leq m$.

(ii) $\{S_i\}_{1\leq i\leq m}$ is a *Type* 2 *family of subalgebras* of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ if $S_i \simeq \langle C_{\alpha+1}, \forall^\alpha\rangle$ for every i, $1 \leq i \leq m$ and some α divisor of r, $\alpha \neq r$.

(iii) $\{S_i\}_{1\leq i\leq m}$ is a *family of subalgebras* of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ if it is a Type 1 or Type 2 family of subalgebras.

It is clear that if $\{S_i\}_{1\leq i\leq m}$ is a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ then $\prod\limits_{i=1}^{m} S_i$, with the U-operator defined as in Example 2.2, is a subalgebra of $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$. In the proofs, unless otherwise specified, we will suppose that the family of subalgebras is Type 1 because it is the most general case.

Let B be a boolean subalgebra of $B(P_{(t_i)_{1\leq i\leq m}})$. Since $B(P_{(t_i)_{1\leq i\leq m}}) \simeq \mathbf{2}^I$ ($\mathbf{2}$ being the Boole algebra with two elements), we have that $B(P_{(t_i)_{1\leq i\leq m}})$ is a Boole algebra with m atoms. It is known that there is a bijective correspondence between subalgebras of $\mathbf{2}^I$ and partitions of the set of their atoms. Moreover, each partition of the set of atoms in $\mathbf{2}^I$ corresponds, in a natural way, with a partition of the set I. Then, there is a bijective correspondence between subalgebras of $B(P_{(t_i)_{1\leq i\leq m}})$ and partitions of I. Let $\Pi(B) = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of the set I determined by the subalgebra B. The elements in B are characterized as the elements $f \in P_{(t_i)_{1\leq i\leq m}}$ such that $f(i) \in \{0, 1\}$ for every $1 \leq i \leq m$ and $f(i) = f(j)$ if $i, j \in \Pi_\nu$.

Let $\{S_i\}_{1\leq i\leq m}$ be a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ and B be a boolean subalgebra of $B(P_{(t_i)_{1\leq i\leq m}})$. We denote by $[\{S_i\}_{1\leq i\leq m}, \Pi(B)]$ the set

$$\left\{ f \in \prod_{i=1}^{m} S_i : \text{if } i \in \Pi_\nu, f(i) \in \bigcap_{j\in\Pi_\nu} S_j, \text{ and } f(i) = f(j) \text{ for all } i, j \in \Pi_\nu \right\}.$$

Lemma 3.2. *Let* $\{S_i\}_{1\leq i\leq m}$ *be a family of subalgebras of*

$$\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$$

and let B *be a boolean subalgebra of* $B(P_{(t_i)_{1\leq i\leq m}})$. *Then,* $[\{S_i\}_{1\leq i\leq m}, \Pi(B)]$ *is a subalgebra of* $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$.

Proof. Routine. □

Conversely, let S be a subalgebra of $\mathcal{P}_{r,(t_i)_{1\le i\le m}}$. For each i, $1 \le i \le m$, let p_i be the projection map on the ith coordinate of $\prod_{i\in I} C_{t_i+1}$. Then $p_i(S)$ is a W-subalgebra of C_{t_i+1} which contains a copy of $K(S)$. Particularly,

$$S \lhd_W \prod_{i=1}^{m} p_i(S) \lhd_W P_{(t_i)_{1\le i\le m}}.$$

If $K(S) \simeq C_{r+1}$, then $p_i(S)$ contains a copy of C_{r+1}, for every i, $1 \le i \le m$. Thus $\langle p_i(S), \forall_i^r\rangle \lhd \langle C_{t_i+1}, \forall_i^r\rangle$, for every i, $1 \le i \le m$. Therefore, $\{p_i(S)\}_{1\le i\le m}$ is a Type 1 family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\le i\le m}$.

If $K(S)$ is a proper subalgebra of C_{r+1}, let us say $K(S) \simeq C_{\alpha+1}$, $\alpha \ne r$, α/r, then $p_i(S) \simeq C_{\alpha+1}$ for every i, $1 \le i \le m$ (because the U-operator is the one determined by r). In this case, $\{p_i(S)\}_{1\le i\le m}$ is a Type 2 family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\le i\le m}$.

Therefore, $\{p_i(S)\}_{1\le i\le m}$ is a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\le i\le m}$.

We say that S is a *Type* 1 (*Type* 2) *subalgebra* of $\mathcal{P}_{r,(t_i)_{1\le i\le m}}$ if $\{p_i(S)\}_{1\le i\le m}$ is a Type 1 (Type 2) family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\le i\le m}$.

The subalgebra $\langle C_{r+1}^m, \forall^r\rangle$ has been included in the class of Type 1 subalgebras, but when it is convenient we will include it in the class of Type 2 subalgebras.

Let $B(S)$ be the boolean subalgebra of $B(P_{(t_i)_{1\le i\le m}})$ determined by S, i.e. $B(S) = S \cap B(P_{(t_i)_{1\le i\le m}})$, and let $\Pi(B(S)) = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of I determined by $B(S)$.

Given $\Pi_\nu \in \Pi(B(S))$ let us consider the element $a \in B(S)$ defined by

$$a(j) = \begin{cases} 1 & \text{if } j \in \Pi_\nu \\ 0 & \text{if } j \notin \Pi_\nu \end{cases}.$$

Since a is an atom in $B(S)$ we have that $D(a) = [a, 1] \cap S$ is a maximal deductive system and $S/D(a)$ is a simple W-algebra.

Lemma 3.3. *$S/D(a) \simeq_W p_i(S)$, for every $i \in \Pi_\nu$.*

Proof. For each $i \in \Pi_\nu$ let $h_i : S\,/\,D(a) \longrightarrow p_i(S)$ be such that $h_i(q_{D(a)}(x)) = p_i(x)$ for all $x \in S$. Let $y \in q_{D(a)}(x)$. Then there is

$b \in D(a)$ such that $x \wedge b = y \wedge b$. For every $i \in \Pi_\nu$ we have that $x(i) = (x \wedge b)(i) = (y \wedge b)(i) = y(i)$ because $b(i) = 1$. Hence, h_i is well defined for each $i \in \Pi_\nu$. It is clear that h_i is a W-epimorphism; moreover $N(h_i) = \{c \in S\,/\,D(a) : h_i(c) = 1\}$ is a deductive system of $S\,/\,D(a)$, which is a simple algebra. Since $q_{D(a)}(0) \notin N(h_i)$ we have that $N(h_i) = \{q_{D(a)}(1)\}$. Then h_i is injective for all $i \in \Pi_\nu$. □

Corollary 3.1. *Let $\Pi_\nu \in \Pi(B(S))$. Then $p_i(S) \simeq_W p_j(S)$ for all $i, j \in \Pi_\nu$.*

Theorem 3.1. *Let S be a subalgebra of $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$. Then there exist $\{p_i(S)\}_{1\leq i\leq m}$, a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$, and $B(S)$, a boolean subalgebra of $B(P_{(t_i)_{1\leq i\leq m}})$, such that $S = [\{p_i(S)\}_{1\leq i\leq m}, \Pi(B(S))]$.*

Proof. Let $\Pi(B(S)) = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of I determined by $B(S)$.

For brevity we will write $\Omega(S)$ instead of $[\{p_i(S)\}_{1\leq i\leq m}, \Pi(B(S))]$.

Suppose that $p_i(S) \simeq \langle C_{z_i+1}, \forall_i^r\rangle$, for each $i \in I$. From (P4) we have that the $(n+1)$-bounded Wajsberg algebra $\prod\limits_{i\in I} p_i(S)$ admits a $(t+1)$-valued Łukasiewicz algebra reduct, where t is the least common multiple of all integers x such that $1 \leq x \leq n$. In fact, $\prod\limits_{i\in I} p_i(S)$ is a W-subalgebra of C_{t+1}^m, which is a $(t+1)$- valued W-algebra and therefore, admits a structure of $(t+1)$-valued Łukasiewicz algebra, where the unary operators $\sigma_1^{t+1}, \ldots, \sigma_t^{t+1}$ are defined pointwise as follows:

> for each $x \in \prod\limits_{i\in I} p_i(S)$ and each $1 \leq k \leq t$, $\sigma_k^{t+1}(x)(i) = \sigma_k^{t+1}(x(i))$, where for $0 \leq a \leq t$,

$$\sigma_k^{t+1}\left(\frac{a}{t}\right) = \begin{cases} 0 & \text{if } k + a \leq t, \\ 1 & \text{otherwise.} \end{cases}$$

Let $s \in S$, $\Pi_\nu \in \mathbf{\Pi}(B(S))$ and $i, j \in \Pi_\nu$. From Corollary 3.1 we can suppose $p_i(S) \simeq \langle C_{z+1}, \forall_i^r\rangle$ for every $i \in \Pi_\nu$. Besides, there is an integer $q \geq 1$ such that $t = qz$.

Let $s(i) = \frac{a}{z} = \frac{aq}{t}$ and $s(j) = \frac{b}{z} = \frac{bq}{t}$, for $0 \leq a,\, b \leq z$.

If $a < b$ then $aq < bq$ and $1 \leq t - bq + 1 \leq t$. Since

$$\sigma_{t-bq+1}^{t+1}(s) \in B = B(S) \cap B(P_{(t_i)_{1\leq i\leq m}}) \simeq B(S) \cap B(C_{t+1}^{m}),$$

we have that $\sigma_{t-bq+1}^{t+1}(s)(i) = \sigma_{t-bq+1}^{t+1}(s)(j)$.

On the other hand, $\sigma_{t-bq+1}^{t+1}(s)(i) = \sigma_{t-bq+1}^{t+1}(\frac{aq}{t}) = 0$ and $\sigma_{t-bq+1}^{t+1}(s)(j) = \sigma_{t-bq+1}^{t+1}(\frac{bq}{t}) = 1$ which is a contradiction. Then, there must be $s(i) = s(j)$ and therefore $s \in \Omega(S)$.

For the other inclusion, let $y \in \Omega(S)$. For each $i \in I = \{1, \ldots, m\}$ we have that $y(i) \in p_i(S)$, i.e. there is $s_i \in S$ such that $p_i(s_i) = y(i)$. Let $\Pi_{\nu_i} \in \mathbf{\Pi}(B(S))$ such that $i \in \Pi_{\nu_i}$. ¿From $S \subseteq \Omega(S)$ we obtain $p_j(s_i) = p_i(s_i)$ for every $j \in \Pi_{\nu_i}$.

Consider the element $x_i \in B(S)$ defined by

$$x_i(j) = \begin{cases} 1 & \text{if } j \in \Pi_{\nu_i}, \\ 0 & \text{if } j \notin \Pi_{\nu_i}. \end{cases}$$

Let $b_i = x_i \wedge s_i$, for each $i \in I$. Then $s = b_1 \vee b_2 \vee \ldots \vee b_m \in S$. For every $j \in \Pi_{\nu_i}$ it is $b_i(j) = (x_i \wedge s_i)(j) = s_i(j) = p_j(s_i) = p_i(s_i) = y(i) = y(j)$, and for all $j \in I - \Pi_{\nu_i}$ it is $b_i(j) = (x_i \wedge s_i)(j) = 0$. Then, $s = y$. □

Let $\{S_i\}_{1\leq i\leq m}$ be a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$, B be a boolean subalgebra of $B(P_{(t_i)_{1\leq i\leq m}})$ and $\Pi(B) = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of I determined by B. To abbreviate we write Ω instead of $[\{S_i\}_{1\leq i\leq m}, \Pi(B)]$. It is clear that $\prod_{i\in I} S_i$ is a subalgebra of $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$ and Ω is a subalgebra of $\prod_{i\in I} S_i$.

For each $\Pi_\nu \in \Pi(B)$ and each $i \in \Pi_\nu$, let $U_i = F(\{i\})$ be the ultrafilter of $\mathbf{2}^I$ generated by $\{i\}$. It is easy to see that $\Pi_\nu \in U_i$. With Θ_i' we denote the W-congruence relation determined by the ultrafilter U_i on $\prod_{i\in I} S_i$, i.e. $(x, y) \in \Theta_i'$ if and only if $\|x = y\| = \{j \in I : x(j) = y(j)\} \in U_i$, for any $x, y \in \prod_{i\in I} S_i$.

Let Θ_i the restriction of Θ_i' to Ω, i.e., $\Theta_i = \Omega^2 \cap \Theta_i'$. With q_i we denote the natural W-epimorphism from Ω onto Ω/Θ_i.

Lemma 3.4. $\Omega/\Theta_i \simeq_W \bigcap_{j\in\Pi_\nu} S_j$, *for every* $i \in \Pi_\nu$ *and each* $\Pi_\nu \in \Pi(B)$.

Proof. Let $\Pi_\nu \in \Pi(B)$, $i \in \Pi_\nu$ and $x \in \Omega$. The function

$$\varphi_i : \Omega/\Theta_i \longrightarrow \bigcap_{j \in \Pi_\nu} S_j$$

such that $\varphi_i(q_i(x)) = x(i)$ is well defined and is an injective W-homomorphism. Besides, φ_i is surjective.

In fact, given $a \in \bigcap_{j \in \Pi_\nu} S_j$, we define $x \in \prod_{j=1}^{m} S_j$ by

$$x(j) = \begin{cases} a & \text{if } j \in \Pi_\nu, \\ 0 & \text{if } j \notin \Pi_\nu. \end{cases}$$

It is clear that $x \in \Omega$ and $\varphi_i(q_i(x)) = a$, for every $i \in \Pi_\nu$. □

Then, we can assure that $\{\Omega/\Theta_i\}_{1 \leq i \leq m}$ is a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r \rangle\}_{1 \leq i \leq m}$ of the same Type that $\{S_i\}_{1 \leq i \leq m}$.

Theorem 3.2. *Suppose that $\{S_i\}_{1 \leq i \leq m}$ is a family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r \rangle\}_{1 \leq i \leq m}$. Let B be a boolean subalgebra of $B(P_{(t_i)_{1 \leq i \leq m}})$ and let $\Pi(B) = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of I determined by B. Then*

$$\Omega \simeq_W \prod_{1 \leq j \leq k} \Omega/\Theta_{i_j},$$

where Θ_{i_j} denotes the W-congruence on Ω determined by $i_j \in \Pi_j$, for each j, $1 \leq j \leq k$.

Proof. For each j, $1 \leq j \leq k$, let $q_j : \Omega \longrightarrow \Omega/\Theta_{i_j}$ the natural W-epimorphism. Let $q : \Omega \longrightarrow \prod_{1 \leq j \leq k} \Omega/\Theta_{i_j}$ be defined by $q(x) = (q_1(x), \ldots, q_k(x))$. It is clear that q is a W-homomorphism. If $q(x) = q(y)$ we have that $(x, y) \in \Theta_{i_j}$ for every j, $1 \leq j \leq k$, then $\Pi_j \subseteq \|x = y\|$ for all j, $1 \leq j \leq k$, from we obtain $x = y$. Therefore q is injective. Let $(q_1(x_1), \ldots, q_k(x_k)) \in \prod_{1 \leq j \leq k} \Omega/\Theta_{i_j}$. For each j, $1 \leq j \leq k$, let $a_j \in \prod_{i \in I} S_i$ defined as follows:

$$a_j(i) = \begin{cases} 1 & \text{if } i \in \Pi_j, \\ 0 & \text{if } i \notin \Pi_j. \end{cases}$$

It is clear that $a_j \in \Omega$ for every j, $1 \leq j \leq k$. Let $b_j = x_j \wedge a_j$, for each j, $1 \leq j \leq k$. Since $i_j \in \Pi_j$ it results $b_j(i_j) = x_j(i_j) \wedge a_j(i_j) = x_j(i_j)$, then holds (2) $(b_j, x_j) \in \Theta_{i_j}$ for each j, $1 \leq j \leq k$. Let $b = b_1 \vee \ldots \vee b_k$. If $i \notin \Pi_j$ we have that $b_j(i) = x_j(i) \wedge a_j(i) = 0$. Thus $b(i_j) = b_1(i_j) \vee \ldots \vee b_k(i_j) = b_j(i_j)$, for any j, $1 \leq j \leq k$. Then holds (3) $(b, b_j) \in \Theta_{i_j}$ for all j, $1 \leq j \leq k$. From (2) and (3) we obtain $q(b) = (q_1(b), \ldots, q_k(b)) = (q_1(x_1), \ldots, q_k(x_k))$, hence q is surjective. □

Corollary 3.2. *Let S be a subalgebra of $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$, $B = B(S)$ and $\mathbf{\Pi} = \{\Pi_1, \Pi_2, \ldots, \Pi_k\}$ be the partition of $\{1, 2, \ldots, m\}$ determined by B. Then there are integers v_j, $1 \leq v_j \leq n$, r/v_j, v_j/t_i, for every j, i, $1 \leq j \leq k$, $i \in \Pi_j$, such that $[\{S_i\}_{1\leq i\leq m}, \mathbf{\Pi}]$ is isomorphic to Type 1 algebra $\mathcal{P}_{r,(v_j)_{1\leq j\leq k}}$; or there is $\alpha \neq r$, α/r such that $[\{S_i\}_{1\leq i\leq m}, \mathbf{\Pi}]$ is isomorphic to Type 2 algebra $\mathcal{P}_{\alpha,(v_j)_{1\leq j\leq k}}$ where $v_j = \alpha$ for all $1 \leq j \leq k$.*

Proof. It follows at once from Theorems 3.1 and 3.2. Note that if $\{p_i(S)\}_{1\leq i\leq m}$ is a Type 1 family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ then for each j, $1 \leq j \leq k$, there is v_j, $1 \leq v_j \leq n$ such that r/v_j and $\Omega/\Theta_{i_j} \simeq_W C_{v_j+1}$; while if $\{p_i(S)\}_{1\leq i\leq m}$ is a Type 2 family of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$, then there is $\alpha \neq r$, such that α/r and for each j, $1 \leq j \leq k$, $\Omega/\Theta_{i_j} \simeq_W C_{\alpha+1}$. □

Remark 3.1 Given a partition $\mathbf{\Pi}$ of the set $\{1, 2, \ldots, m\}$, the families $\{\Omega/\Theta_i\}_{1\leq i\leq m}$ and $\{S_i\}_{1\leq i\leq m}$ of subalgebras of $\{\langle C_{t_i+1}, \forall_i^r\rangle\}_{1\leq i\leq m}$ are equivalent in the sense that they determine the same subalgebra of $\mathcal{P}_{r,\{t_i\}_{i\in I}}$.

In what follows we will describe a way to find all the subalgebras of a given algebra $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$. With the current notation for $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$, it is explicit the total number of axes of the algebra, m, and that each axis can be a chain with at most $n+1$ elements that it contains to C_{r+1} as W-subalgebra. We need to change the notation so as to make explicit the number of axes of certain length. Then, instead of $\mathcal{P}_{r,(t_i)_{1\leq i\leq m}}$ we will write

Notation: $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}} = \left\langle P_{(m_i)_{1\leq i\leq n}}, \forall^r \right\rangle$, being now the product $P_{(t_i)_{1\leq i\leq m}}$ denoted by $P_{(m_i)_{1\leq i\leq n}} = \prod\limits_{i=1}^{n} C_{i+1}^{m_i}$, where $m_i \geq 0$, $m = \sum\limits_{i=1}^{n} m_i$ is the total number of axes, and $m_i \neq 0$ implies r divides i, for each $1 \leq i \leq n$.

If S is a Type 1 subalgebra of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$ then we will use the above notation; if S is a Type 2 subalgebra of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$ we will write $S = \mathcal{P}_{\alpha,k} = \langle C^k_{\alpha+1}, \forall^{\alpha}\rangle$, where $1 \leq k \leq m$, α divides r and $\alpha \neq r$.

Determination of the subalgebras of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$

A q-partition of $I = \{1, 2, \ldots, m\}$ is a partition of I with q subsets. It is known that there are $S(m, q)$ q-partitions of I, therefore the number of partitions of I is $\pi(m) = \sum_{q=1}^{m} S(m, q)$, where

$$S(m, q) = \frac{1}{q!} \sum_{j=0}^{q-1} (-1)^j \begin{pmatrix} q \\ j \end{pmatrix} (q - j)^m.$$

For each integer q, $1 \leq q \leq m$, the algebra $B(P_{(m_i)_{1\leq i\leq n}}) \simeq \mathbf{2}^m$ has $S(m, q)$ subalgebras isomorphic to $\mathbf{2}^q$. Since the subalgebras of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$ can be Type 1 or Type 2, we consider the following cases.

(i) Determination of the Type 1 subalgebras of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$

For each i, $1 \leq i \leq n$ such that $m_i \neq 0$, let y_i be the number of Type 1 subalgebras of $\langle C_{i+1}, \forall^r\rangle$; i.e., y_i is the cardinal of the set $\{j : r/j \text{ y } j/i\}$. Consider all the families of subalgebras $\{S_j\}_{1\leq j\leq m}$ integrated by m_i Type 1 subalgebras of $\langle C_{i+1}, \forall^r\rangle$, for each $m_i \neq 0$. This is equivalent to select m_i unordered objects from a set of cardinal y_i with allowed repetitions, that can be made in $\begin{pmatrix} m_i + y_i - 1 \\ y_i - 1 \end{pmatrix}$ different ways. Therefore the number of obtainable families is

$$F = \prod_{\{i:1\leq i\leq n, m_i\neq 0\}} \begin{pmatrix} m_i + y_i - 1 \\ y_i - 1 \end{pmatrix}.$$

If for each family and each partition we consider the corresponding subalgebra, according to Remark 3.1 we could be counting several times the same subalgebra.

Example 3.1. Consider the algebra

$$\mathcal{P}_{2,(0,1,0,2,0,1)} = \left\langle C_{2+1} \times C^2_{4+1} \times C_{6+1}, \forall^2 \right\rangle.$$

In this case $m = 4$, $m_1 = m_3 = m_5 = 0$, $m_2 = m_6 = 1$ and $m_4 = 2$; also $y_2 = 1$, $y_4 = 2$ and $y_6 = 2$. The only subalgebra of $\langle C_{2+1}, \forall^2 \rangle$ is C_{2+1}; subalgebras of $\langle C_{4+1}, \forall^2 \rangle$ are C_{2+1} and C_{4+1}; and subalgebras of $\langle C_{6+1}, \forall^2 \rangle$ are C_{2+1} and C_{6+1}. The number of different families of subalgebras that can be formed is

$$F = \begin{pmatrix} 1+1-1 \\ 1-1 \end{pmatrix} \times \begin{pmatrix} 2+2-1 \\ 2-1 \end{pmatrix} \times \begin{pmatrix} 1+2-1 \\ 2-1 \end{pmatrix} = 6.$$

They are:

$$\begin{array}{lllll} F_1: & C_{2+1}, & C_{2+1}, & C_{2+1}, & C_{2+1}. \\ F_2: & C_{2+1}, & C_{2+1}, & C_{4+1}, & C_{2+1}. \\ F_3: & C_{2+1}, & C_{4+1}, & C_{4+1}, & C_{2+1}. \\ F_4: & C_{2+1}, & C_{2+1}, & C_{2+1}, & C_{6+1}. \\ F_5: & C_{2+1}, & C_{2+1}, & C_{4+1}, & C_{6+1}. \\ F_6: & C_{2+1}, & C_{4+1}, & C_{4+1}, & C_{6+1}. \end{array}$$

Any family that we take with the partition $\Pi_1 = \{\{1, 2, 3, 4\}\}$ give rise to the same subalgebra of $\mathcal{P}_{2,(0,1,0,2,0,1)}$, because anyway, the intersection of the four algebras is C_{2+1}. Thus $[F_1, \Pi_1] = [F_2, \Pi_1] = [F_3, \Pi_1] = [F_4, \Pi_1] = [F_5, \Pi_1] = [F_6, \Pi_1] = \{(0, 0, 0, 0), (\frac{1}{2}, \frac{2}{4}, \frac{2}{4}, \frac{3}{6}), (1, 1, 1, 1)\}$.

We avoid this problem in the following way. Each family contains at most m different subalgebras. For each j, $1 \leq j \leq F$, let λ_{j_h} be the number of times that the algebra C_{h+1} repeats in the family $\mathcal{F}_j$. It is clear that $0 \leq \lambda_{j_h} \leq m$ for every j, $1 \leq j \leq F$ and all h, $1 \leq h \leq n$, and $\sum_{h=1}^{n} \lambda_{j_h} = m$. The number of subalgebras that the family F_j provides is $\prod_{\{i:1 \leq i \leq n, \lambda_{j_i} \neq 0\}} \pi(\lambda_{j_i})$, where $\pi(\lambda_{j_i}) = \sum_{q=1}^{\lambda_{j_i}} S_i(\lambda_{j_i}, q)$ and, for each i, q is the number of axes of the subalgebra isomorphic to C_{i+1}. Therefore, the number of Type 1 subalgebras of $\mathcal{P}_{r,(m_i)_{1 \leq i \leq n}}$ is

$$N_1 = \sum_{j=1}^{F} \prod_{\{i:1 \leq i \leq n, \lambda_{j_i} \neq 0\}} \pi(\lambda_{j_i}) = \sum_{j=1}^{F} \prod_{\{i:1 \leq i \leq n, \lambda_{j_i} \neq 0\}} \left(\sum_{q=1}^{\lambda_{j_i}} S_i(\lambda_{j_i}, q) \right).$$

For each j, $1 \leq j \leq F$, let $I^j = \{i : 1 \leq i \leq n, \lambda_{j_i} \neq 0\}$ and for each $i \in I^j$ let $Q_i^j = \{q : 1 \leq q \leq \lambda_{j_i}\}$; thus applying the distributive property in expression above we obtain

$$\begin{aligned}\prod_{\{i:1\leq i\leq n,\lambda_{j_i}\neq 0\}} \left(\sum_{q=1}^{\lambda_{j_i}} S_i(\lambda_{j_i}, q)\right) &= \prod_{i\in I^j}\left(\sum_{q\in Q_i^j} S_i(\lambda_{j_i}, q)\right) \\ &= \sum_{h\in H}\left(\prod_{i\in I^j} S_i(\lambda_{j_i}, h(i))\right),\end{aligned}$$

where $H = \prod_{i\in I^j} Q_i^j$. Each term of this sum indicates the number of subalgebras obtained from the family F_j isomorphic to $\left\langle \prod_{i\in I^j} C_{i+1}^{h(i)}, \forall^r \right\rangle$, where $\sum_{i\in I^j} h(i)$ is the total number of axes of the subalgebra. Remark that isomorphic algebras can be obtained from different families.

Example 3.2. We will find all the Type 1 subalgebras of

$$\mathcal{P}_{2,(0,1,0,2,0,1)} = \left\langle C_{2+1} \times C_{4+1}^2 \times C_{6+1}, \forall^2 \right\rangle.$$

From Example 3.1 we know that there are 6 families of Type 1 subalgebras. The values of λ_{j_i}, for $1 \leq i, j \leq 6$, are the following:

	λ_{j_1}	λ_{j_2}	λ_{j_3}	λ_{j_4}	λ_{j_5}	λ_{j_6}
F_1:	0	4	0	0	0	0
F_2:	0	3	0	1	0	0
F_3:	0	2	0	2	0	0
F_4:	0	3	0	0	0	1
F_5:	0	2	0	1	0	1
F_6:	0	1	0	2	0	1

Therefore, the number of subalgebras that each family provides is:

F_1: $S_2(4,1) + S_2(4,2) + S_2(4,3) + S_2(4,4)$,

F_2: $(S_2(3,1) + S_2(3,2) + S_2(3,3)) \cdot S_4(1,1)$,

F_3: $(S_2(2,1) + S_2(2,2)) \cdot (S_4(2,1) + S_4(2,2))$,

F_4: $(S_2(3,1) + S_2(3,2) + S_2(3,3)) \cdot S_6(1,1)$,

F_5: $(S_2(2,1) + S_2(2,2)) \cdot S_4(1,1) \cdot S_6(1,1)$,

F_6: $S_2(1,1) \cdot (S_4(2,1) + S_4(2,2)) \cdot S_6(1,1)$.

Applying the distributive property and grouping isomorphic algebras coming from different families, it follows that

$$\mathcal{P}_{2,(0,1,0,2,0,1)} = \left\langle C_{2+1} \times C_{4+1}^2 \times C_{6+1}, \forall^2 \right\rangle$$

has:

1 subalgebra isomorphic to $\left\langle C_{2+1}, \forall^2 \right\rangle$;

7 subalgebras isomorphic to $\left\langle C_{2+1}^2, \forall^2 \right\rangle$;

2 subalgebras isomorphic to $\left\langle C_{2+1} \times C_{4+1}, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1} \times C_{6+1}, \forall^2 \right\rangle$;

6 subalgebras isomorphic to $\left\langle C_{2+1}^3, \forall^2 \right\rangle$;

4 subalgebras isomorphic to $\left\langle C_{2+1}^2 \times C_{4+1}, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1} \times C_{4+1}^2, \forall^2 \right\rangle$;

3 subalgebras isomorphic to $\left\langle C_{2+1}^2 \times C_{6+1}, \forall^2 \right\rangle$;

2 subalgebras isomorphic to $\left\langle C_{2+1} \times C_{4+1} \times C_{6+1}, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1}^4, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1}^3 \times C_{4+1}, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1}^2 \times C_{4+1}^2, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1}^3 \times C_{6+1}, \forall^2 \right\rangle$;

1 subalgebra isomorphic to $\left\langle C_{2+1}^2 \times C_{4+1} \times C_{6+1}, \forall^2 \right\rangle$,

1 subalgebra isomorphic to $\langle C_{2+1} \times C_{4+1}^2 \times C_{6+1}, \forall^2 \rangle$.

(ii) Determination of the Type 2 subalgebras of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$

Let $\mathcal{Z}(r) = \{\alpha/r : \alpha \neq r\}$. Then, from Corollary 3.2, for each $\alpha \in \mathcal{Z}(r)$ there are $S(m,q)$ subalgebras isomorphic to $\mathcal{P}_{\alpha,q} = \langle C_{\alpha+1}^q, \forall^\alpha \rangle$. Hence, the number of Type 2 subalgebras of $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$ is

$$N_2 = \sum_{z=1}^{m} S(m,z)\,|\mathcal{Z}(r)| = |\mathcal{Z}(r)|\,\pi(m).$$

Example 3.3. There are $\pi(4) = 15$ Type 2 subalgebras of

$$\mathcal{P}_{2,(0,1,0,2,0,1)} = \langle C_{2+1} \times C_{4+1}^2 \times C_{6+1}, \forall^2 \rangle$$

because $\mathcal{Z}(2) = \{1\}$; specifically there are $S(4,q)$ subalgebras isomorphic to $\mathcal{P}_{1,q} = \langle C_{1+1}^q, \forall^1 \rangle$ for each q, $1 \leq q \leq 4$.

4. Finitely generated algebras

Lemma 4.1. *The algebras $\mathcal{P}_{r,\{t_i\}_{i\in I}}$ are locally finite.*

Proof. Let $G \subset \mathcal{P}_{r,\{t_i\}_{i\in I}}$ a finite set such that $\mathcal{P}_{r,\{t_i\}_{i\in I}} = [G]_{UW_{n+1}}$. It is easy to see that $\mathcal{P}_{r,\{t_i\}_{i\in I}} = \left[G \cup K\left(\mathcal{P}_{r,\{t_i\}_{i\in I}}\right)\right]_{W_{n+1}}$. Since $K\left(\mathcal{P}_{r,\{t_i\}_{i\in I}}\right)$ is isomorphic to C_{r+1}, the set $G \cup K\left(\mathcal{P}_{r,\{t_i\}_{i\in I}}\right)$ is finite; thus $\mathcal{P}_{r,\{t_i\}_{i\in I}}$ is finite because the W_{n+1}-algebras are locally finite [18]. □

Corollary 4.1. *Every simple finitely generated UW_{n+1}-algebra is finite.*

Theorem 4.1. *Every finitely generated UW_{n+1}-algebra is finite.*

Proof. Let A be a UW_{n+1}-algebra and G be a finite set of generators of A; suppose that the cardinal of G is k. From Theorem 2.2 we have that A is isomorphic to a subalgebra of $\prod_{M\in\mathcal{M}_U(A)} A/M$, where A/M is simple for each $M \in \mathcal{M}_U(A)$. Since A/M is generated by $q_M(G)$, from Corollary 4.1 it results that A/M is finite for each $M \in \mathcal{M}_U(A)$. On the other hand, from Corollary 3.2 we have that (a) there are integers $1 \leq r \leq n$ and $m_i \geq 0$, where for all $1 \leq i \leq n$ it is $m_i = 0$ if r does not divide i, such that

$$A/M \simeq \mathcal{P}_{r,(m_i)_{1\leq i\leq n}} = \left\langle \prod_{i=1}^{n} C_{i+1}^{m_i}, \forall^r \right\rangle,$$

or (b) there are integers $1 \leq r \leq n$ and $1 \leq m$ such that $A/M \simeq \mathcal{P}_{z,m} = \langle C_{z+1}^{m}, \forall^z \rangle$ for some integer z divisor of r, $z \neq r$. In the case (a), if for every $i \neq r$ it is $m_i = 0$ then $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}} = \langle C_{r+1}^{m_r}, \forall^r \rangle$ is an algebra that can be considered together with those of the case (b).

We will prove that $\mathcal{M}_U(A)$ is a finite union of finite sets. Specifically,

$$\mathcal{M}_U(A) = \bigcup_{r=1}^{n} \left(\left(\bigcup_{(m_i)_{1\leq i\leq n} \in S(k,r)} E_{k,r,(m_i)_{1\leq i\leq n}} \right) \bigcup \left(\bigcup_{z/r} \bigcup_{m=1}^{(z+1)^k} E_{k,r,z,m} \right) \right),$$

where

$$E_{k,r,z,m} = \{M \in \mathcal{M}_U(A) : A/M \simeq \langle C_{z+1}^{m}, \forall^z \rangle\},$$

$$E_{k,r,(m_i)_{1\leq i\leq n}} = \{M \in \mathcal{M}_U(A) : A/M \simeq \mathcal{P}_{r,(m_i)_{1\leq i\leq n}}\},$$

and $(m_i)_{1\leq i\leq n} \in S(k,r)$ if and only if it is a solution of system:

$$\begin{cases} x_i = 0 \text{ if } r \text{ does not divide } i, \\ x_i \neq 0 \text{ for some } i \neq r, \\ 0 \leq x_i \leq w_{k+1}(i+1) = (i+1)^{k+1} - \sum\limits_{j/i,\ j\neq i} w_{k+1}(j+1). \end{cases}$$

Let $M \in \mathcal{M}_U(A)$. Suppose that we are in the case (a) and $A/M \simeq \mathcal{P}_{r,(m_i)_{1\leq i\leq n}} \neq \langle C_{r+1}^{m_r}, \forall^r \rangle$, i.e., there is an integer i, $1 \leq i \leq n$, $i \neq r$ such that i is multiple of r and $m_i \neq 0$. Moreover there exists a set of cardinal at most k which generates $\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}$ into the class of UW_{n+1}-algebras. Since $K(\mathcal{P}_{r,(m_i)_{1\leq i\leq n}}) \simeq C_{r+1}$ can be generated as W_{n+1}-algebra by an only element, let g be a generator of $K(\mathcal{P}_{r,(m_i)_{1\leq i\leq n}})$. Then

$$\mathcal{P}_{r,(m_i)_{1\leq i\leq n}} = [G']_{UW_{n+1}} = [G' \cup K(\mathcal{P}_{r,(m_i)_{1\leq i\leq n}})]_{W_{n+1}} = [G' \cup \{g\}]_{W_{n+1}},$$

and from (P3) it must be $m_i \leq w_{k+1}(i+1)$ for every $1 \leq i \leq n$; hence $(m_i)_{1\leq i\leq n} \in S(k,r)$.

In the case (b) we have that A/M is isomorphic to C^m_{z+1} for some z divisor of r. From (P5) it results that

$$C^m_{z+1} = [G']_{UW_{n+1}} = [G' \cup K(C^m_{z+1})]_{W_{n+1}} = [G' \cup K(C^m_{z+1})]_{P_{z+1}} = [G']_{P_{z+1}}.$$

Therefore $m \leq (z+1)^k$ and $M \in E_{k,r,z,m}$.

The sets $E_{k,r,(m_i)_{1\leq i\leq n}}$ and $E_{k,r,z,m}$ are finite (routine). Therefore A is finite.(It easy to see that the sets $E_{k,r,z,m} \neq \emptyset$ but the sets $E_{k,r,(m_i)_{1\leq i\leq n}}$ could be the empty set). □

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